Curve Modeling
B-Spline Curves

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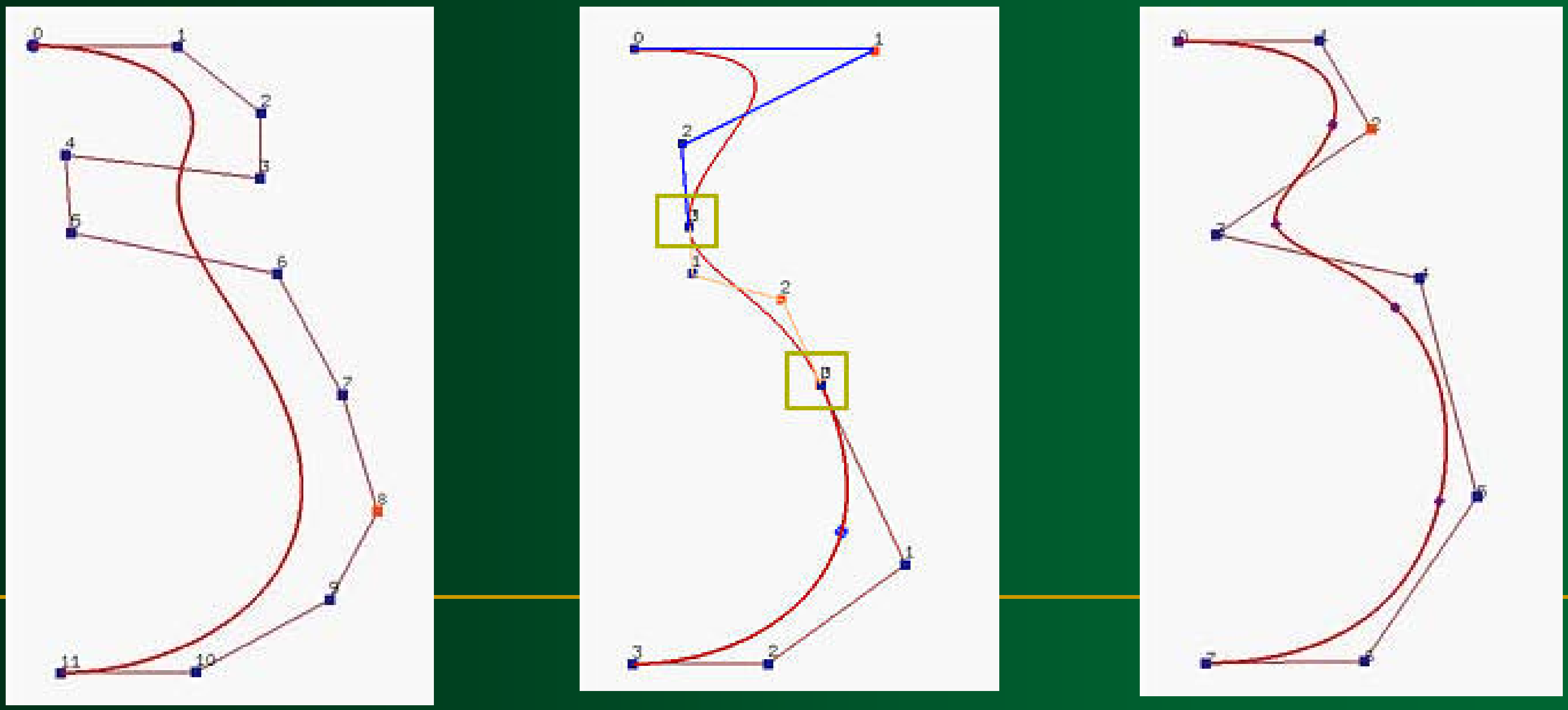
Motivation

B-Spline Basis: Motivation

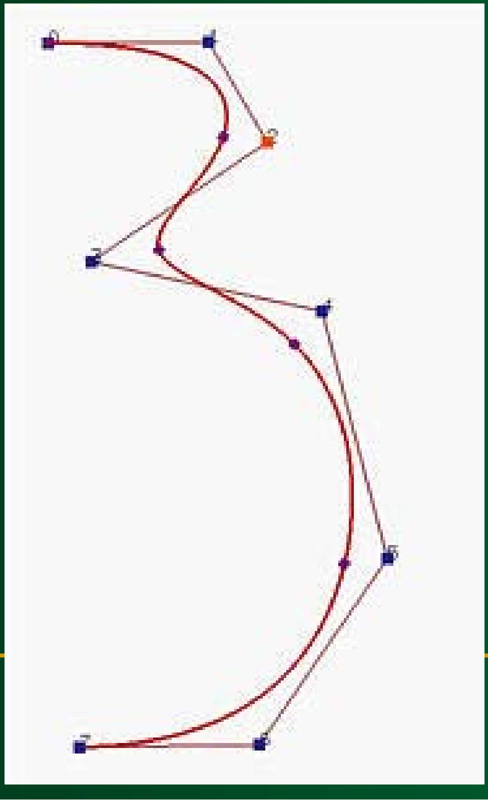
Consider designing the profile of a vase.

- The left figure below is a Bézier curve of degree 11; but, it is difficult to bend the "neck" toward the line segment **P4P5**.
- The middle figure above uses this idea. It has three Bézier curve segments of degree 3 with joining points marked with yellow rectangles.
- The right figure above is a B-spline curve of degree 3 defined by 8 control points .



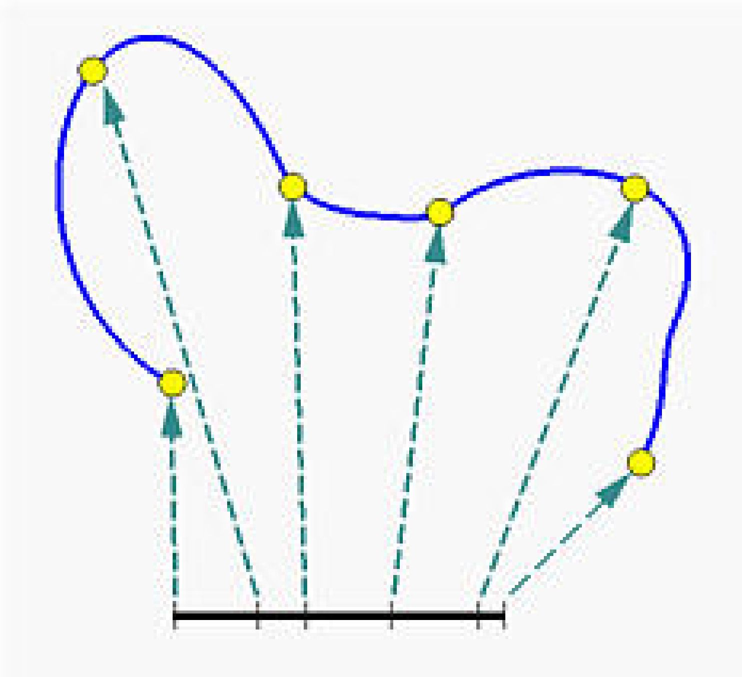
B-Spline Basis: Motivation

- Those little dots subdivide the B-spline curve into curve segments.
- One can move control points for modifying the shape of the curve just like what we do to Bézier curves.
- We can also modify the subdivision of the curve. Therefore, B-spline curves have higher degree of freedom for curve design.

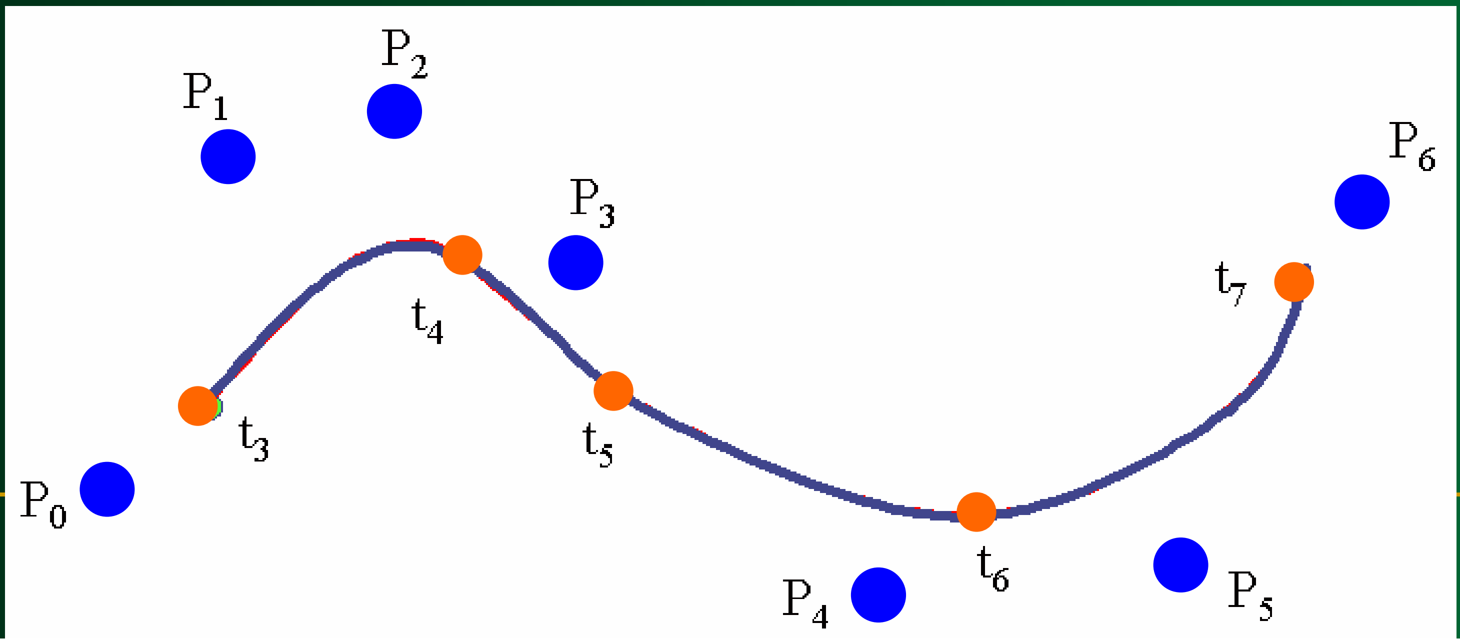


B-Spline Basis: Motivation

- Subdividing the curve directly is difficult to do. Instead, we subdivide the domain of the curve.
- The domain of a curve is [0,1], this closed interval is **subdivided** by points called ***knots***.
- These knots be $\mathbf{0 <= u_0 <= u_1 <= \ldots <= u_m <= 1}$.
- Modifying the subdivision of [0,1] changes the shape of the curve.



- **In summary**: to design a B-spline curve, we need a set of control points, a set of knots and a set of coefficients, one for each control point, so that all curve segments are joined together satisfying certain continuity condition.



B-Spline Basis: Motivation

- The computation of the coefficients is perhaps the most complex step because they must ensure certain continuity conditions.

B-Spline Curves

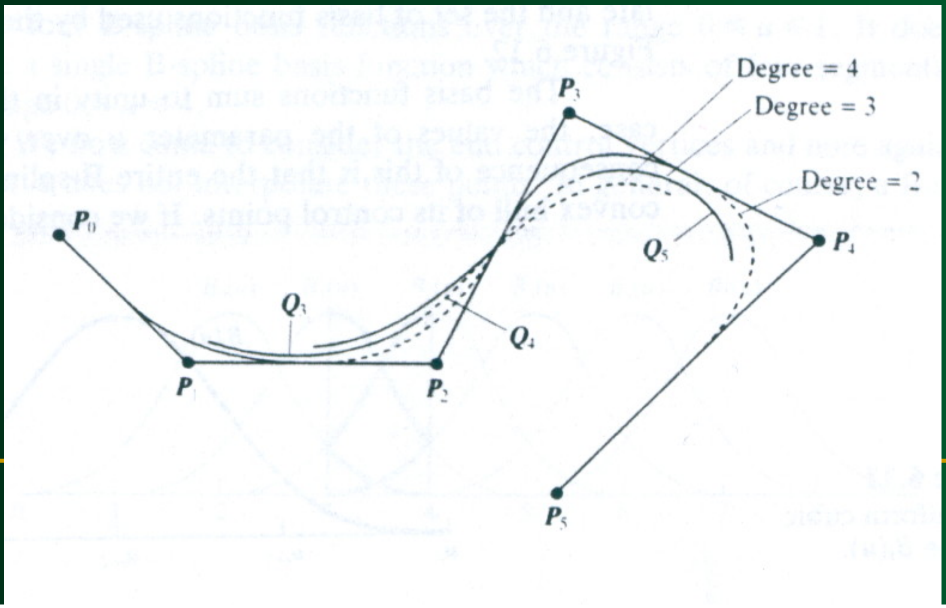
B-Spline Curves
(Two Advantages)

1. The degree of a B-spline polynmial can be set independently of the number of control points.

2. B-splines allow local control over the shape of a spline curve (or surface)

B-Spline Curves
(Two Advantages)

