FUNDAMENTALS OF STRUCTURAL ANALYSIS

3rd Edition

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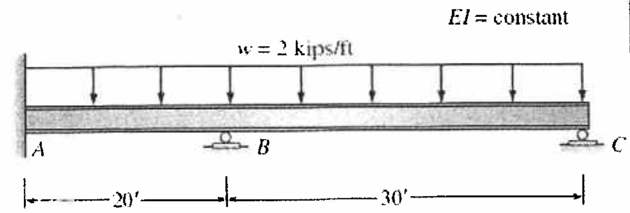
SOLUTIONS MANUAL

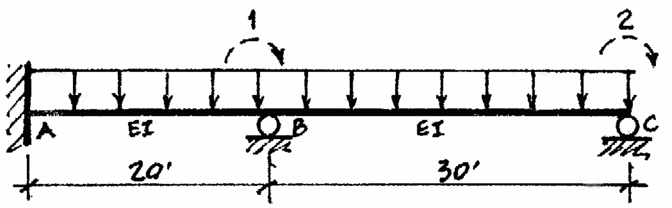
CHAPTER 18:

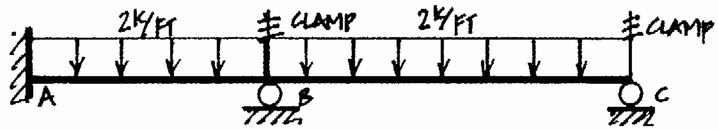
MATRIX ANALYSIS OF BEAMS AND FRAMES BY THE DIRECT STIFFNESS METHOD

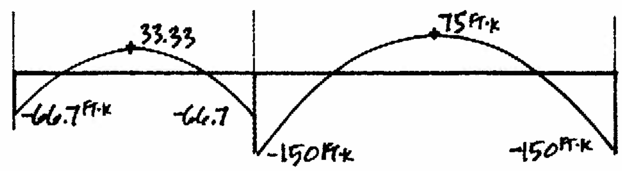
P18.1 Using the stiffness method, analyze the two-span continuous beam shown in Figure P18.1 and draw the shear and moment diagrams. EI is constant.

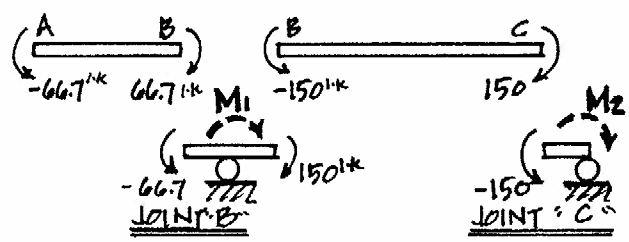


P18.1

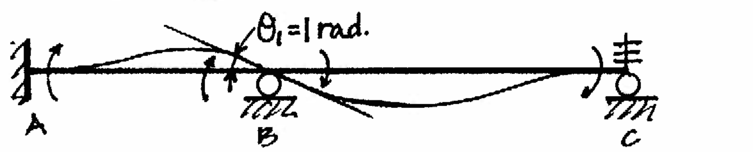


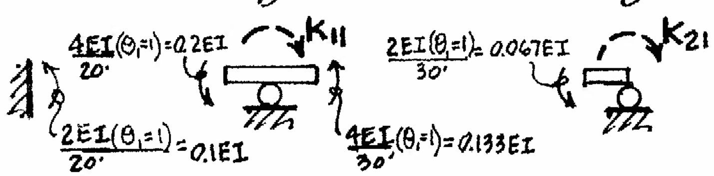




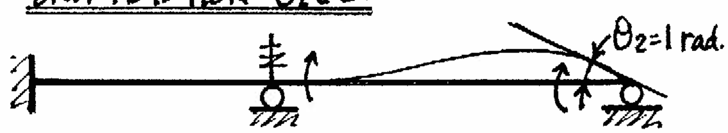


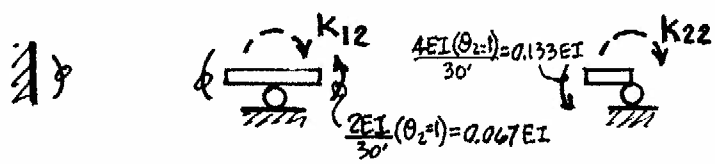
UNIT ROTATION θ_1 @ B:





UNIT ROTATION θ_2 @ C:





RESTRAINING MOMENTS:

JT. B: $M_1 - 66.67 + 150 = 0$

$M_1 = -83.33$ FT·K

JT. C: $M_2 - 150 = 0$

$M_2 = 150$ FT·K

FORCE VECTOR $F = \begin{bmatrix} 83.33 \text{ FT·K} \\ -150 \text{ FT·K} \end{bmatrix}$

STIFFNESS MATRIX:

UNIT ROTATION @ B:

JOINT "B": $\Sigma M_B = 0$;

$K_{11} - 0.2EI - 0.133EI = 0$

$K_{11} = 0.333EI$

JOINT "C": $\Sigma M_C = 0$;

$K_{21} - 0.067EI = 0$

$K_{21} = 0.067EI$

UNIT ROTATION @ C:

JOINT "B": $\Sigma M_B = 0$;

$K_{12} - 0.067EI = 0$

$K_{12} = 0.067EI$

JOINT "C": $\Sigma M_C = 0$;

$K_{22} - 0.133EI = 0$

$K_{22} = 0.133EI$

$$K = EI \begin{bmatrix} 0.33 & 0.067 \\ 0.067 & 0.133 \end{bmatrix}$$

P18.1 Continued

P18.1 Continued

$K \cdot \Delta = F$

$$\begin{bmatrix} 0.33 & 0.067 \\ 0.067 & 0.133 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 83.33 \\ -150 \end{bmatrix}$$

$$K^{-1} = \frac{1}{0.33(0.133) - (0.067)^2} \begin{bmatrix} 0.133 & -0.067 \\ -0.067 & 0.33 \end{bmatrix} = 25 \begin{bmatrix} 0.133 & -0.067 \\ -0.067 & 0.33 \end{bmatrix}$$

