CHAPTER 6

1. The energy eigenvalues are $\varepsilon_k = \frac{\hbar^2}{2m}k^2$. The mean value over the volume of a sphere in k space is

$$<\varepsilon> = \frac{\hbar^2}{2m}\frac{\int k^2 dk \cdot k^2}{\int k^2 dk} = \frac{3}{5}\cdot\frac{\hbar^2}{2m}k_F^{\ 2} = \frac{3}{5}\varepsilon_F.$$

The total energy of N electrons is

$$U_0 = N\cdot\frac{3}{5}\varepsilon_F.$$

2a. In general p = –∂U/∂V at constant entropy. At absolute zero all processes are at constant entropy (the Third Law), so that $p = -dU_0/dV$, where $U_0 = \frac{3}{5}N\varepsilon_F \quad = \frac{3}{5}N\frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3}$, whence $p = \frac{2}{3}\cdot\frac{U_0}{V}$. (b) Bulk modulus

$$B = -V\frac{dp}{dV} = V\left(-\frac{2}{3}\frac{U_0}{V^2} + \frac{2}{3V}\frac{dU_0}{dV}\right) = \frac{2}{3}\cdot\frac{U_0}{V} + \left(\frac{2}{3}\right)^2\frac{U_0}{V} = \frac{10}{9}\frac{U_0}{V}.$$

(c) For Li,

$$\begin{aligned}\frac{U_0}{V} &= \frac{3}{5}(4.7\times10^{22}\ \text{cm}^{-3})(4.7\ \text{eV})(1.6\times10^{-12}\ \text{erg/eV})\\ &= 2.1\times10^{11}\ \text{erg cm}^{-3} = 2.1\times10^{11}\ \text{dyne cm}^{-2},\end{aligned}$$

whence B = 2.3 × 10^{11} dyne cm^{-2}. By experiment (Table 3.3), B = 1.2 × 10^{11} dyne cm^{-2}.

3. The number of electrons is, per unit volume, $n = \int_0^\infty d\varepsilon\ D(\varepsilon)\cdot\frac{1}{e^{(\varepsilon-\mu)/\tau}+1}$, where D(ε) is the density of orbitals. In two dimensions

$$\begin{aligned}n &= \frac{m}{\pi\hbar^2}\int_0^\infty d\varepsilon\frac{1}{e^{(\varepsilon-\mu)/\tau}+1}\\ &= \frac{m}{\pi\hbar^2}(\mu + \tau\ \log(1 + e^{-\mu/\tau})),\end{aligned}$$

where the definite integral is evaluated with the help of Dwight [569.1].

4a. In the sun there are $\frac{2\times10^{33}}{1.7\times10^{-24}} \simeq 10^{57}$ nucleons, and roughly an equal number of electrons. In a white dwarf star of volume

$$\frac{4\pi}{3}(2\times10^9)^3 \approx 3\times10^{28}\ \text{cm}^3$$

the electron concentration is $\approx \dfrac{10^{57}}{3\times10^{28}} \approx 3\times10^{28}\ \text{cm}^{-3}$. Thus

$\varepsilon_F = \dfrac{\hbar^2}{2m}(3\pi^2 n)^{2/3} \approx \dfrac{1}{2}10^{-27}\cdot10^{20} \approx \dfrac{1}{2}10^{-7}$ ergs, or $\approx 3.10^4$ eV. (b) The value of k_F is not affected by relativity and is $\approx n^{1/3}$, where n is the electron concentration. Thus $\varepsilon_F \simeq \hbar c k_F \simeq \hbar c^3 \sqrt{n}$. (c) A change of radius to 10 km = 10^6 cm makes the volume $\approx 4\times10^{18}$ cm^3 and the concentration $\approx 3\times10^{38}$ cm^{-3}. Thus $\varepsilon_F \approx 10^{-27}(3.10^{10})(10^{13}) \approx 2.10^{-4}$ erg $\approx 10^8$ eV. (The energy is relativistic.)

5. The number of moles per cm^3 is $81\times10^{-3}/3 = 27\times10^{-3}$, so that the concentration is 16×10^{21} atoms cm^{-3}. The mass of an atom of He^3 is (3.017) (1.661) $\times$ 10^{-24} = 5.01 $\times$ 10^{-24} g. Thus $\varepsilon_F \simeq [(1.1\times10^{-54})/10^{-23}][(30)(16)\times10^{21}]^{2/3} \approx 7\times10^{-16}$ erg, or $T_F \approx 5K$.