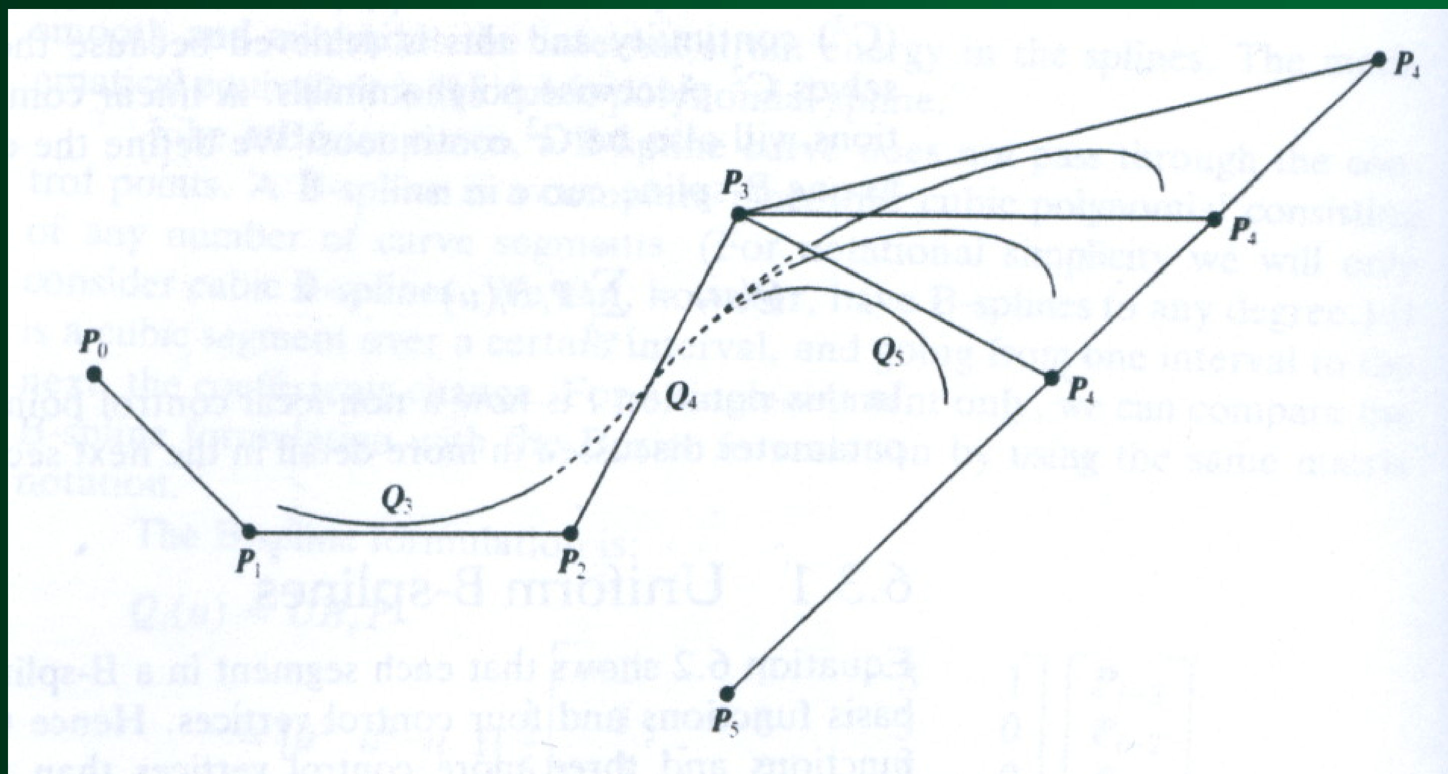
- A B-spline curve that is defined by 6 control point, and shows the effect of varying the degree of the polynomials (2,3, and 4)
- Q_3 is defined by P_0,P_1,P_2,P_3
- Q_4 is defined by P_1,P_2,P_3,P_4
- Q_5 is defined by P_2,P_3,P_4,P_5



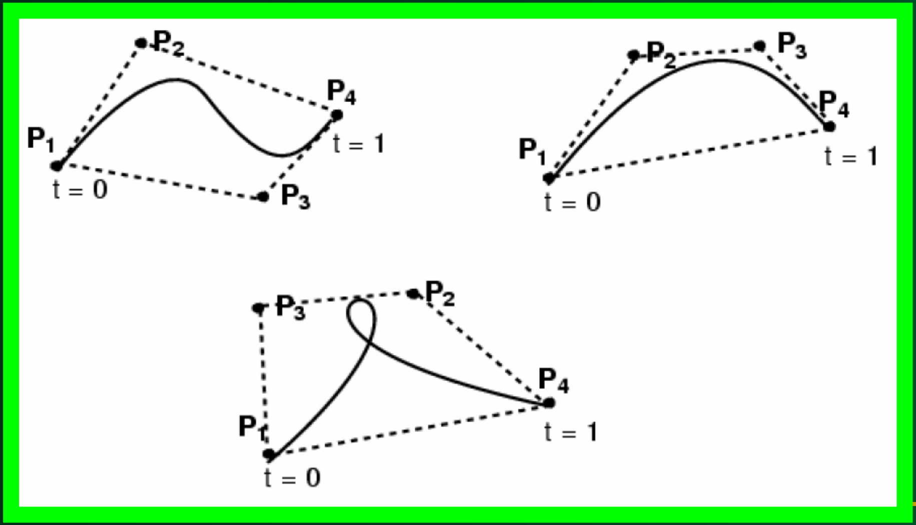
- Each curve segment shares control points.

B-Spline Curves
(Two Advantages)

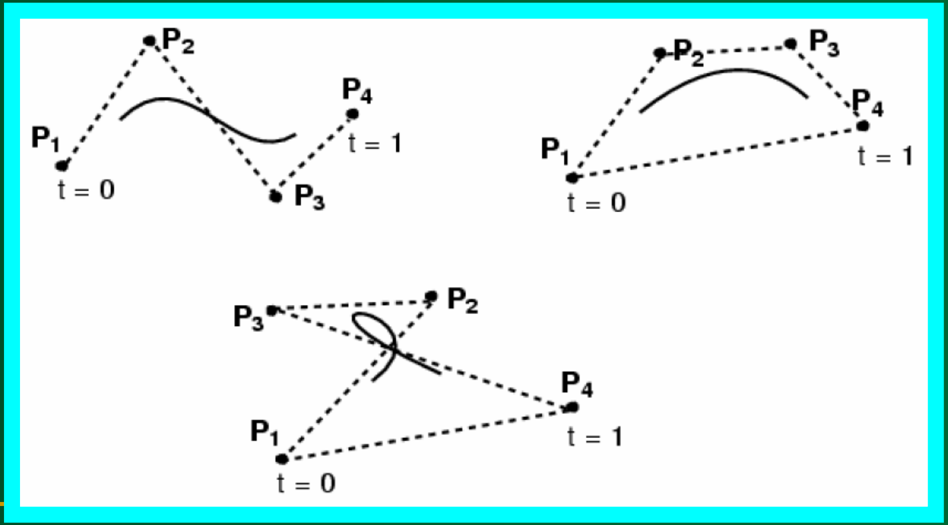
- The effect of changing the position of control point **P_4 (locality property)**.



B-Spline Curves



Bézier Curve



B-Spline Curve

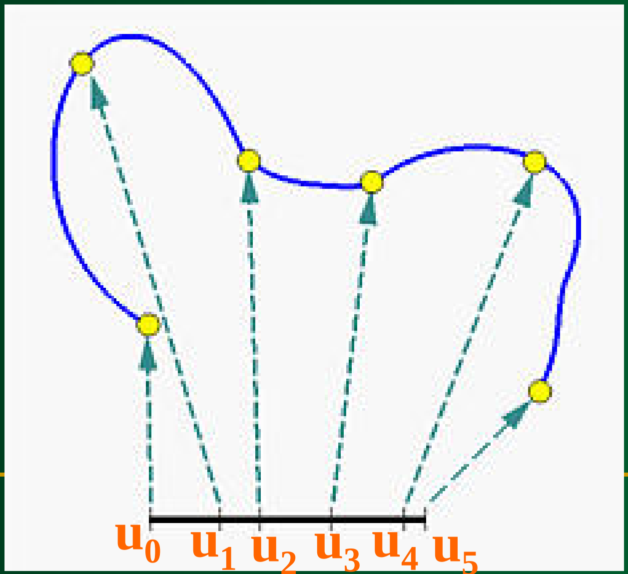
B-Spline

Basis Functions

B-Spline Basis Functions
(Knots, Knot Vector)

- Let U be a set of $m + 1$ non-decreasing numbers, $u_0 <= u_2 <= u_3 <= \ldots <= u_m$. The u_i's are called ***knots***,
- The set U is the ***knot vector***.

$$\boxed{U = \{u_0, u_1, u_2, \ldots, u_m\}}$$



B-Spline Basis Functions (Knots, Knot Vector)

$$U = \{u_0, u_1, u_2, \ldots, u_m\}$$

- The half-open interval $[u_i, u_{i+1})$ is the *i-th* ***knot span***.
- Some u_i's may be equal, some knot spans may not exist.

B-Spline Basis Functions (Knots)

$$U = \{u_0, u_1, u_2, \ldots, u_m\}$$

- If a knot u_i appears k times (i.e., $u_i = u_{i+1} = \ldots = u_{i+k-1}$), where $k > 1$, u_i is a ***multiple knot*** of multiplicity k, written as $u_i(k)$.
- If u_i appears only once, it is a ***simple knot***.
- If the knots are equally spaced (i.e., $u_{i+1} - u_i$ is a constant for $0 <= i <= m - 1$), the knot vector or the knot sequence is said ***uniform***; otherwise, it is ***non-uniform***.

B-Spline Basis Functions

All B-spline basis functions are supposed to have their domain on $[u_0, u_m]$.

- We use $u_0 = 0$ and $u_m = 1$ frequently so that the domain is the closed interval [0,1].

B-Spline Basis Functions

- To define B-spline basis functions, we need one more parameter.
- The degree of these basis functions, p. The i-th B-spline basis function of degree p, written as $N_{i,p}(u)$, is defined recursively as follows:

$$N_{i,0}(0) = \begin{cases} 1 & \text{if } u_i \le u < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

B-Spline Basis Functions

$$N_{i,0}(0) = \begin{cases} 1 & \text{if } u_i \le u < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