THUS $$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \frac{25}{EI} \begin{bmatrix} 0.133 & -0.067 \\ -0.067 & 0.33 \end{bmatrix} \begin{bmatrix} 83.33 \\ -150 \end{bmatrix} = \begin{bmatrix} 527.8/EI \\ -1388.9/EI \end{bmatrix}$$

SPAN "BC" END MOMENTS:

$M_{BC} = M'_{BC} + M''_{BC} = -150 - 22.2 \qquad = \underline{-172.2 \text{ FT·K}}$

WHERE $M'_{BC} = -150 \text{ FT·K}$

AND $M''_{BC} = \theta_1 (0.133 EI) + \theta_2 (0.067 EI) = -22.2 \text{ FT·K}$

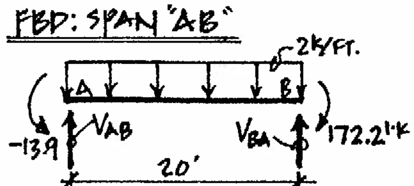
$M_{CB} = M'_{CB} + M''_{CB} = +150 + (\theta_1 (0.067 EI) + \theta_2 (0.133 EI)) = \underline{0}$

SPAN "AB" END MOMENTS:

$M_{AB} = M'_{AB} + M''_{AB} = (-66.67) + (\theta_1 (0.1 EI) + \theta_2 (0)) \qquad = \underline{-13.9 \text{ FT·K}}$

$M_{BA} = M'_{BA} + M''_{BA} = (66.67) + (\theta_1 (0.2 EI) + \theta_2 (0)) \qquad = \underline{172.2 \text{ FT·K}}$

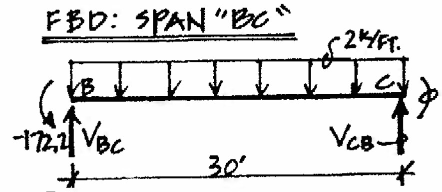
FBD: SPAN "AB"



$\Sigma M_A = 0; \ -13.9 + 172.2 - V_{BA}(20) + 2(20)(10) = 0$

$\underline{V_{BA} = 27.92^K \uparrow}$ & $\underline{V_{AB} = 12.08^K \uparrow}$

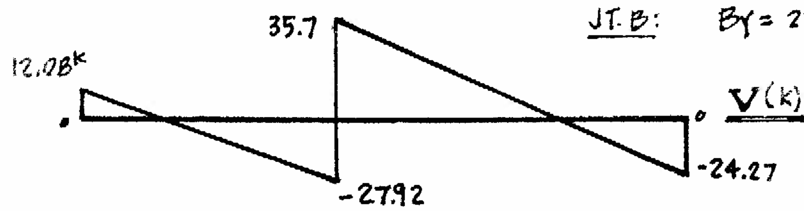
FBD: SPAN "BC"

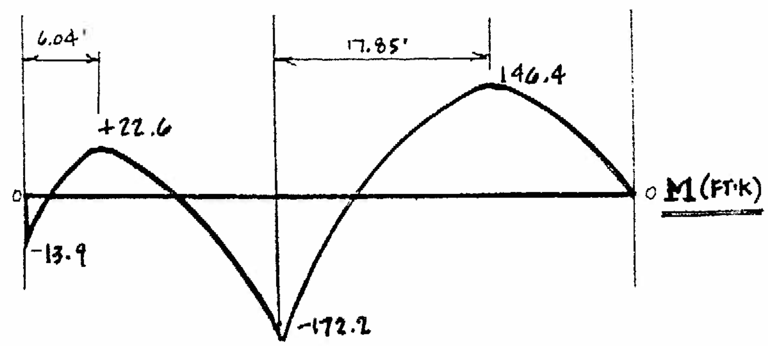


$\Sigma M_B = 0; \ -172.2 + 2(30)(15) - V_{CB}(30) = 0$

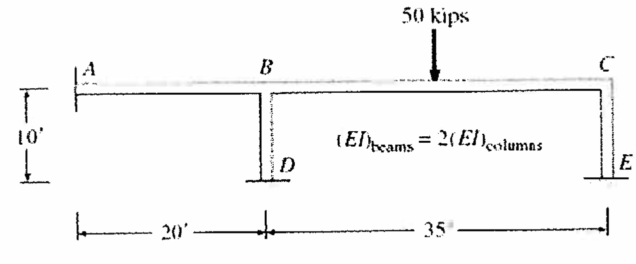
$\underline{V_{CB} = 24.26^K \uparrow}$ & $\underline{V_{BC} = 35.74^K \uparrow}$

JT. B: $B_Y = 27.92 + 35.74 = \underline{63.66^K} \uparrow$

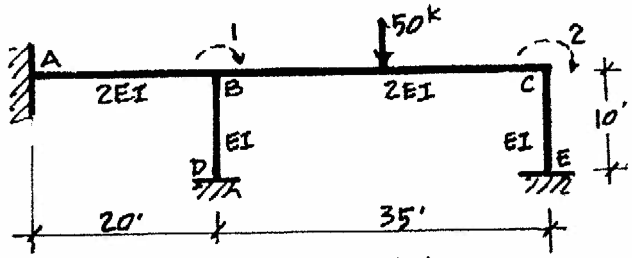


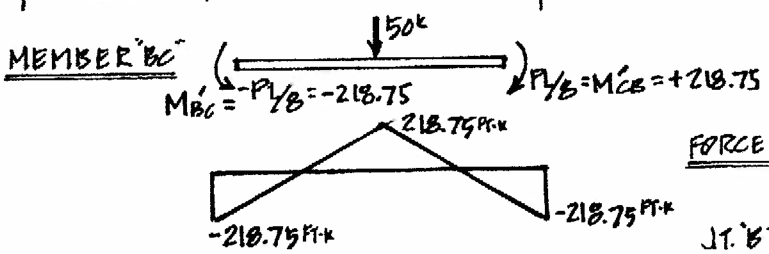


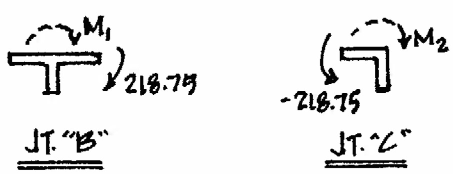
P18.2 Neglecting axial deformations, find the end moments in the frame shown in Figure P18.2.