6. Let E, v vary as e^{-iwt}. Then

$$v = -\frac{eE/m}{-i\omega + (1/\tau)} = -\frac{e\tau E}{m}\cdot\frac{1 + i\omega\tau}{1+(\omega\tau)^2},$$

and the electric current density is

$$j = n(-e)v = \frac{ne^2\tau}{m}\cdot\frac{1 + i\omega\tau}{1+(\omega\tau)^2}E.$$

7. (a) From the drift velocity equation

$$i\omega v_x = (e/m)E_x + \omega_c v_y\ ;\ i\omega v_y = (e/m)E_y - \omega_c v_x.$$

We solve for v_x, v_y to find

$$({\omega_c}^2 - \omega^2)v_x = i\omega(e/m)E_x + \omega_c(e/m)E_y\ ;$$
$$({\omega_c}^2 - \omega^2)v_y = i\omega(e/m)E_y + \omega_c(e/m)E_x\ .$$

We neglect the terms in ω_c^2. Because j = n(–e)v = σE, the components of σ come out directly. (b) From the electromagnetic wave equation

$$c^2\nabla^2 E = \varepsilon\partial^2 E/\partial t^2,$$

we have, for solutions of the form $e^{i(kz-\omega t)}$, the determinantal equation

$$\begin{vmatrix} \varepsilon_{xx}\omega^2 - c^2k^2 & \varepsilon_{xy}\omega^2 \\ \varepsilon_{yx}\omega^2 & \varepsilon_{yy}\omega^2 - c^2k^2 \end{vmatrix} = 0.$$

Here $\varepsilon_{xx} = \varepsilon_{yy} = 1 - \omega_P{}^2/\omega^2$ and $\varepsilon_{xy} = -\varepsilon_{yx} = i\,\omega_c\omega_p{}^2/\omega^3$. The determinantal equation gives the dispersion relation.

8. The energy of interaction with the ion is

$$e\int_0^{r_0} (\rho/r)4\pi r^2 dr = -3e^2/2r_0 ,$$

where the electron charge density is $-e(3/4\pi r_0{}^3)$. (b) The electron self-energy is

$$\rho^2\int_0^{r_0} dr\left(4\pi r^3/3\right)\left(4\pi r^2\right)r^{-1} = 3e^2/5r_0 .$$

The average Fermi energy per electron is $3\varepsilon_F/5$, from Problem 6.1; because $N/V = 3/4\pi r_0{}^3$, the average is $3(9\pi/4)^{2/3}\hbar^2/10\,mr_0{}^2$. The sum of the Coulomb and kinetic contributions is

$$U = -\frac{1.80}{r_s} + \frac{2.21}{r_s{}^2}$$

which is a minimum at

$$\frac{1.80}{r_s{}^2} = \frac{4.42}{r_s{}^3},\ \text{or}\ r_s = 4.42/1.80 = 2.45.$$

The binding energy at this value of r_s is less than 1 Ry; therefore separated H atoms are more stable.

9. From the magnetoconductivity matrix we have

$$j_y = \sigma_{yx}\,E_x = \frac{\omega_c\tau}{1+(\omega_c\tau)^2}\sigma_0\,E_x .$$

For $\omega_c\tau >> 1$, we have $\sigma_{yx} \cong \sigma_0/\omega_c\tau = \left(ne^2\tau/m\right)\left(mc/eB\tau\right) = neB/c$.

10. For a monatomic metal sheet one atom in thickness, $n \approx 1/d^3$, so that

$$R_{sq} \approx mv_F/nd^2e^2 \approx mv_F d/e^2 .$$

If the electron wavelength is d, then $mv_F d \approx \hbar$ by the de Broglie relation and

$$R_{sq} \approx \hbar/e^2 = 137/c$$

in Gaussian units. Now

$$
\begin{aligned}
R_{sq}(\text{ohms}) &= 10^{-9}\, c^2\, R_{sq}(\text{gaussian}) \\
&\approx (30)(137)\,\text{ohms} \\
&\approx 4.1\,\text{k}\Omega\ .
\end{aligned}
$$