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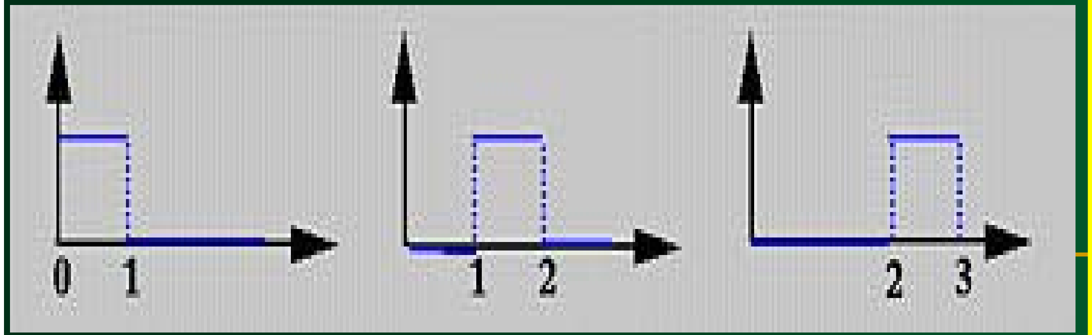
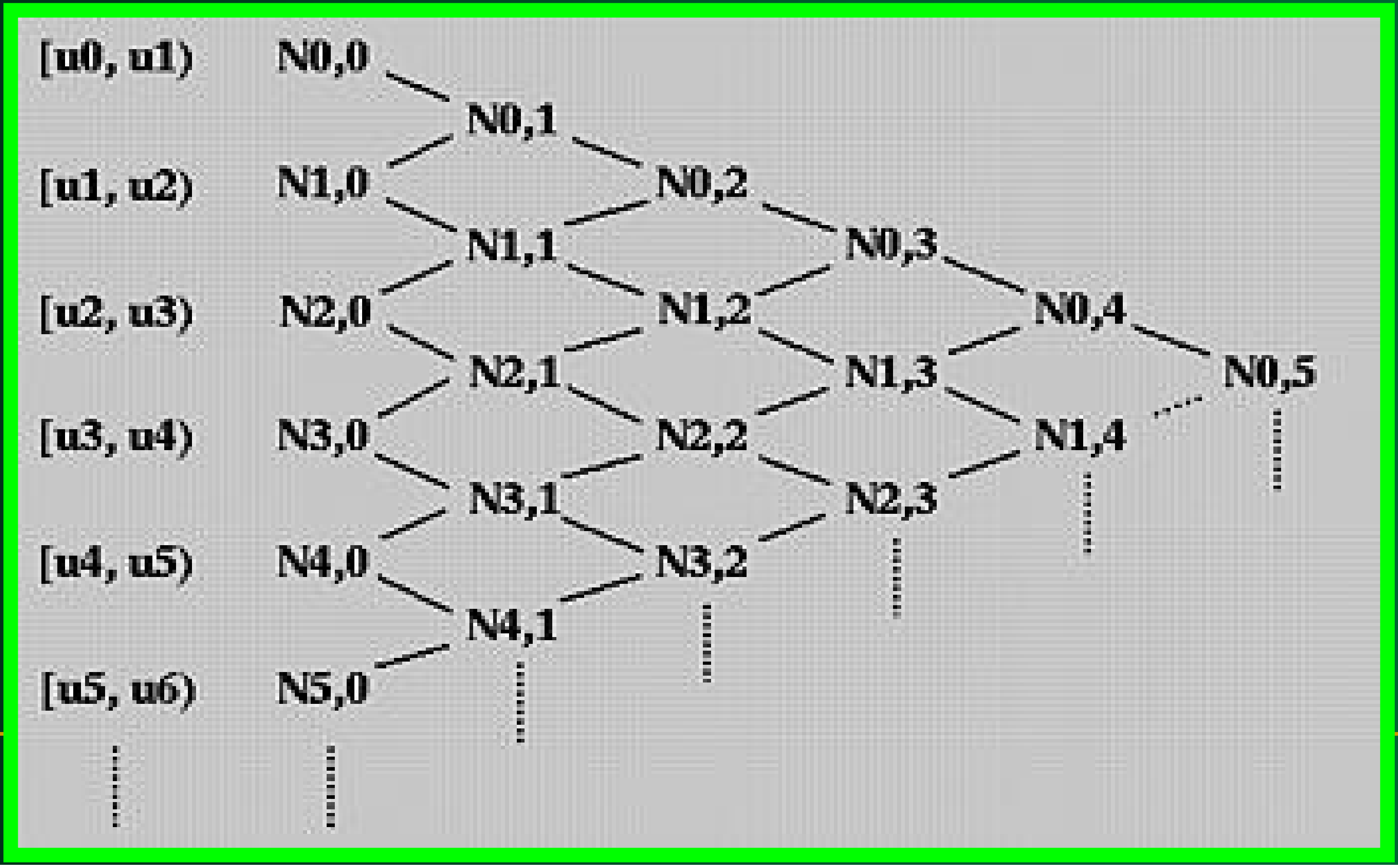
- The above is usually referred to as the Cox-de Boor recursion formula.
- If the degree is zero (i.e., $p = 0$), these basis functions are all step functions .
- basis function $N_{i,0}(u)$ is 1 if u is in the i-th knot span $[u_i, u_{i+1})$.



- We have four knots $u_0 = 0$, $u_1 = 1$, $u_2 = 2$ and $u_3 = 3$, knot spans 0, 1 and 2 are $[0,1)$, $[1,2)$, $[2,3)$ and the basis functions of degree 0 are $N_{0,0}(u) = 1$ on $[0,1)$ and 0 elsewhere, $N_{1,0}(u) = 1$ on $[1,2)$ and 0 elsewhere, and $N_{2,0}(u) = 1$ on $[2,3)$ and 0 elsewhere.

B-Spline Basis Functions

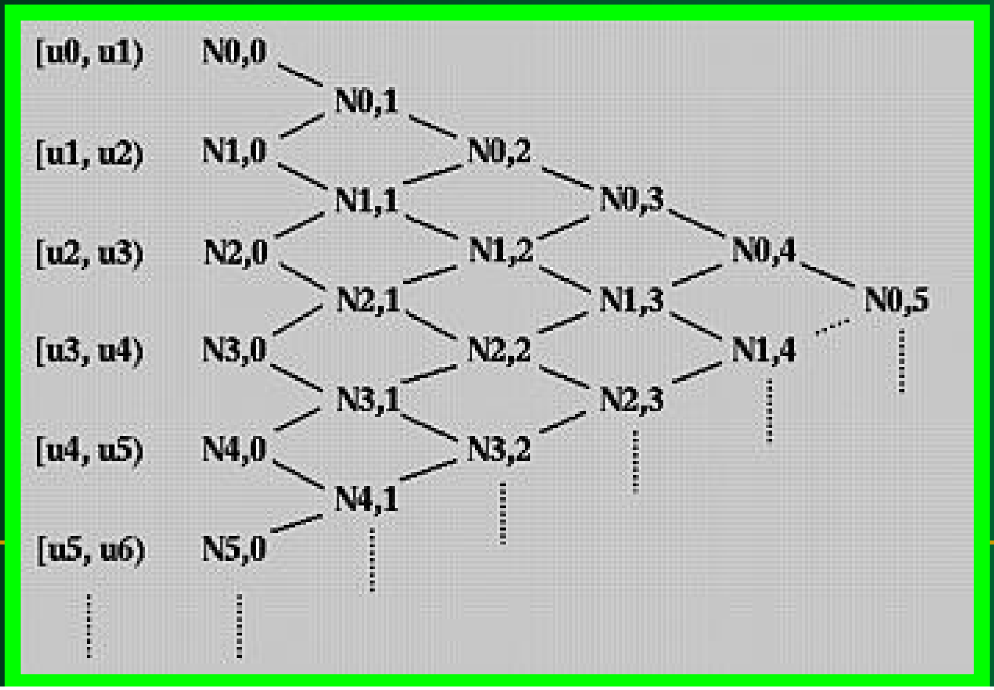
- To understand the way of computing $N_{i,p}(u)$ for p greater than 0, we use the triangular computation scheme.



B-Spline Basis Functions

- To compute $N_{i,1}(u)$, $N_{i,0}(u)$ and N_{i+1},0(u) are required. Therefore, we can compute $N_{0,1}(u)$, $N_{1,1}(u)$, $N_{2,1}(u)$, $N_{3,1}(u)$ and so on. All of these $N_{i,1}(u)$'s are written on the third column. Once all $N_{i,1}(u)$'s have been computed, we can compute $N_{i,2}(u)$'s and put them on the fourth column. This process continues until all required $N_{i,p}(u)$'s are computed.

$$N_{0,1}(u) = \frac{u - u_0}{u_1 - u_0} N_{0,0}(u) + \frac{u_2 - u}{u_2 - u_1} N_{1,0}(u)$$

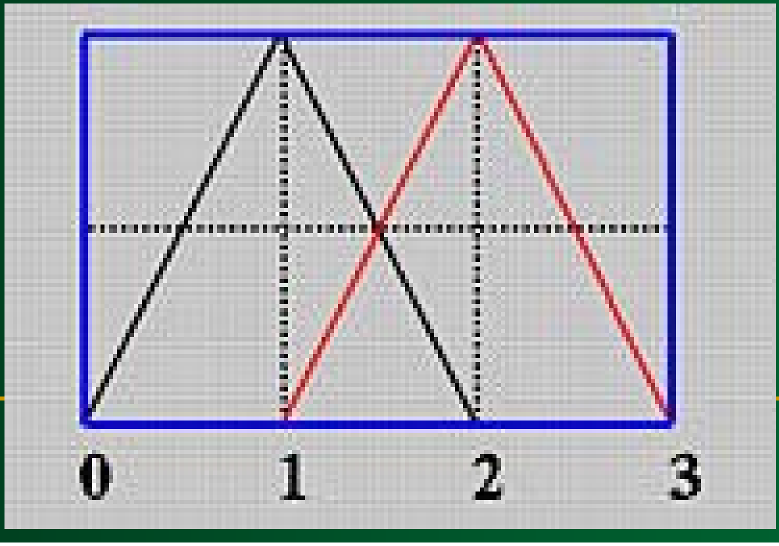


B-Spline Basis Functions

$$N_{0,1}(u) = \frac{u - u_0}{u_1 - u_0} N_{0,0}(u) + \frac{u_2 - u}{u_2 - u_1} N_{1,0}(u)$$

- Since $u0 = 0$, $u1 = 1$ and $u2 = 2$, the above becomes

$$N_{0,1}(u) = u N_{0,0}(u) + (2 - u) N_{1,0}(u)$$



B-Spline Basis Functions

$$N_{0,2}(u) = \frac{u - u_0}{u_2 - u_0} N_{0,1}(u) + \frac{u_3 - u}{u_3 - u_1} N_{1,1}(u)$$

- u is in [0,1): In this case, only $N0,1(u)$ contributes to the value of $N0,2(u)$. Since $N0,1(u)$ is u, we have

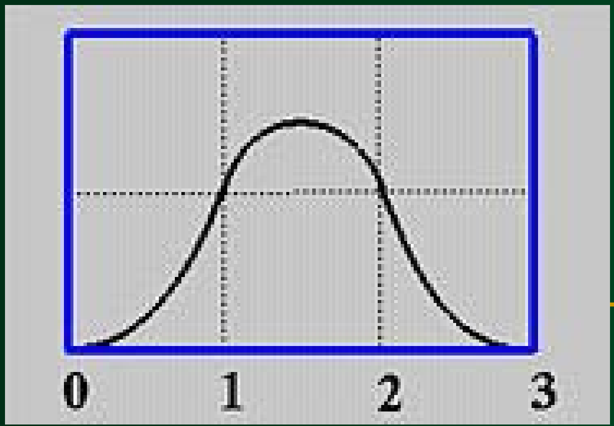
$$N_{0,2}(u) = 0.5u^2$$

- u is in [1,2): In this case, both $N0,1(u)$ and $N1,1(u)$ contribute to $N0,2(u)$. Since $N0,1(u) = 2 - u$ and $N1,1(u) = u - 1$ on [1,2), we have

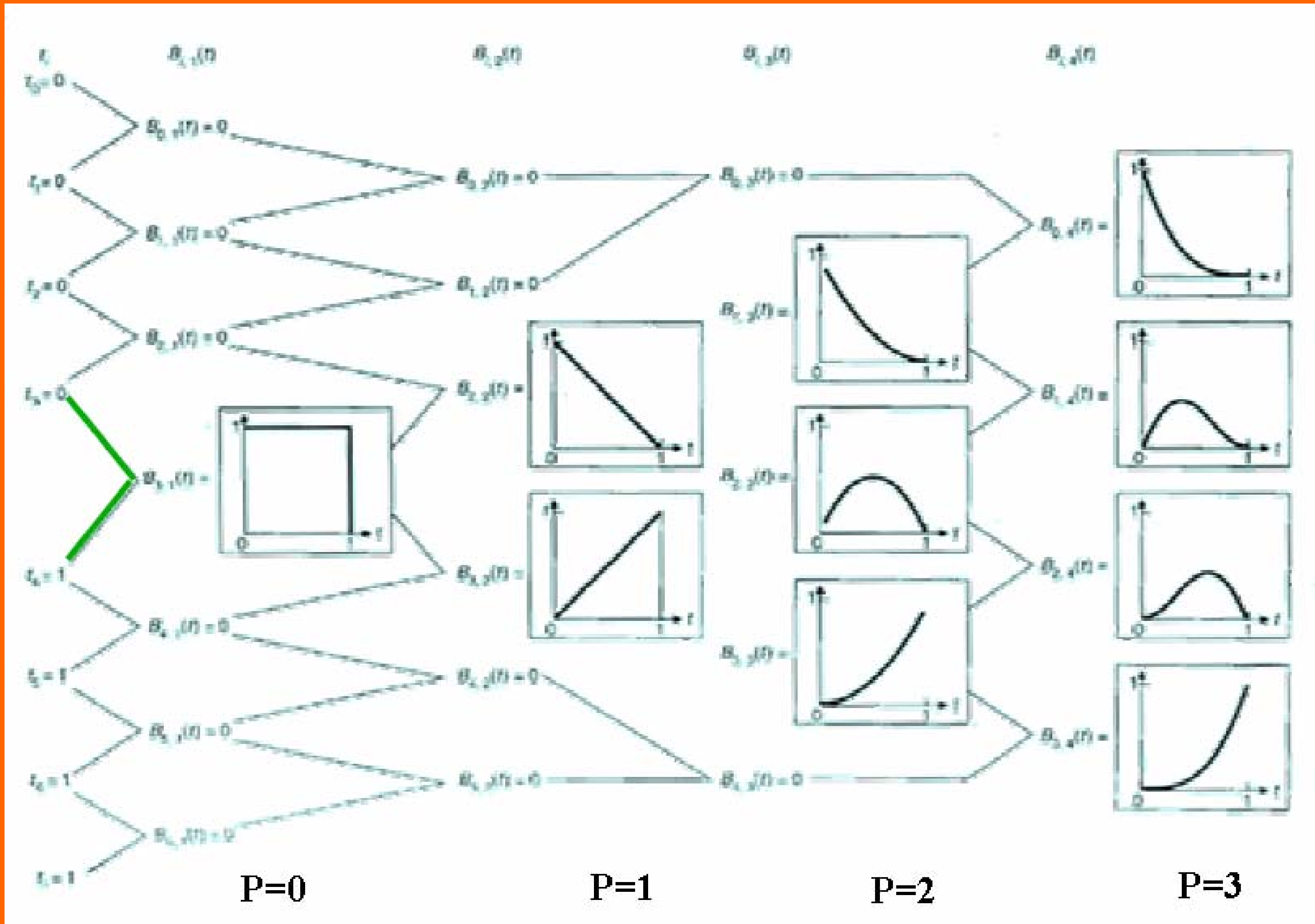
$$N_{0,2}(u) = (0.5u)(2 - u) + 0.5(3 - u)(3 - u) = 0.5(-3 + 6u - 2u^2)$$