P18.2







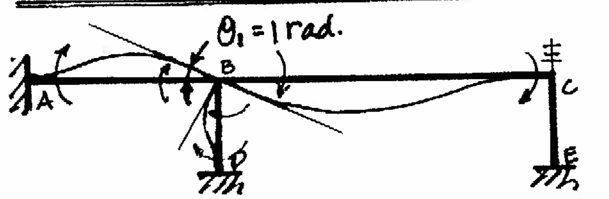
FORCE VECTOR F:

JT. "B": $M_1 + 218.75 = 0$ $\qquad M_1 = -218.75^{1\cdot K}$

JT. "C": $M_2 - 218.75 = 0$ $\qquad M_2 = +218.75^{1\cdot K}$

$$[F] = \begin{bmatrix} +218.75 \\ -218.75 \end{bmatrix}$$

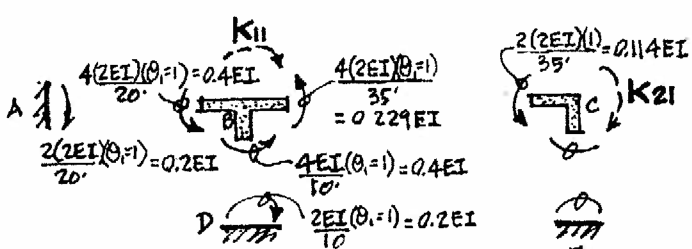
STIFFNESS MATRIX: UNIT ROTATION @ JT. "B" $\theta_1 = 1$



$\Sigma M_B = 0$ @ JT. "B":

$K_{11} - 0.4EI - 0.4EI - 0.229EI = 0$

$K_{11} = 1.03EI$

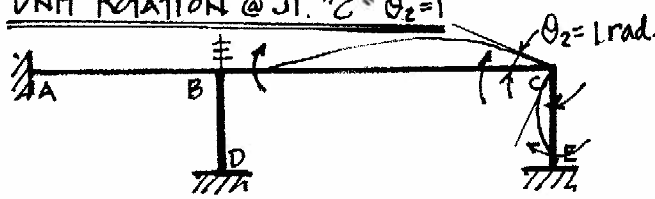


$\Sigma M_C = 0$ @ JT. "C":

$K_{21} - 0.114EI = 0$

$K_{21} = 0.114EI$

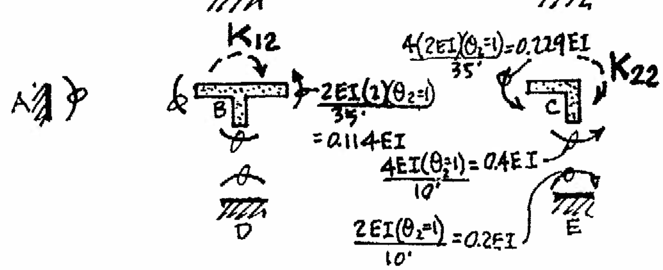
UNIT ROTATION @ JT. "C" $\theta_2 = 1$



$\Sigma M_B = 0$ @ JT. "B":

$K_{12} - 0.114EI = 0$

$K_{12} = 0.114EI$



$\Sigma M_C = 0$ @ JT. "C":

$K_{22} - 0.229EI - 0.4EI = 0$

$K_{22} = 0.629EI$

P18.2 Continued

P18.2 Continued

$$K = \begin{bmatrix} 1.03 & 0.114 \\ 0.114 & 0.629 \end{bmatrix} EI$$

$\underline{K\Delta = F}$

$$K^{-1} = \frac{1}{1.03(0.629) - (0.114)^2} \begin{bmatrix} 0.629 & -0.114 \\ -0.114 & 1.03 \end{bmatrix} = \frac{1.575}{EI} \begin{bmatrix} 0.629 & -0.114 \\ -0.114 & 1.03 \end{bmatrix}$$

$$KK^{-1}\Delta = K^{-1}F$$

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \frac{1.575}{EI} \begin{bmatrix} 0.629 & -0.114 \\ -0.114 & 1.03 \end{bmatrix} \begin{bmatrix} 218.75 \\ -218.75 \end{bmatrix} = \begin{bmatrix} 256/EI \\ -394.1/EI \end{bmatrix}$$

SPAN "BC" END MOMENTS:

$M_{BC} = M'_{BC} + M''_{BC} = -218.75 + 13.7 = \underline{\underline{-205.1 \text{ FT·K}}}$

WHERE $M'_{BC} = -218.75$ FT·K

AND $M''_{BC} = \theta_1(0.229EI) + \theta_2(0.114EI) = 13.68$ FT·K

$M_{CB} = M'_{CB} + M''_{CB} = 218.75 - 61.06 = \underline{\underline{157.7 \text{ FT·K}}}$

WHERE $M'_{CB} = +218.75$

AND $M''_{CB} = \theta_1(0.114EI) + \theta_2(0.229EI) = -61.06$ FT·K

SPAN "AB" END MOMENTS:

$M_{AB} = \cancel{M'_{AB}}^{\,0} + M''_{AB} = 0 + \theta_1(0.2EI) + \theta_2(0) = \underline{\underline{51.2 \text{ FT·K}}}$

