CHAPTER 10

1a. $\frac{d^2B}{dx^2} = \frac{1}{\lambda^2}B$; this is the London equation. The proposed solution is seen directly to satisfy this and to satisfy the boundary conditions $B\left(\pm\frac{1}{2}\delta\right) = B_a$. (b) For $\delta << \lambda_L$,

$$\cosh\frac{x}{\lambda} = 1 + \frac{1}{2}\left(\frac{x}{\lambda_L}\right)^2 + \dots$$

$$\cosh\frac{\delta}{2\lambda} = 1 + \frac{1}{2}\left(\frac{\delta}{2\lambda}\right)^2 + \dots$$

therefore $B(x) = B_a - B_a\left(1/8\lambda^2\right)\left(\delta^2 - 4x^2\right)$.

2a. From (4), $dF_S = -\mathbf{M}d\mathbf{B}_a$ at $T = 0$. From Problem 1b,

$$M(x) = -\frac{1}{4\pi}\cdot\frac{1}{8\lambda^2}B_a\cdot\left(\delta^2 - 4x^2\right),$$

whence

$$F_S(x,B_a) - F_S(0) = \frac{1}{64\pi\lambda^2}\left(\delta^2 - 4x^2\right)B_a^{\ 2}.$$

b. The average involves

$$\frac{\int_0^{1/2\delta}\left(\delta^2 - 4x^2\right)dx}{\frac{1}{2}\delta} = \frac{\frac{1}{2}\delta^3 - \frac{4}{3}\cdot\frac{\delta^3}{8}}{\frac{1}{2}\delta} = \frac{2}{3}\delta^2,$$

whence

$$\langle\Delta F\rangle = \frac{1}{96\pi}B_a^{\ 2}\left(\frac{\delta}{\lambda}\right)^2, \text{ for } \delta << \lambda.$$

c. Let us set

$$\frac{1}{96\pi}B_{af}^2\left(\frac{\delta}{\lambda}\right)^2 = \frac{1}{8\pi}B_{ac}^2,$$

where B_{af} is the critical field for the film and B_{ac} is the bulk critical field. Then

$$B_{af} = \sqrt{12} \cdot \frac{\lambda}{\delta} B_{ac}.$$

3a. (CGS) $\mathrm{curl}\, H = \frac{4\pi j}{c} + \frac{1}{c}\frac{\partial E}{\partial t} = \frac{4\pi}{c}\left(\sigma_0 E - \frac{c}{4\pi\lambda^2} A\right) + \frac{1}{c}\frac{\partial E}{\partial t}.$

$$\mathrm{curl}\,\mathrm{curl}\, H = -\nabla^2 H = \frac{4\pi}{c}\left(\sigma_0\, \mathrm{curl}\, E - \frac{c}{4\pi\lambda^2} B\right) + \frac{1}{c}\frac{\partial\, \mathrm{curl}\ E}{\partial t}.$$

Now in CGS in nonmagnetic material B and H are identical. We use this and we use the Maxwell equation

$$\mathrm{curl}\, E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

to obtain

$$\nabla^2 B - \frac{1}{\lambda^2} B \frac{1}{c^2}\frac{\partial^2 B}{\partial t^2} - \frac{4\pi\sigma_0}{c^2}\frac{\partial B}{\partial t} = 0.$$

If $B \sim e^{i}(\mathbf{k} \cdot \mathbf{r} - \omega t)$, then

$$-k^2 - \frac{1}{\lambda^2} + \frac{\omega^2}{c^2} + \frac{4\pi i \sigma_0 \omega}{c^2} = 0. \qquad \text{Q.E.D.}$$

b. $\frac{1}{\lambda^2} = \frac{\omega_p{}^2}{c^2} \gg \frac{\omega^2}{c^2}$; also, $\omega^2 << 4\pi\sigma_0\omega$ and $\frac{1}{\lambda^2} >> \frac{4\pi\sigma_0\omega}{c^2}$.

Thus the normal electrons play no role in the dispersion relation in the low frequency range.

4. The magnetic influence of the core may be described by adding the two-dimensional delta function $\Phi_0\delta(\rho)$, where ϕ_0 is the flux quantum. If the magnetic field is parallel to the z axis and div $\mathbf{B}$ = 0, then

$$B - \lambda^2\nabla^2 B = \Phi_0\delta(\rho),$$

or

$$\lambda^2\left(\frac{\partial^2 B}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial B}{\partial\rho}\right)-B=-\Phi_0\delta(\rho).$$

This equation has the solution $B(\rho)=\left(\Phi_0/2\pi\lambda^2\right)K_0(\rho/\lambda)$, where K_0 is a hyperbolic Bessel function* infinite at the origin and zero at infinity:

$$(\rho<<\lambda)\,B(\rho)\simeq\left(\Phi_0/2\pi\lambda^2\right)\ell n(\lambda/\rho);$$
$$(\rho>>\lambda)\,B(\rho)\simeq\left(\Phi_0/2\pi\lambda^2\right)(\pi\lambda/2\rho)^{1/2}\exp(-\rho/\lambda).$$

The total flux is the flux quantum:

$$2\pi\int_0^\infty d\rho\;\rho\;B(\rho)=\Phi_0\int_0^\infty dx\;x\;K_0(x)=\Phi_0.$$

5. It is a standard result of mechanics that $\mathbf{E}=-\mathrm{grad}\,\varphi-c^{-1}\partial\mathbf{A}/\partial t.$ If grad φ = 0, when we differentiate the London equation we obtain $\partial j/\partial t=\left(c^2/4\pi\lambda_L{}^2\right)E.$ Now j = nqv and $\partial j/\partial t=nq\,\partial v/\partial t=\left(nq^2/m\right)E.$ Compare the two equations for $\partial j/\partial t$ to find $c^2/4\pi\lambda_L{}^2=nq^2/m.$

*Handbook of mathematical functions, U.S. National Bureau of Standards AMS 55, sec. 9.6.

6. Let x be the coordinate in the plane of the junction and normal to B, with $-w/2\le x\le w/2.$ The flux through a rectangle of width 2x and thickness T is 2xTB = ϕ (x). The current through two elements at x and –x, each of width dx is

$$dJ=\left(J_0/w\right)\cos\left[e\Phi(x)/\hbar c\right]dx=\left(J_0/w\right)\cos\left(2xTeB/\hbar c\right)dx,$$

and the total current is

$$J=\left(J_0/w\right)\int_0^{w/2}\cos\left(xTe\,B/\hbar c\right)dx=J_0\frac{\sin\left(wTBe/\hbar c\right)}{\left(wTBe/\hbar c\right)}.$$

7a. For a sphere $H(\text{inside})=B_a-4\pi M/3;$ for the Meissner effect $H(\text{inside})=-4\pi M,$ whence $B_a=-8\pi M/3.$

b. The external field due to the sphere is that of a dipole of moment μ = MV, when V is the volume. In the equatorial plane at the surface of the sphere the field of the sphere is $-\mu/a^3=-4\pi M/3=B_a/2.$ The total field in this position is $3B_a/2$.