- u is in [2,3): In this case, only N1,1(u) contributes to N0,2(u). Since N1,1(u) = 3 u on [2,3), we have

$$N_{0,2}(u) = 0.5(3 - u)(3 - u) = 0.5(3 - u)^2$$



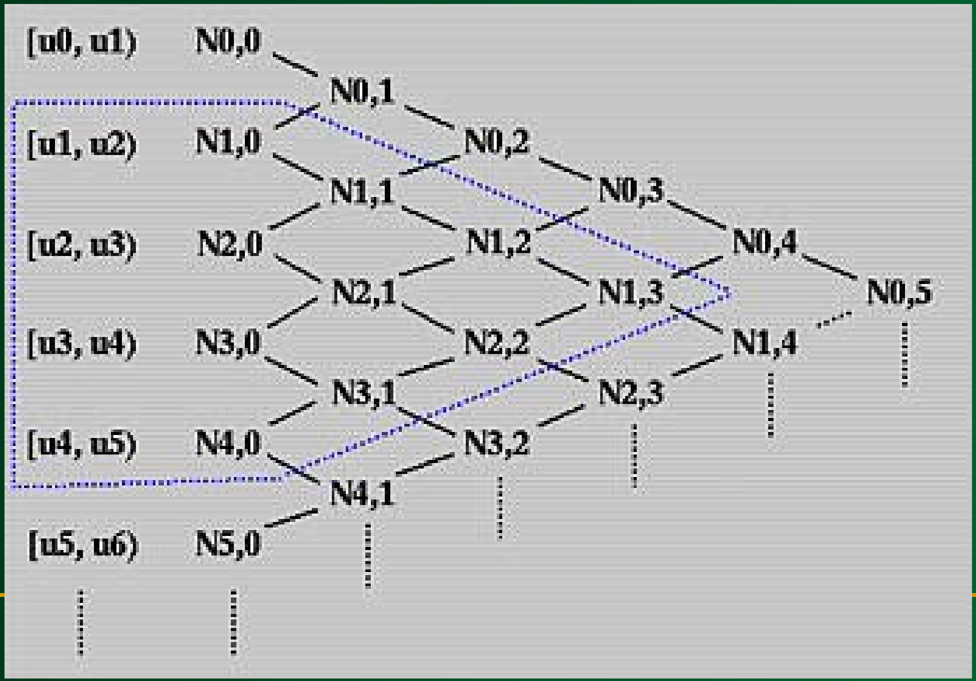
B-Spline Basis Functions



Two Important Observation

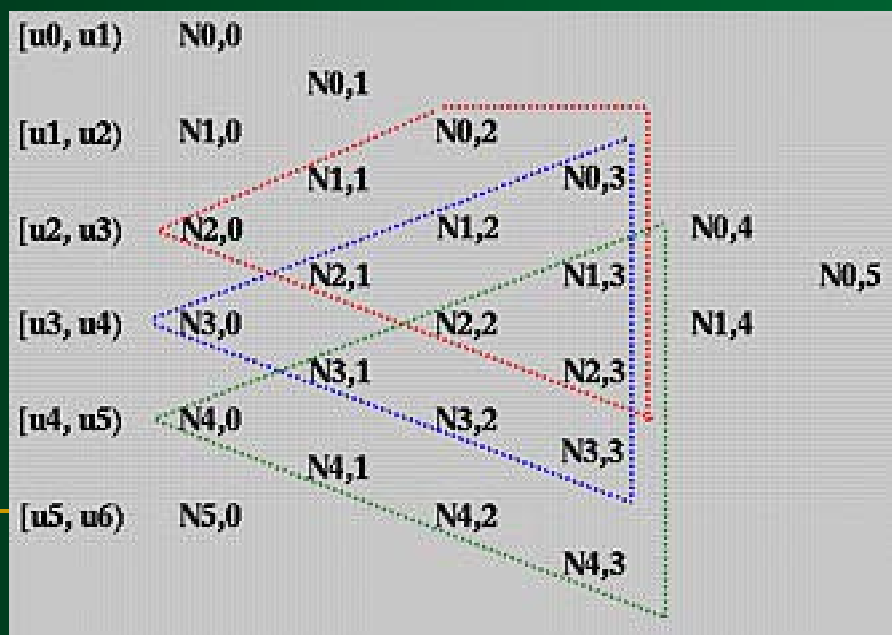
Two Important Observation

- **Basis function $N_{i,p}(u)$ is non-zero on $[u_i, u_{i+p+1})$. Or, equivalently, $N_{i,p}(u)$ is non-zero on p+1 knot spans $[u_i, u_{i+1})$, $[u_{i+1}, u_{i+2})$, ..., $[u_{i+p}, u_{i+p+1})$.**



Two Important Observation

- On any knot span $[u_i, u_{i+1})$, at most p+1 degree p basis functions are non-zero, namely: $N_{i-p,p}(u)$, $N_{i-p+1,p}(u)$, $N_{i-p+2,p}(u)$, ..., $N_{i-1,p}(u)$ and $N_{i,p}(u)$,



B-Spline Basis Functions
(Important Properties)

B-Spline Basis Functions (Important Properties)

$$N_{i,0}(0) = \begin{cases} 1 & \text{if } u_i \le u < u_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(u) = \frac{u - u_i}{u_{i+p} - u_i} N_{i,p-1}(u) + \frac{u_{i+p+1} - u}{u_{i+p+1} - u_{i+1}} N_{i+1,p-1}(u)$$

1. $N_{i,p}(u)$ is a degree p polynomial in u.
2. Nonnegativity -- For all i, p and u, $N_{i,p}(u)$ is non-negative
3. Local Support -- $N_{i,p}(u)$ is a non-zero polynomial on $[u_{i,}u_{i+p+1})$

B-Spline Basis Functions (Important Properties)

4. On any span $[u_i, u_{i+1})$, at most $p+1$ degree p basis functions are non-zero, namely: $N_{i-p,p}(u)$, $N_{i-p+1,p}(u)$, $N_{i-p+2,p}(u)$, ..., and $N_{i,p}(u)$.
5. The sum of all non-zero degree p basis functions on span $[u_i, u_{i+1})$ is 1.
6. If the number of knots is $m+1$, the degree of the basis functions is p, and the number of degree p basis functions is $n+1$, then $\boldsymbol{m = n + p + 1}$

B-Spline Basis Functions (Important Properties)

7. Basis function $N_{i,p}(u)$ is a composite curve of degree p polynomials with joining points at knots in $[u_i, u_{i+p+1})$

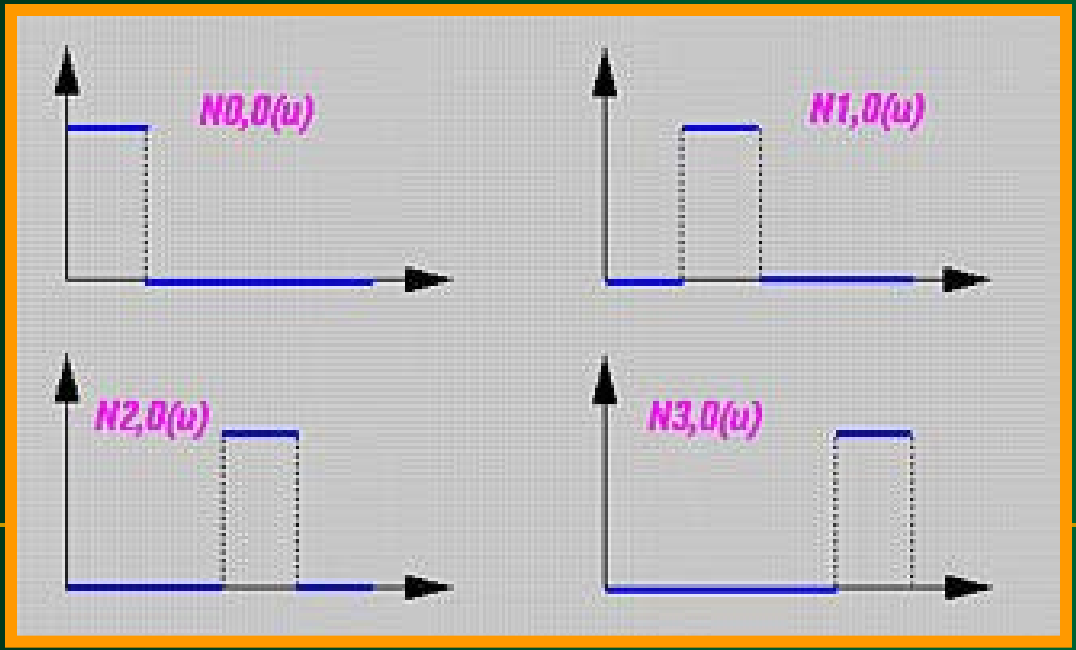
8. At a knot of multiplicity k, basis function $N_{i,p}(u)$ is C^{p-k} continuous.

Increasing multiplicity decreases the level of continuity, and increasing degree increases continuity.

B-Spline Basis Functions (Computation Examples)

Simple Knots

- Suppose the knot vector is $U = \{ 0, 0.25, 0.5, 0.75, 1 \}$.
- Basis functions of degree **0**: $N_{0,0}(u)$, $N_{1,0}(u)$, $N_{2,0}(u)$ and $N_{3,0}(u)$ defined on knot span [0,0.25,), [0.25,0.5), [0.5,0.75) and [0.75,1), respectively.



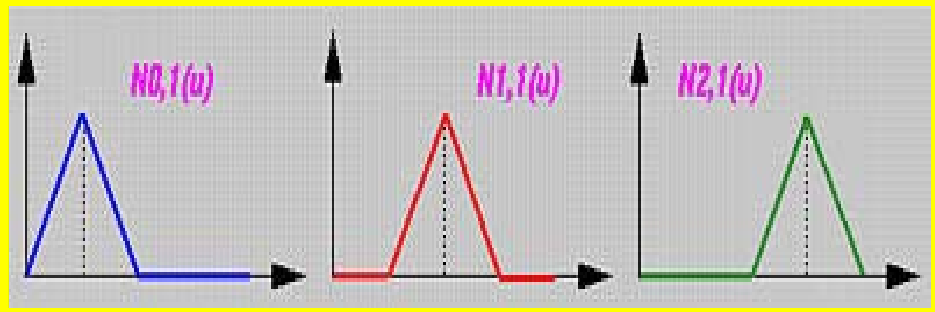
B-Spline Basis Functions (Computation Examples)

All *Ni*,1(*u*)'s ($U = \{ 0, 0.25, 0.5, 0.75, 1 \}$):

$$N_{0,1}(u) = \begin{cases} 4u & \text{for } 0 \le u < 0.25 \\ 2(1-2u) & \text{for } 0.25 \le u < 0.5 \end{cases}$$

$$N_{1,1}(u) = \begin{cases} 4u-1 & \text{for } 0.25 \le u < 0.5 \\ 3-u & \text{for } 0.5 \le u < 0.75 \end{cases}$$

$$N_{2,1}(u) = \begin{cases} 2(2u-1) & \text{for } 0.5 \le u < 0.75 \\ 4(1-u) & \text{for } 0.75 \le u < 1 \end{cases}$$



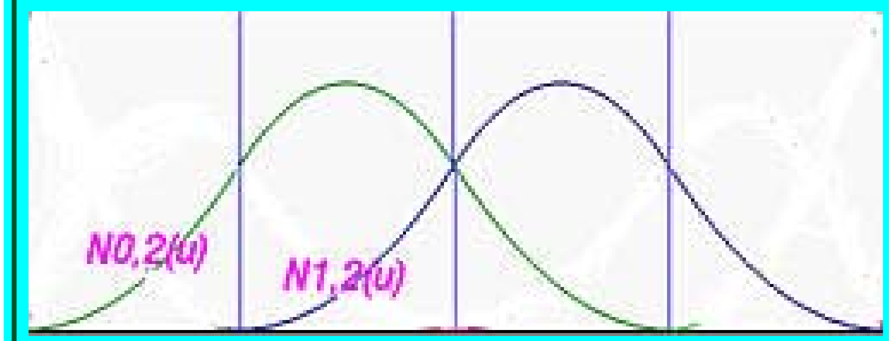
- Since the internal knots 0.25, 0.5 and 0.75 are all simple (*i.e.*, $k = 1$) and $p = 1$, there are $p - k + 1 = 1$ non-zero basis function and three knots. Moreover, $N_{0,1}(u)$, $N_{1,1}(u)$ and $N_{2,1}(u)$ are C^0 continuous at knots 0.25, 0.5 and 0.75, respectively.