$M_{BA} = \cancel{M'_{BA}}^{\,0} + M''_{BA} = 0 + \theta_1(0.4EI) + \theta_2(0) = \underline{\underline{102.5 \text{ FT·K}}}$

SPAN "DB" COLUMN END MOMENTS:

$M_{BD} = \cancel{M'_{BD}}^{\,0} + M''_{BD} = 0 + \theta_1(0.4EI) + \theta_2(0) = \underline{\underline{102.5 \text{ FT·K}}}$

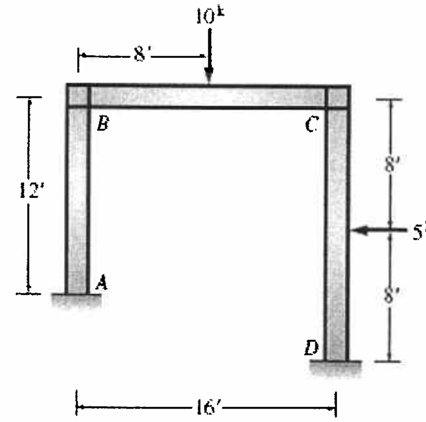
$M_{DB} = \cancel{M'_{DB}}^{\,0} + M''_{DB} = 0 + \theta_1(0.2EI) + \theta_2(0) = \underline{\underline{51.2 \text{ FT·K}}}$

SPAN "CE" COLUMN END MOMENTS:

$M_{CE} = \cancel{M'_{CE}}^{\,0} + M''_{CE} = \theta_1(0) + \theta_2(0.4EI) = \underline{\underline{-157.6 \text{ FT·K}}}$

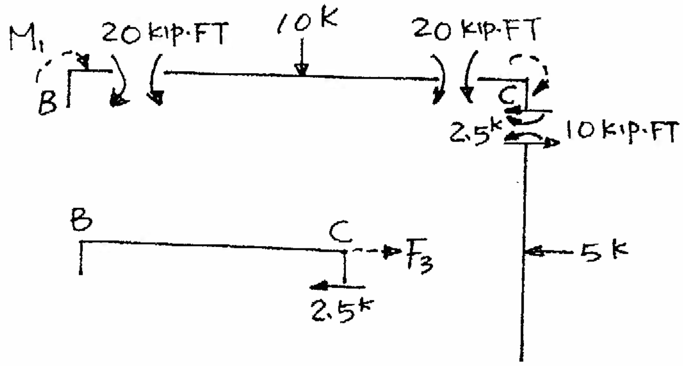
$M_{EC} = \cancel{M'_{EC}}^{\,0} + M''_{EC} = \theta_1(0) + \theta_2(0.2EI) = \underline{\underline{-78.8 \text{ FT·K}}}$

P18.3 Using the stiffness method, analyze the frame in Figure P18.3 and draw the shear and moment diagrams for the members. Neglect axial deformations. *EI* is constant.



P18.3

FORCE VECTOR

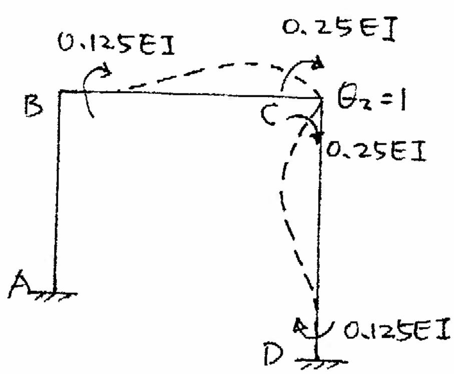


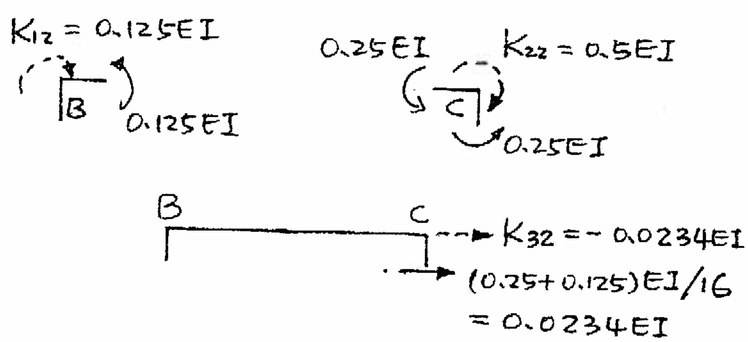
JT. "B"	JT. "C"	FREEBODY "BC"
$M_1 + 20 = 0$	$M_2 - 20 + 10 = 0$	$F_3 - 2.5 = 0$
$M_1 = -20$	$M_2 = 10$	$F_3 = 2.5^k$

$$[F] = \begin{bmatrix} 20 \\ -10 \\ -2.5 \end{bmatrix}$$

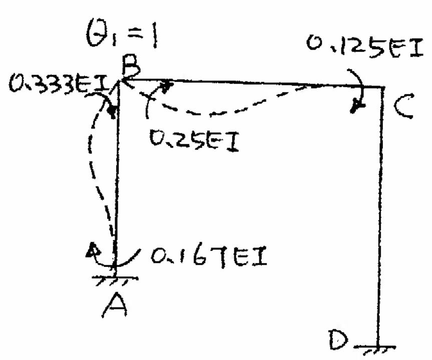
STIFFNESS MATRIX

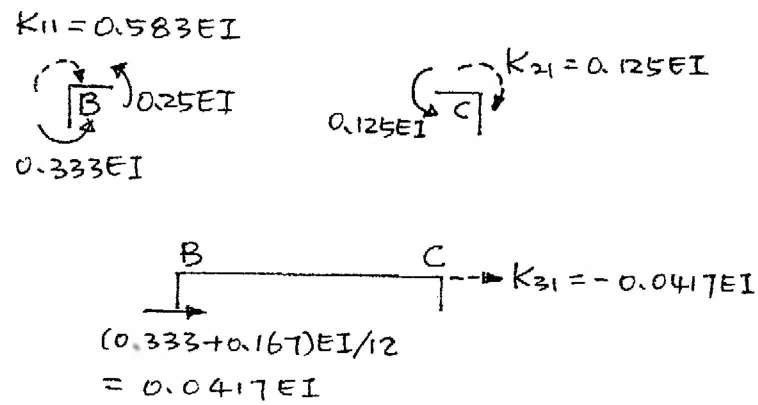
UNIT ROTATION AT JT. "C"