B-Spline Basis Functions (Computation Examples)

- From $N_{i,1}(u)$'s, one can compute the basis functions of degree 2. Since $m = 4$, $p = 2$, and $m = n + p + 1$, we have $n = 1$ and there are only two basis functions of degree 2: $N_{0,2}(u)$ and $N_{1,2}(u)$. ($U = \{ 0, 0.25, 0.5, 0.75, 1 \}$):

$$N_{0,2}(u) = \begin{cases} 8u^2 & \text{for } 0 \le u < 0.25 \\ -1.5 + 12u - 16u^2 & \text{for } 0.25 \le u < 0.5 \\ 4.5 - 12u + 8u^2 & \text{for } 0.5 \le u < 0.75 \end{cases}$$

$$N_{1,2}(u) = \begin{cases} 0.5 - 4u + 8u^2 & \text{for } 0.25 \le u < 0.5 \\ -1.5 + 8u - 8u^2 & \text{for } 0.5 \le u < 0.75 \\ 8(1-u)^2 & \text{for } 0.75 \le u < 1 \end{cases}$$



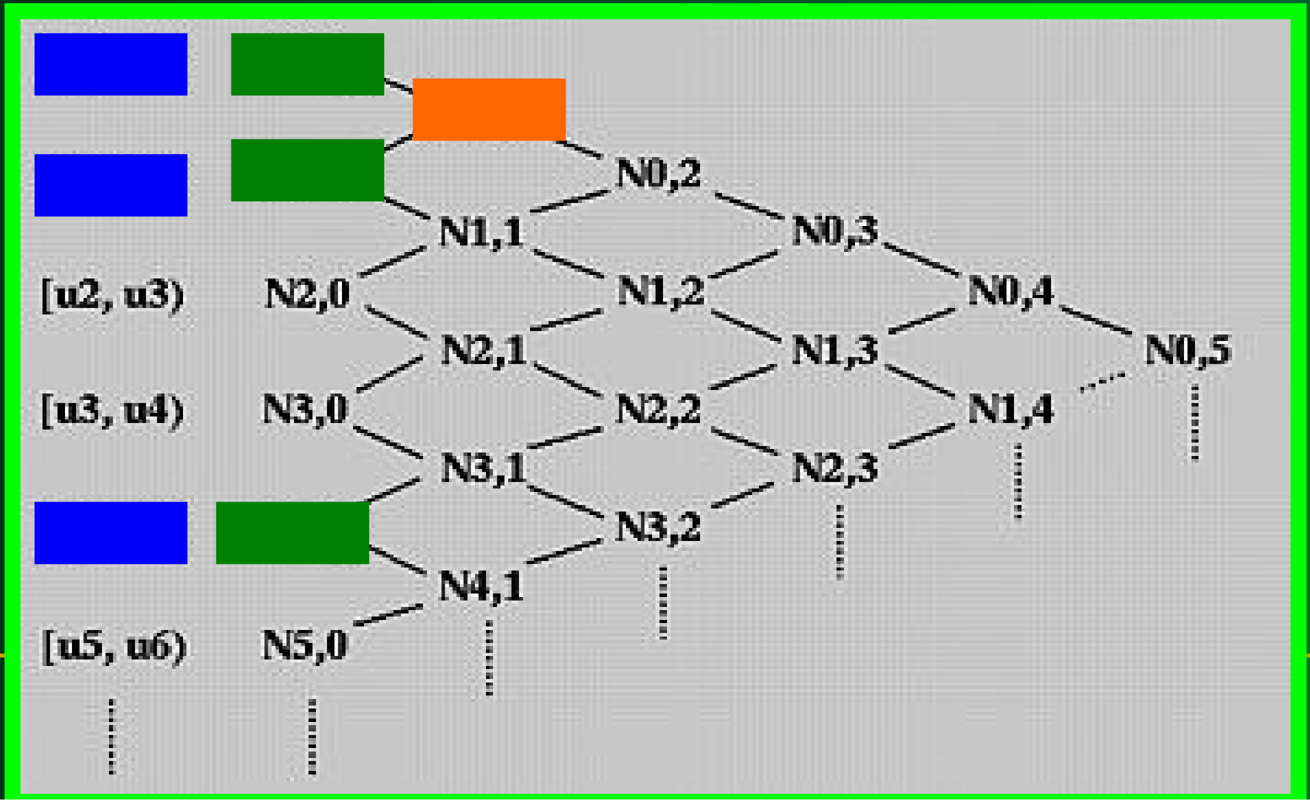
- each basis function is a composite curve of three degree 2 curve segments.
- composite curve is of C^1 continuity

B-Spline Basis Functions (Computation Examples)

Knots with Positive Multiplicity :

Suppose the knot vector is U = { 0, 0, 0, 0.3, 0.5, 0.5, 0.6, 1, 1, 1}

- Since $m = 9$ and $p = 0$ (degree 0 basis functions), we have $n = m - p - 1 = 8$. there are only four non-zero basis functions of degree 0: $N_{2,0}(u)$, $N_{3,0}(u)$, $N_{5,0}(u)$ and $N_{6,0}(u)$.



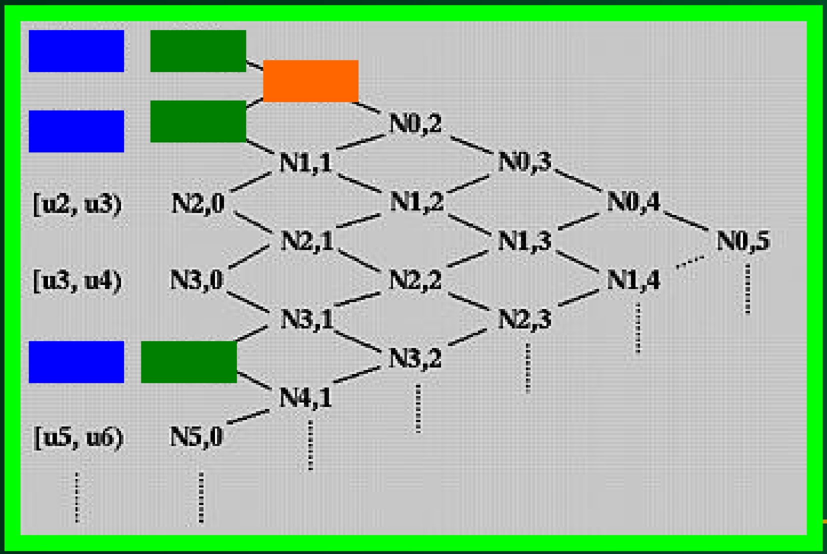
B-Spline Basis Functions (Computation Examples)

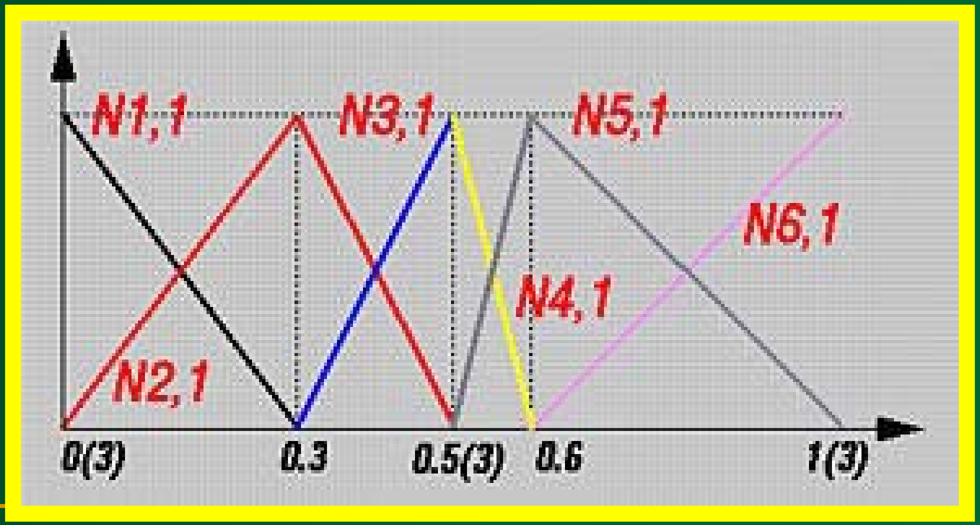
- Basis functions of degree 1: Since p is 1, $n = m - p - 1 = 7$. The following table shows the result

[illegible]	[illegible]	[illegible]
$N_{0,1(u)}$	all u	0
$N_{1,1(u)}$	[0, 0.3)	1 - (10/3)u
$N_{2,1(u)}$	[0, 0.3)	(10/3)u
	[0.3, 0.5)	2.5(1 - 2u)
$N_{3,1(u)}$	[0.3, 0.5)	5u - 1.5
$N_{4,1(u)}$	[0.5, 0.6)	6 - 10u
$N_{5,1(u)}$	[0.5, 0.6)	10u - 5
	[0.6, 1)	2.5(1 - u)
$N_{6,1(u)}$	[0.6, 1)	2.5u - 1.5
$N_{7,1(u)}$	all u	0

B-Spline Basis Functions (Computation Examples)

- Basis functions of degree 1:





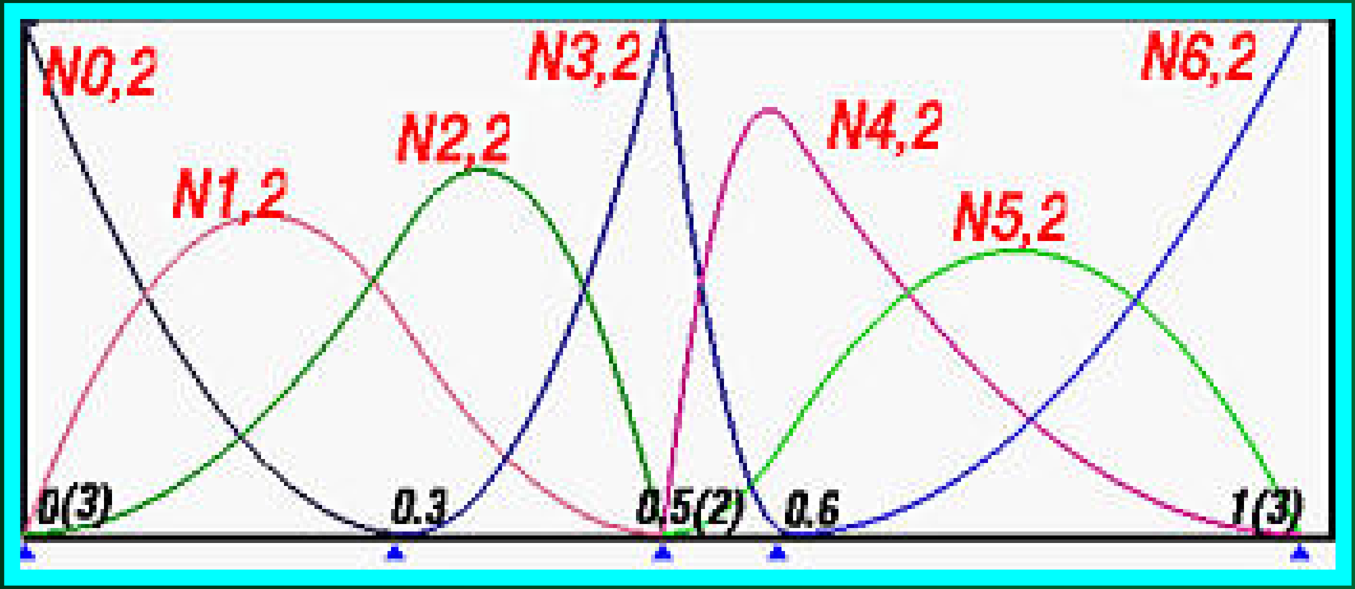
B-Spline Basis Functions (Computation Examples)

- Since $p = 2$, we have $n = m - p - 1 = 6$. The following table contains all $N_{i,2}(u)$'s:

□□□□□□□□	□□□□□	□□□□□□□□
$N_{0,2}(u)$	[0, 0.3)	$(1 - (10/3)u)^2$
$N_{1,2}(u)$	[0, 0.3)	$(20/3)(u - (8/3)u^2)$
	[0.3, 0.5)	$2.5(1 - 2u)^2$
$N_{2,2}(u)$	[0, 0.3)	$(20/3)u^2$
	[0.3, 0.5)	$-3.75 + 25u - 35u^2$
$N_{3,2}(u)$	[0.3, 0.5)	$(5u - 1.5)^2$
	[0.5, 0.6)	$(6 - 10u)^2$
$N_{4,2}(u)$	[0.5, 0.6)	$20(-2 + 7u - 6u^2)$
	[0.6, 1)	$5(1 - u)^2$
$N_{5,2}(u)$	[0.5, 0.6)	$12.5(2u - 1)^2$
	[0.6, 1)	$2.5(-4 + 11.5u - 7.5u^2)$
$N_{6,2}(u)$	[0.6, 1)	$2.5(9 - 30u + 25u^2)$

B-Spline Basis Functions (Computation Examples)

- Basis functions of degree 2: U = { 0, 0, 0, 0.3, 0.5, 0.5, 0.6, 1, 1, 1}



- Since its multiplicity is 2 and the degree of these basis functions is 2, basis function $N_{3,2}(u)$ is C^0 continuous at 0.5(2). This is why $N_{3,2}(u)$ has a sharp angle at 0.5(2).
- For knots not at the two ends, say 0.3 and 0.6, C^1 continuity is maintained since all of them are simple knots.