UNIT ROTATION AT JT. "B"

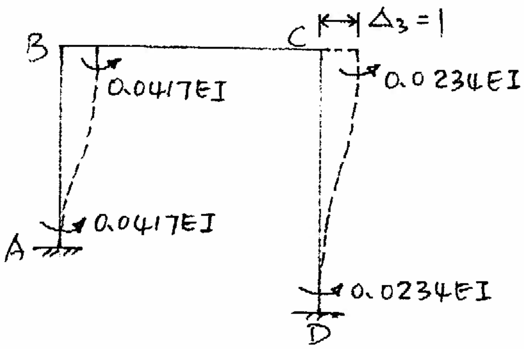




P18.3 Continued

P18.3 Continued

UNIT HORIZ. DISPL. AT JT. "C"

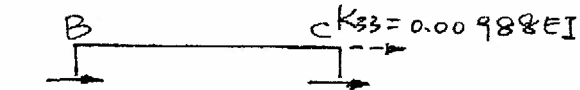


$K_{13} = -0.0417EI$

0.0417EI

$K_{23} = -0.0234EI$

0.0234EI



$2(0.417EI)/12 = 0.00695EI$ $\quad$ $2(0.0234EI)/16 = 0.00293EI$

$$\therefore [K] = EI\begin{bmatrix} 0.583 & 0.125 & -0.0417 \\ 0.125 & 0.5 & -0.0234 \\ -0.0417 & -0.0234 & 0.00988 \end{bmatrix}$$

SOLVE $[K][\Delta] = [F]$:
$$\begin{bmatrix} \theta_1 \\ \theta_2 \\ \Delta_3 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 25.62 \\ -37.32 \\ -233.30 \end{bmatrix}$$

EFFECTS OF JOINT DISPLACEMENTS $\qquad$ EQ. 18.36

$$\begin{bmatrix} M_A'' \\ M_B'' \\ V_A'' \\ V_B'' \end{bmatrix} = \frac{2EI}{12}\begin{bmatrix} 2 & 1 & 1/4 & -1/4 \\ 1 & 2 & 1/4 & -1/4 \\ 1/4 & 1/4 & 1/24 & -1/24 \\ -1/4 & -1/4 & -1/24 & 1/24 \end{bmatrix}\begin{bmatrix} \theta_A = 0 \\ \theta_B = 25.62/EI \\ \Delta_A = 0 \\ \Delta_B = -233.30/EI \end{bmatrix} = \begin{bmatrix} 14.0 \\ 18.3 \\ 2.7 \\ -2.7 \end{bmatrix}$$ (FOR MEMBER AB)

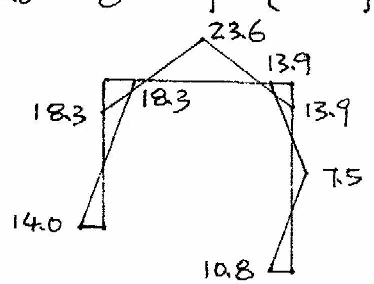
$$\begin{bmatrix} M_B'' \\ M_C'' \\ V_B'' \\ V_C'' \end{bmatrix} = \frac{2EI}{16}\begin{bmatrix} 2 & 1 & 3/16 & -3/16 \\ 1 & 2 & 3/16 & -3/16 \\ 3/16 & 3/16 & 3/128 & -3/128 \\ -3/16 & -3/16 & -3/128 & 3/128 \end{bmatrix}\begin{bmatrix} \theta_B = 25.62/EI \\ \theta_C = -37.32/EI \\ \Delta_B = 0 \\ \Delta_C = 0 \end{bmatrix} = \begin{bmatrix} 1.7 \\ -6.1 \\ -0.3 \\ 0.3 \end{bmatrix}$$ (FOR MEMBER BC)

$$\begin{bmatrix} M_C'' \\ M_D'' \\ V_C'' \\ V_D'' \end{bmatrix} = \frac{2EI}{16}\begin{bmatrix} 2 & 1 & 3/16 & -3/16 \\ 1 & 2 & 3/16 & -3/16 \\ 3/16 & 3/16 & 3/128 & -3/128 \\ -3/16 & -3/16 & -3/128 & 3/128 \end{bmatrix}\begin{bmatrix} \theta_C = -37.32/EI \\ \theta_D = 0 \\ \Delta_C = 233.30/EI \\ \Delta_D = 0 \end{bmatrix} = \begin{bmatrix} -3.9 \\ 0.8 \\ -0.2 \\ 0.2 \end{bmatrix}$$ (FOR MEMBER CD)

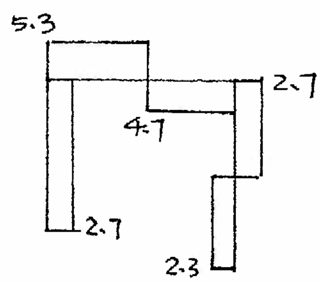
NOTE: Δ_C IS POSITIVE HERE

FINAL RESULTS

$M_{AB} = 0 + 14.0 = 14.0$ kip·FT
$M_{BA} = 0 + 18.3 = 18.3$
$M_{BC} = -20 + 1.7 = -18.3$
$M_{CB} = 20 - 6.1 = 13.9$
$M_{CD} = -10 - 3.9 = -13.9$
$M_{DC} = 10 + 0.8 = 10.8$

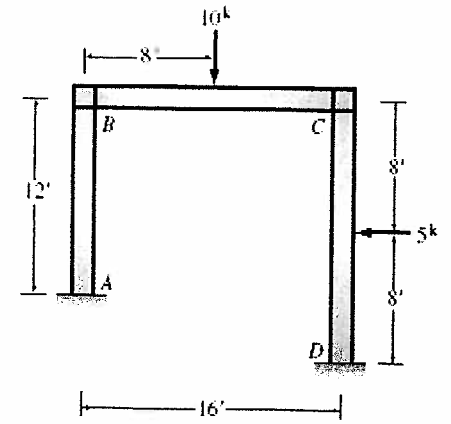


M DIAGRAM (FT·K)

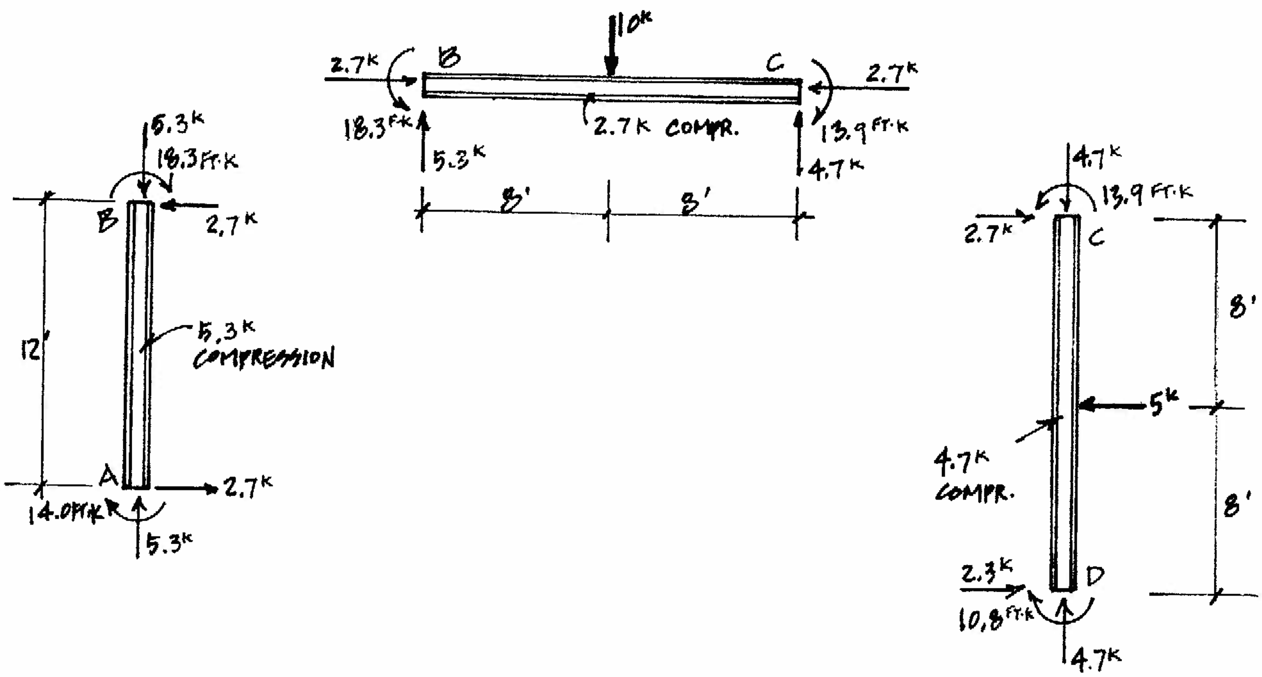


V DIAGRAM (KIPS)

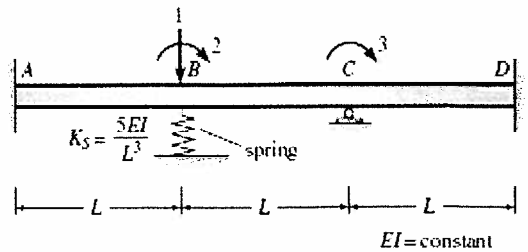
P18.4 Using the solution of Problem P18.3, calculate the axial forces in the members of the frame. (Use freebody diagrams that relate axial loads in one member with shears in another.)



P18.3



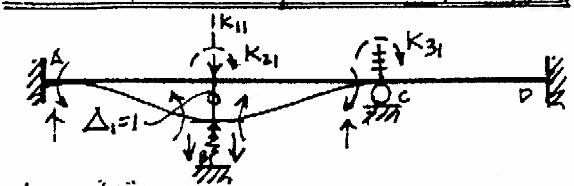
P18.5 Write the stiffness matrix corresponding to the degrees of freedom 1, 2, and 3 of the continuous beam shown in Figure P18.5.



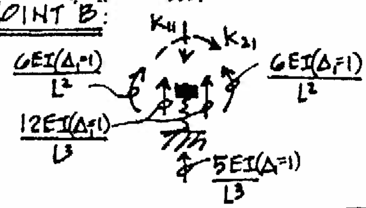
P18.5

STIFFNESS MATRIX K

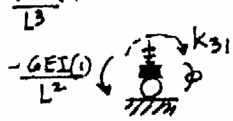
UNIT VERTICAL DISPLACEMENT $\Delta_1 = 1$



JOINT "B":



JOINT "C":



JOINT "B":

$+\uparrow\Sigma F_Y = 0;\quad \frac{2(12EI)}{L^3} + \frac{5EI}{L^3} - K_{11} = 0$