B-Spline

Curves

B-Spline Curves (Definition)

- Given $n + 1$ control points $P_0, P_1, \ldots, P_n$ and a knot vector $U = \{ u_0, u_1, \ldots, u_m \}$, the B-spline curve of degree p defined by these control points and knot vector U is

$$\boxed{\mathbf{C}(u) = \sum_{i=0}^{n} N_{i,p}(u)\mathbf{p}_i, \quad u_0 \le u \le u_m \quad p = m - n - 1}$$

- The point on the curve that corresponds to a knot u_i, $\mathbf{C}(u_i)$, is referred to as a ***knot point***.
- The knot points **divide** a B-spline curve into curve segments, each of which is defined on a knot span.

B-Spline Curves (Definition)

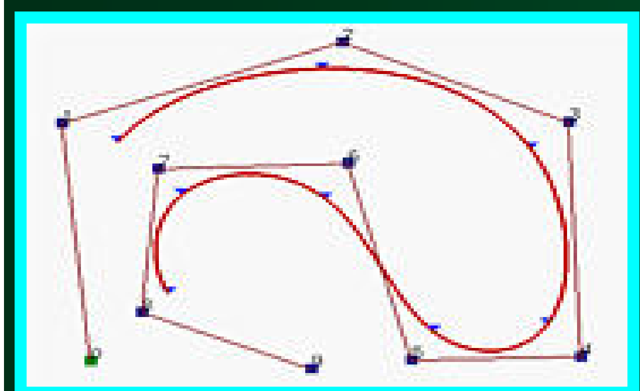
$$\mathbf{C}(u)=\sum_{i=0}^{n} N_{i,p}(u)\mathbf{p}_i, \quad u_0 \le u \le u_m \quad p=m-n-1$$

- The degree of a B-spline basis function is an input.
- To ***change*** the ***shape*** of a B-spline curve, one can modify one or more of these control parameters:

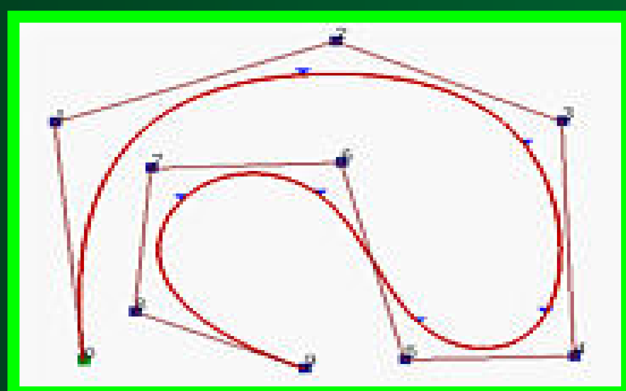
1. The positions of control points
2. The positions of knots
3. The degree of the curve

(Open, Clamped & Closed B-Spline Curves)

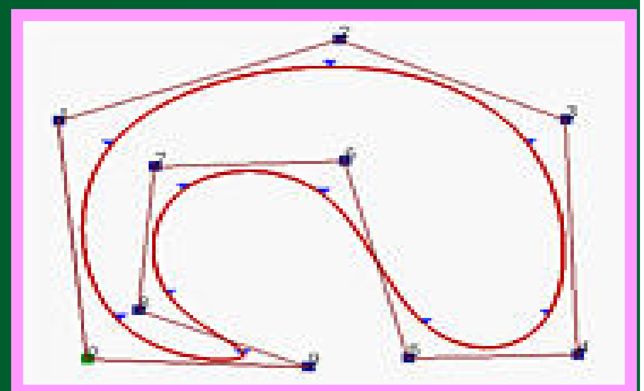
- **Open B-spline curves**: If the knot vector does not have any particular structure, the generated curve will not touch the first and last legs of the control polyline.
- **Clamped B-spline curve:** If the first knot and the last knot multiplicity $p+1$, curve is tangent to the first and the last legs at the first and last control polyline, as a Bézier curve does.
- **Closed B-spline curves:** By repeating some knots and control points, the generated curve can be a *closed* one. In this case, the start and the end of the generated curve join together forming a closed loop.



Open B-Spline



Clamped B-Spline



Closed B-Spline

control points (n=9) and p = 3. m must be 13 so that the knot vector has 14 knots. To have the clamped effect, the first $p+1 = 4$ and the last 4 knots must be identical. The remaining 14 - (4 + 4) = 6 knots can be anywhere in the domain. In fact, the curve is generated with knot vector $U = \{ 0, 0, 0, 0, 0.14, 0.28, 0.42, 0.57, 0.71, 0.85, 1, 1, 1, 1 \}$.

Open
B-Spline Curves

Open B-Spline Curves

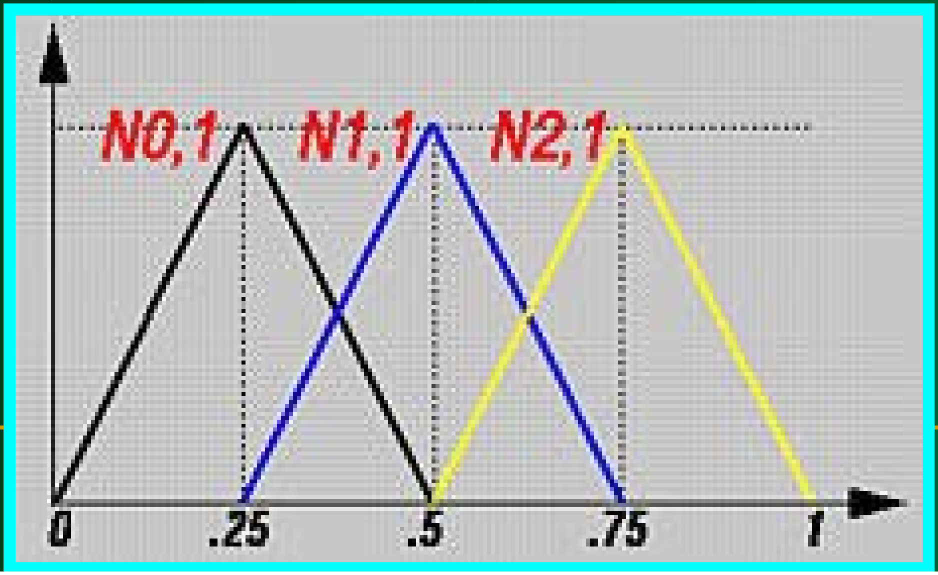
- Recall from the B-spline basis function property that on a knot span $[u_i, u_{i+1})$, there are at most $p+1$ non-zero basis functions of degree p.

For open B-spline curves, the domain is $[u_p, u_{m-p}]$.

Open B-Spline Curves

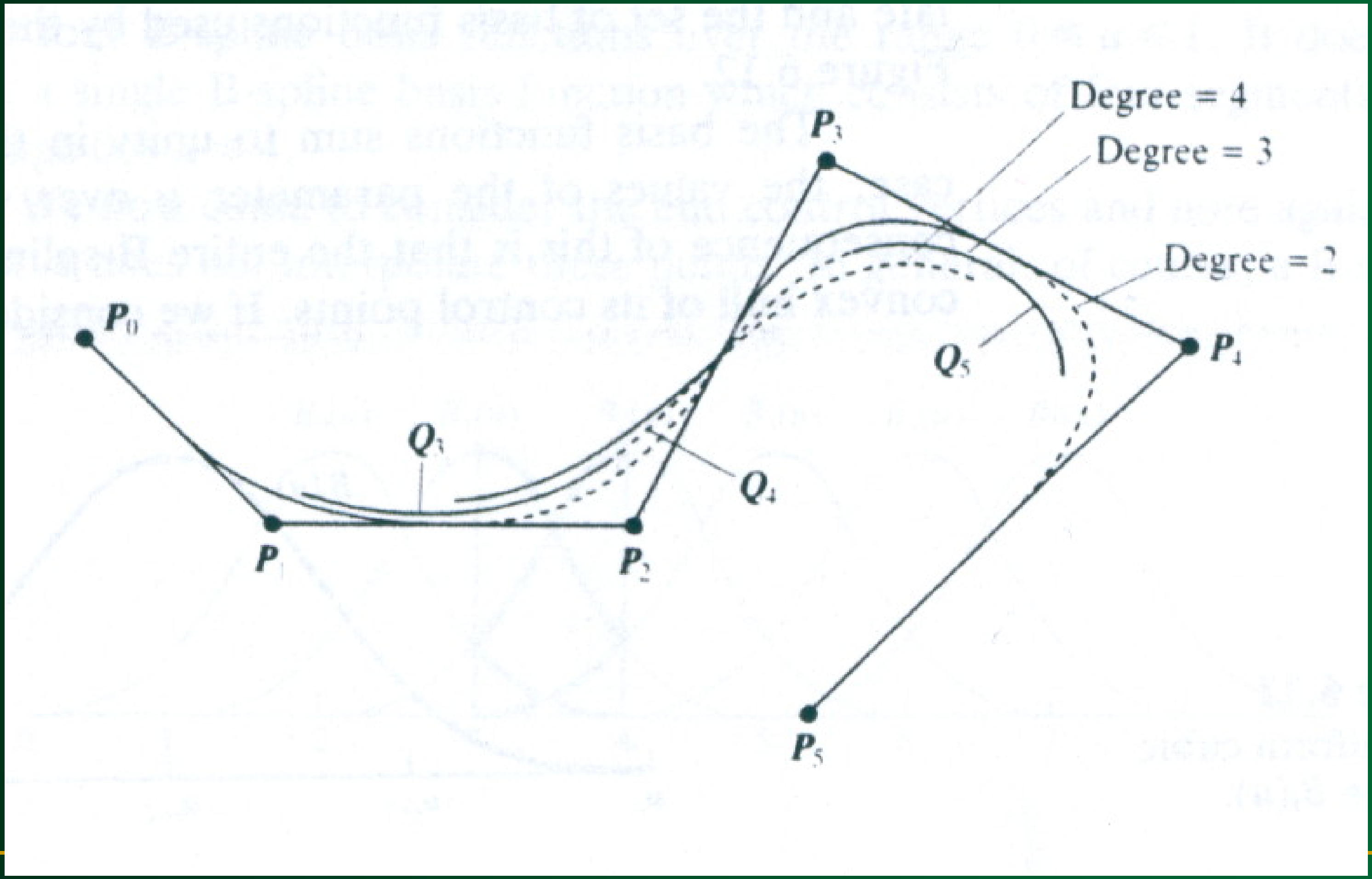
Example 1:

- knot vector $U = \{ 0, 0.25, 0.5, 0.75, 1 \}$, where $m = 4$. If the basis functions are of degree 1 (*i.e.*, $p = 1$), there are three basis functions $N_{0,1}(u)$, $N_{1,1}(u)$ and $N_{2,1}(u)$.
- Since this knot vector is not clamped, the first and the last knot spans (*i.e.*, [0, 0.25) and [0.75, 1)) have only one non-zero basis functions while the second and third knot spans (*i.e.*, [0.25, 0.5) and [0.5, 0.75)) have two non-zero basis functions.