$K_{11} = \frac{29EI}{L^3}$

$\Sigma M_B = 0;\quad -K_{21} - 6EI/L^2 + 6EI/L^2 = 0$

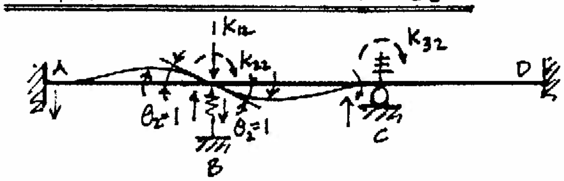
$K_{21} = 0$

JOINT "C":

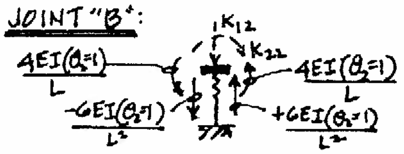
$\Sigma M_C = 0;\quad K_{31} - 6EI/L^2 = 0$

$K_{31} = 6EI/L^2$

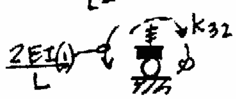
UNIT ROTATION @ JOINT "B" $\theta_2 = 1$



JOINT "B":



JOINT "C":



JOINT "B":

$+\uparrow\Sigma F_Y = 0;\quad -K_{12} - \frac{6EI}{L^2} + \frac{6EI}{L^2} = 0$

$K_{12} = 0$

$\Sigma M_B = 0;\quad K_{22} - 2(4EI)/L = 0$

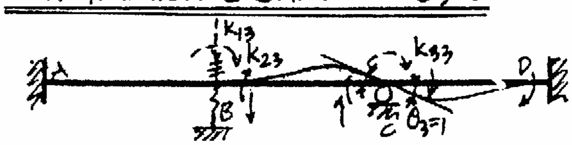
$K_{22} = 8EI/L$

JOINT "C":

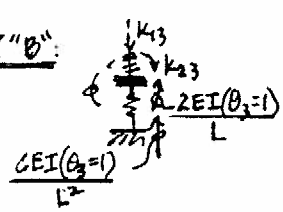
$\Sigma M_C = 0;\quad K_{32} - 2EI/L = 0$

$K_{32} = 2EI/L$

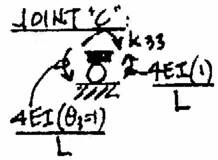
UNIT ROTATION @ JOINT "C" $\theta_3 = 1$



JOINT "B":



JOINT "C":



JOINT "B"

$+\uparrow\Sigma F_Y = 0;\quad -K_{13} + 6EI/L^2 = 0$

$K_{13} = 6EI/L^2$

$\Sigma M_B = 0;\quad K_{23} - 2EI/L$

$K_{23} = 2EI/L$

JOINT "C":

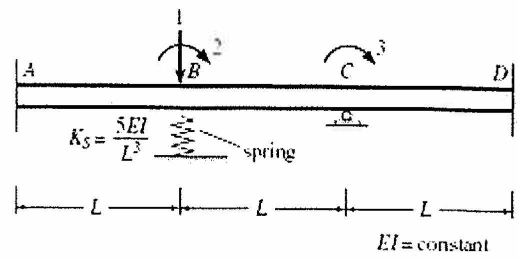
$\Sigma M_C = 0;\quad K_{33} - 2(4EI)/L = 0$

$K_{33} = 8EI/L$

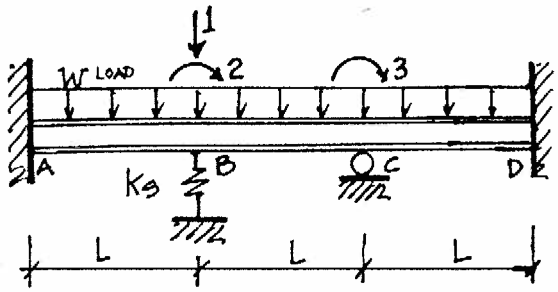
STIFFNESS MATRIX:

$$K = \frac{EI}{L}\begin{bmatrix} 29/L^2 & 0 & 6/L \\ 0 & 8 & 2 \\ 6/L & 2 & 8 \end{bmatrix}$$

P18.6 In Problem P18.5, find the force in the spring located at B if beam $ABCD$ supports a downward uniform load w along its entire length.



P18.5



[K] FROM 18.5

$$\{F\} = \begin{Bmatrix} -WL \\ 0 \\ 0 \end{Bmatrix}$$

$$\frac{EI}{L}\left[\begin{array}{c|cc} 29/L^2 & 0 & \frac{6}{L} \\ \hline 0 & 8 & 2 \\ \frac{6}{L} & 2 & 8 \end{array}\right] \begin{Bmatrix} \Delta \\ \hline \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} WL \\ \hline 0 \\ 0 \end{Bmatrix}$$

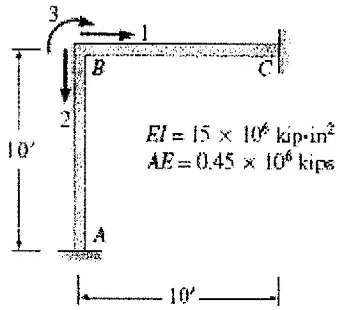
$$\begin{Bmatrix} 0 \\ 6/L \end{Bmatrix} \Delta + \begin{bmatrix} 8 & 2 \\ 2 & 8 \end{bmatrix} \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{Bmatrix} \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 5/24 \\ -5/6 \end{Bmatrix} \frac{\Delta}{L}$$

$$EI \frac{29}{L^3} \Delta + \begin{bmatrix} 0 & \frac{6}{L} \end{bmatrix} \begin{Bmatrix} 5/24 \\ -5/6 \end{Bmatrix} \frac{\Delta}{L} \frac{EI}{L} = \{WL\}$$

$\Delta = 0.042\ WL^4/EI$

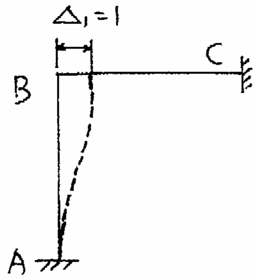
FORCE IN SPRING $= K_S \Delta = 0.208\ WL$

P18.7 For the frame shown in Figure P18.7, write the stiffness matrix in terms of the 3 degrees of freedom indicated. Use both the method of introducing unit displacements and the member stiffness matrix of Equation 18.36.

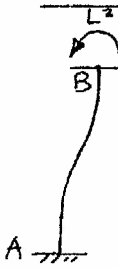


P18.7

$\Delta_1 = 1$

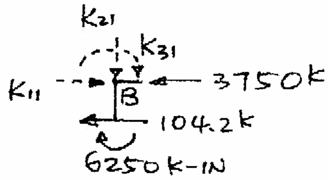


$\frac{6EI(1)}{L^2} = \frac{6(15\times10^6)}{(10\times12)^2} = 6250$ k-in



$\frac{12EI(1)}{L^3} = \frac{12(15\times10^6)}{(10\times12)^3} = 104.2$ k

$\frac{EA(1)}{L} = \frac{(0.45\times10^6)(1)}{10\times12} = 3750^k$

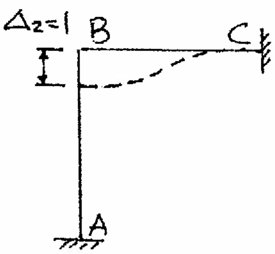


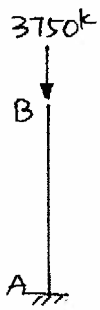
$K_{11} = 3750 + 104.2 = 3854.2$

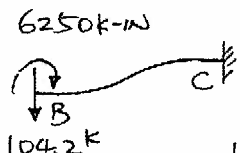
$K_{21} = 0$

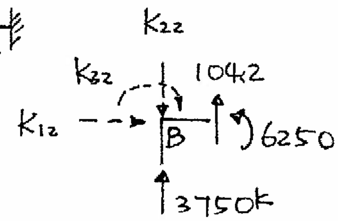
$K_{31} = -6250$

$\Delta_2 = 1$







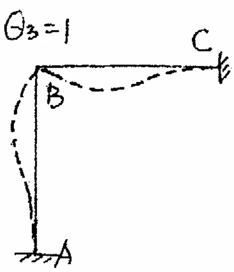


$K_{12} = 0$

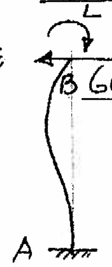
$K_{22} = 3750 + 104.2 = 3854.2$

$K_{32} = 6250$

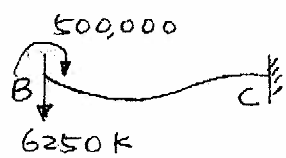
$\theta_3 = 1$

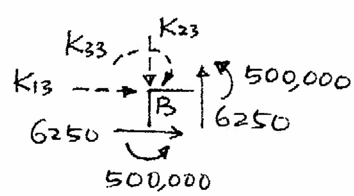


$\frac{4EI(1)}{L} = \frac{4(15\times10^6)}{10\times12} = 500{,}000$ k-in



$\frac{6EI(1)}{L^2} = \frac{6(15\times10^6)}{(10\times12)^2} = 6250$ k





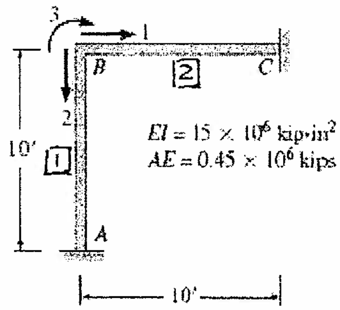
$K_{13} = -6250$

$K_{23} = 6250$

$K_{33} = 500{,}000 + 500{,}000 = 1{,}000{,}000$

$$[K] = \begin{bmatrix} 3854.2 & 0 & -6250 \\ 0 & 3854.2 & 6250 \\ -6250 & 6250 & 1{,}000{,}000 \end{bmatrix}$$

P18.8 Solve Problem P18.7 using the direct summation of global element stiffness matrices.



P18.7

FOR MEMBERS 1 AND 2:

$$N = \frac{A}{I} = \frac{EA}{EI} = \frac{0.45\times10^6}{15\times10^6} = 0.03 \text{ IN}^{-2}$$

$$P = \frac{12}{L^2} = \frac{12}{(10\times12)^2} = 8.33\times10^{-4} \text{ IN}^{-2}$$

$$Q = \frac{6}{L} = \frac{6}{10\times12} = 0.05 \text{ IN}^{-1}$$

$$\frac{EI}{L} = \frac{15\times10^6}{10\times12} = 125{,}000 \text{ KIP-IN}$$

FOR MEMBER 1 $\quad \phi = 90°, \quad s = \sin\phi = 1, \quad c = \cos\phi = 0$

$$[k_F^1] = \frac{EI}{L}\begin{bmatrix} Nc^2+Ps^2 & sc(-N+P) & -Qs \\ & Ns^2+Pc^2 & -Qc \\ \text{SYM.} & & 4 \end{bmatrix}$$

$$= 125{,}000\begin{bmatrix} 8.33\times10^{-4} & 0 & -0.05 \\ 0 & 0.03 & 0 \\ -0.05 & 0 & 4 \end{bmatrix}$$

FOR MEMBER 2 $\quad \phi = 0°, \quad s = \sin\phi = 0, \quad c = \cos\phi = 1$

$$[k_N^2] = 125{,}000\begin{bmatrix} 0.03 & 0 & 0 \\ 0 & 8.33\times10^{-4} & -0.05 \\ 0 & -0.05 & 4 \end{bmatrix}$$

$$[K] = [k_F^1] + [k_N^2]$$

$$= \begin{bmatrix} 3854.1 & 0 & -6250 \\ 0 & 3854.1 & -6250 \\ -6250 & -6250 & 1{,}000{,}000 \end{bmatrix}$$