Open B-Spline Curves

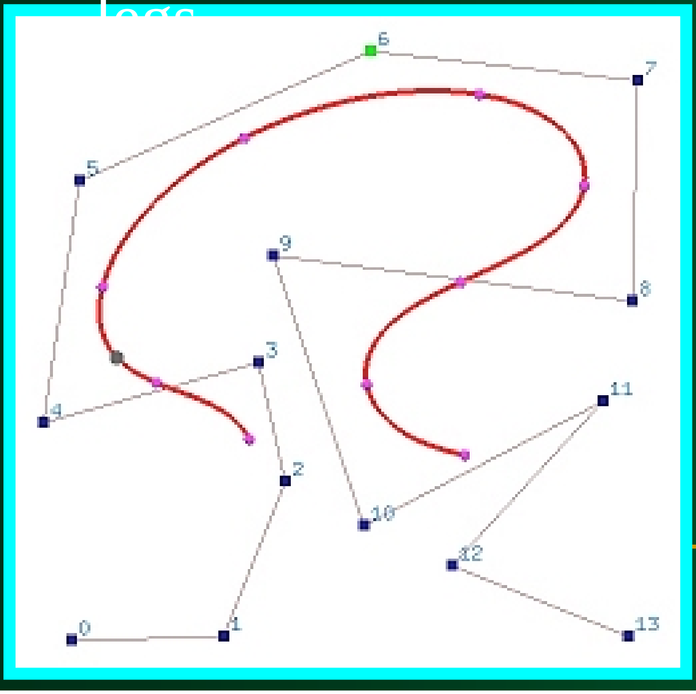
Example 2:

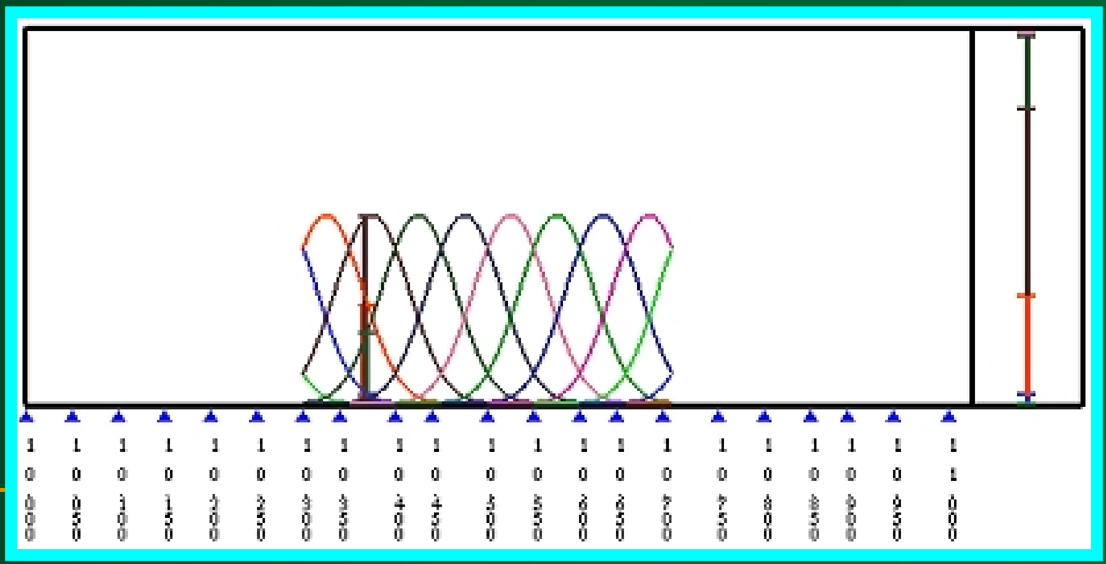


Example 3:

Open B-Spline Curves

- A B-spline curve of degree 6 (*i.e.*, $p = 6$) defined by 14 control points (*i.e.*, $n = 13$). The number of knots is 21 (*i.e.*, $m = n + p + 1 = 20$).
- If the knot vector is *uniform*, the knot vector is {0, 0.05, 0.10, 0.15, ..., 0.90, 0.95,10}. The *open* curve is defined on $[u_p, u_{m-p}] = [u_6, u_{14}] = [0.3, 0.7]$ and is not tangent to the first and last legs.

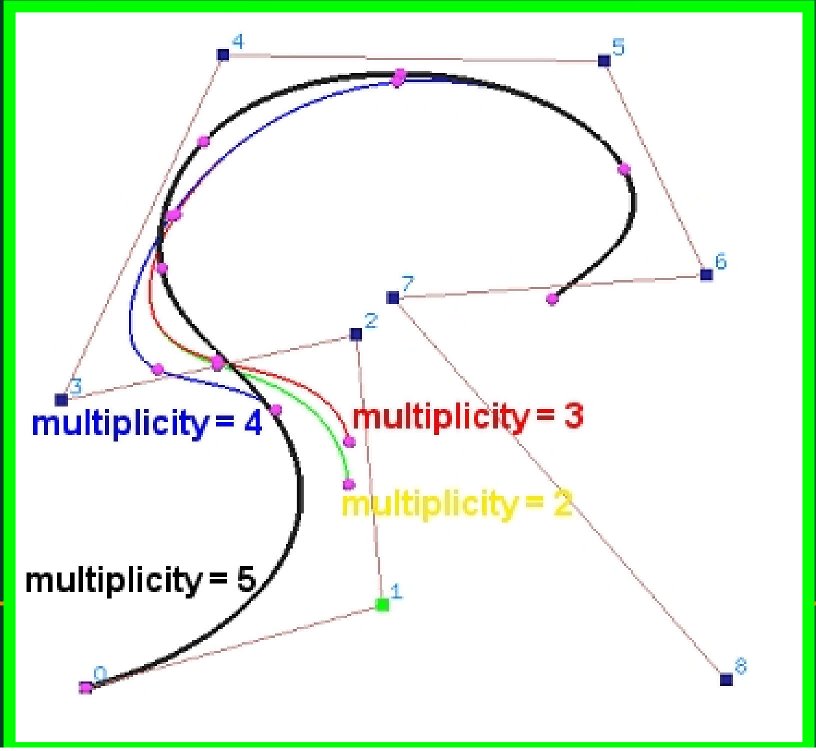




Clamped
B-Spline Curves

Clamped B-Spline Curves

- **We use an exampleuse** to illustrate the change between an open curve and a clamped one:
- An open B-spline curve of degree 4 , n = 8 and a uniform knot vector { 0, 1/13, 2/13, 3/13, ..., 12/13, 1 }.
- **Multiplicity 5** (*i.e.*, p+1),(second, third, fourth and fifth knot to 0) the curve not only passes through the first control point but also is tangent to the first leg of the control polyline.



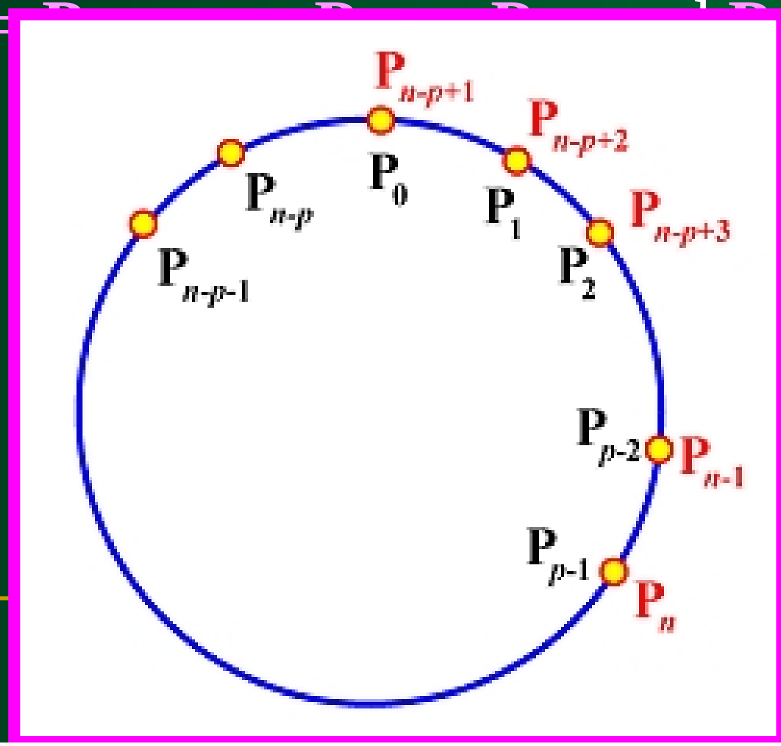
Closed
B-Spline Curves

Closed B-Spline Curves

- To construct a **closed B-spline** curve $\mathbf{C}(u)$ of degree p defined by n+1 control points ,the number of knots is m+1, ***We must***:

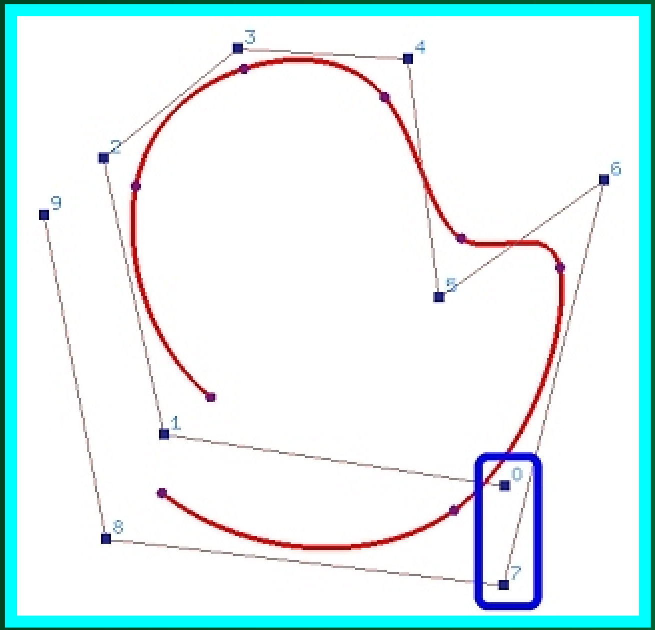
1. Design an uniform knot sequence of m+1 knots: $u_0 = 0$, $u_1 = 1/m$, $u_2 = 2/m$, ..., $u_m = 1$. Note that the domain of the curve is $[u_p, u_{n-p}]$.

2. ***Wrap*** the first p and last p control points. More precisely, let $\mathbf{P}_0 = \mathbf{P}_{n-p+1}$, $\mathbf{P}_1 =$ … $_{-1} = \mathbf{P}_n$.



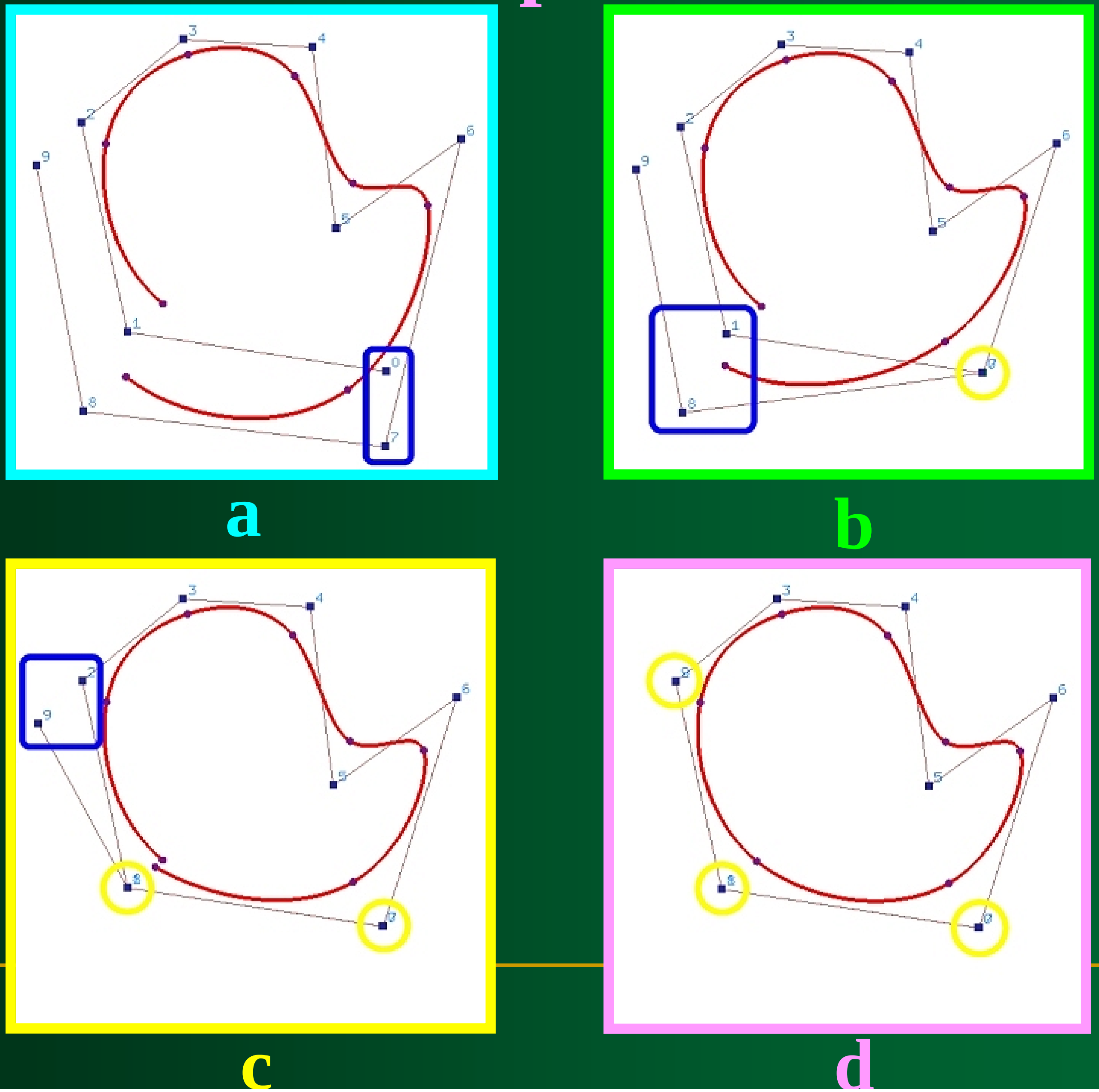
Closed B-Spline Curves

- **Example**. Figure (a) shows an open B-spline curve of degree 3 defined by 10 (n = 9) control points and a uniform knot vector.
- In the figure, control point pairs 0 and 7, Figure (b), 1 and 8, Figure (c), and 2 and 9, Figure (d) are placed close to each other to illustrate the construction.



a

Closed B-Spline Curves



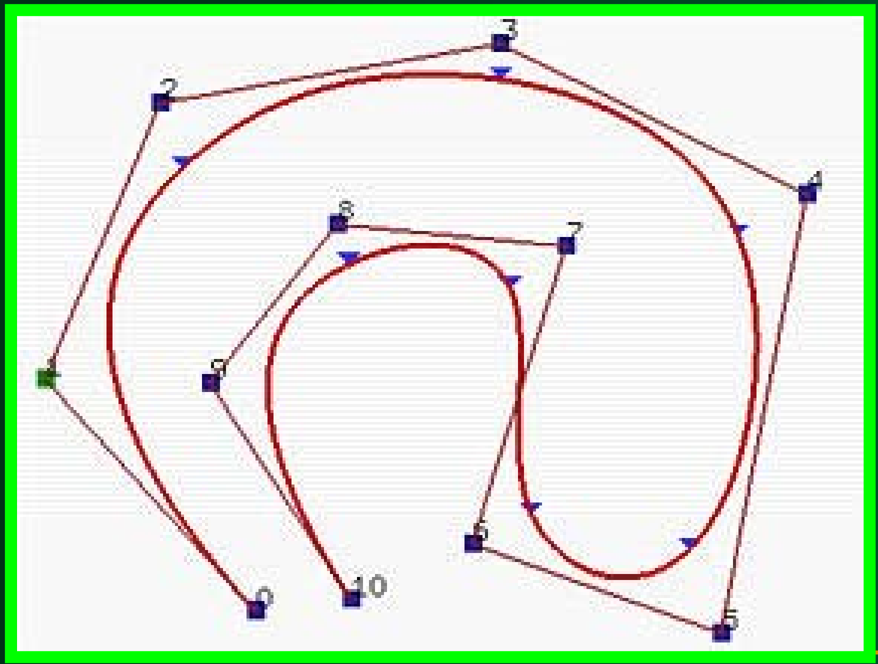
B-Spline Curves

Important Properties

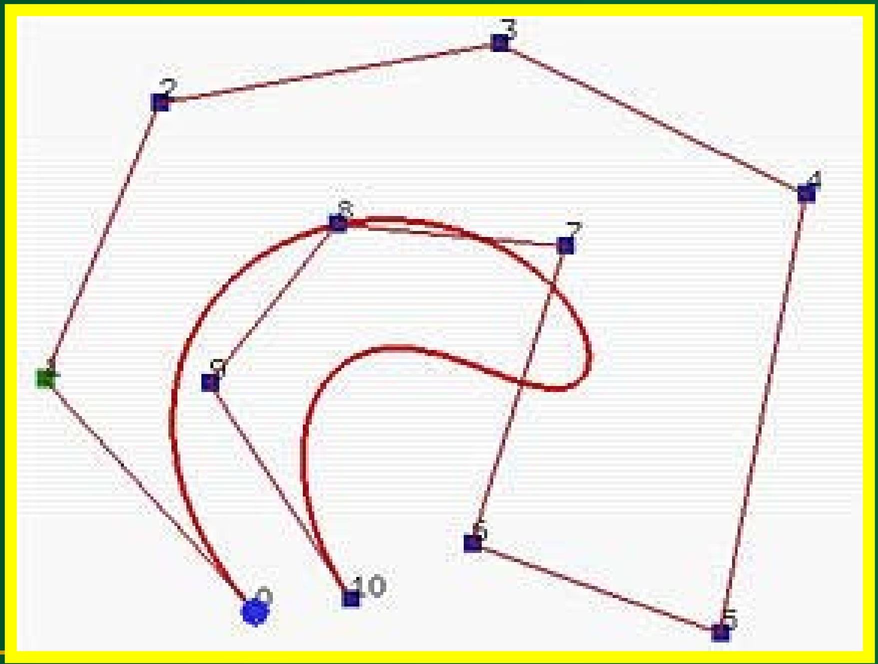
B-Spline Curves Important Properties

1. B-spline curve C(u) is a piecewise curve with each component a curve of degree p.

- **Example:** where n = 10, m = 14 and p = 3, the first four knots and last four knots are clamped and the 7 internal knots are uniformly spaced. There are 8 knot spans, each of which corresponds to a curve segment.



Clamped B-Spline Curve



Bézier Curve (degree 10!)

B-Spline Curves Important Properties

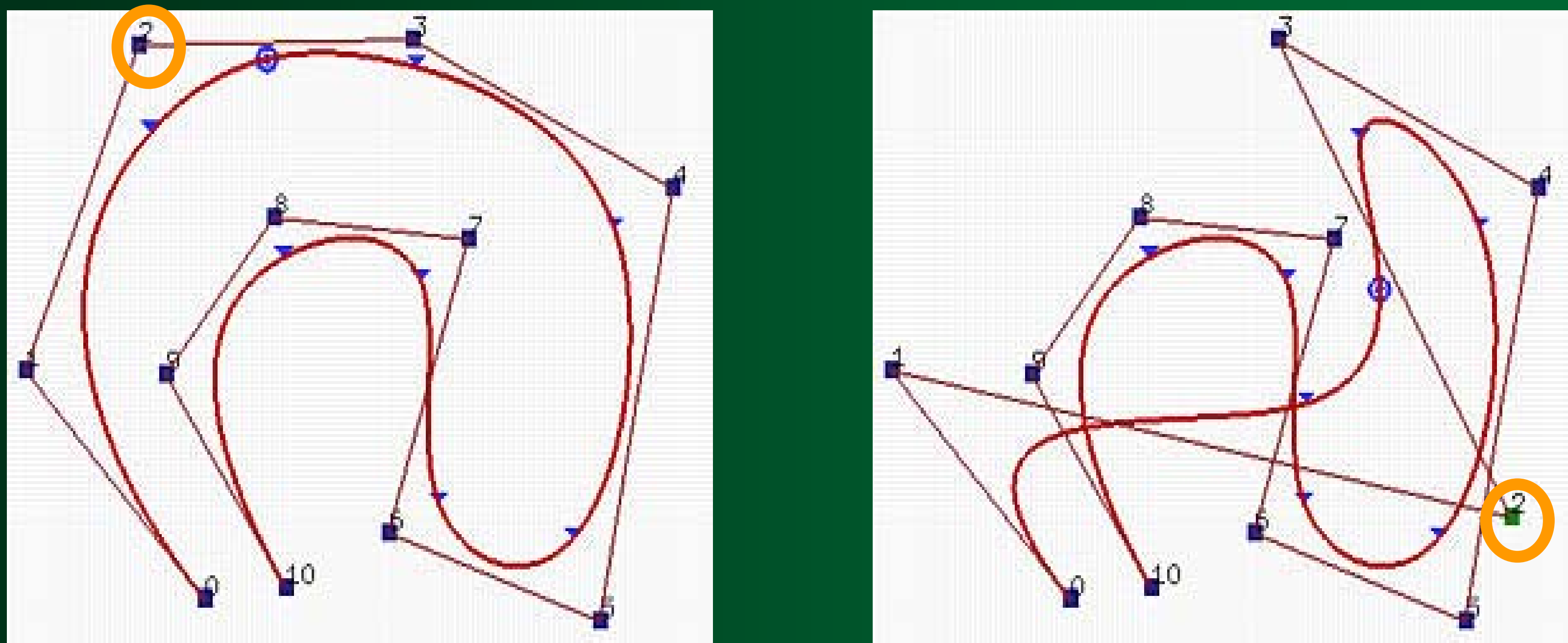
2. Equality $m = n + p + 1$ must be satisfied.

3. **Clamped B-spline** curve $C(u)$ passes through the two end control points P_0 and P_n.

4. **Strong Convex Hull Property:** A B-spline curve is contained in the convex hull of its control polyline.

B-Spline Curves Important Properties

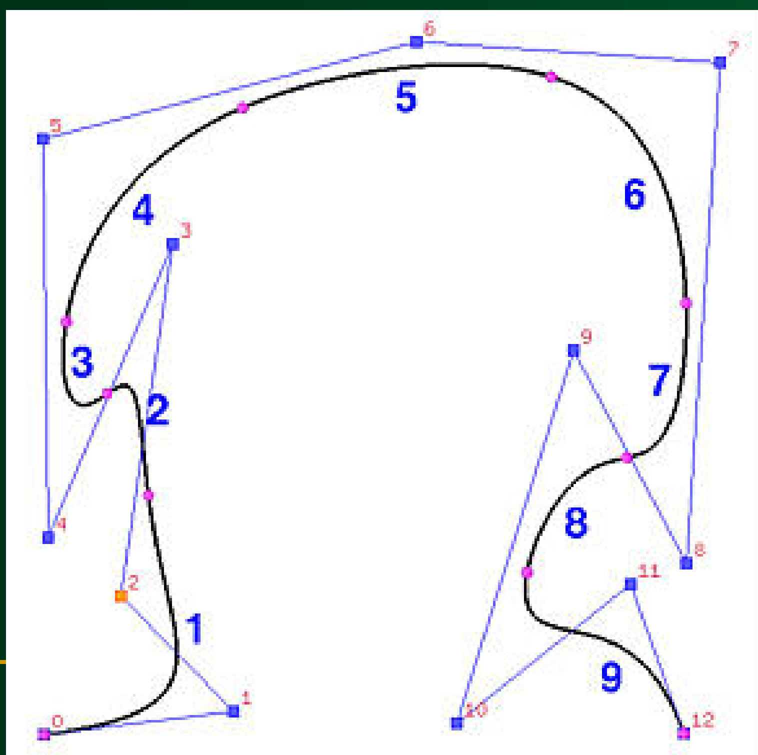
5. **Local Modification Scheme:** changing the position of control point Pi only affects the curve C(u) on interval $[u_i, u_{i+p+1})$.

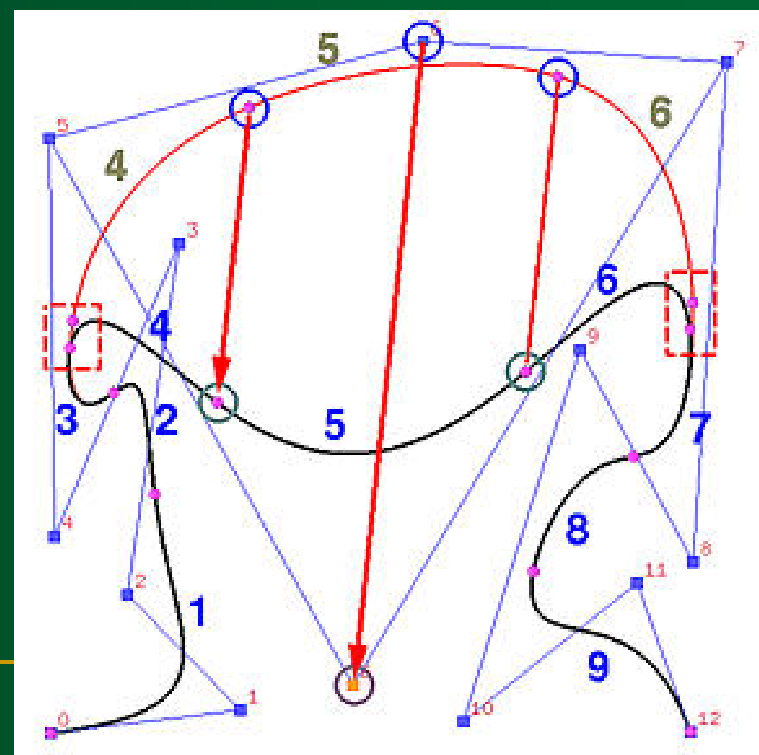


- The right figure shows the result of moving P_2 to the lower right corner. Only the first, second and third curve segments change their shapes and all remaining curve segments stay in their original place without any change.

B-Spline Curves Important Properties

- A B-spline curve of degree 4 defined by 13 control points and 18 knots .
- Move P_6.
- The coefficient of P_6 is $N_{6,4}(u)$, which is non-zero on $[u_6, u_{11})$. Thus, moving P6 affects curve segments 3, 4, 5, 6 and 7. Curve segments 1, 2, 8 and 9 are not affected.





B-Spline Curves Important Properties

6. C(u) is C^{p-k} continuous at a knot of multiplicity k

7. Affine Invariance