CHAPTER 11

1. From Eq. (10),

$$\chi = -N \frac{e^2}{6mc^2} < r^2 > .$$

Here $< r^2 > = \frac{1}{\pi a_0^{\ 3}} \cdot 4\pi \int_0^\infty r^2 \ dr \cdot e^{-2r/a_0} = 3a_0^{\ 2}$.

The numerical result follows on using N = 6.02 × 10^{23} mol^{-1}.

2a. Eu^{++} has a half-filled f shell. Thus S = 7 × 1/2 = 7/2. The orbitals m_L = 3, 2, 1, 0, –1, –2, –3 have one spin orientation filled, so that L = Σm_L = 0. Also J = L + S = 7/2. Hence the ground state is $^8S_{7/2}$.

b. Yb^{+++} has 13 electrons in the f shell, leaving one hole in the otherwise filled shell. Thus L = 3, S = 1/2, J = 7/2 -- we add S to L if the shell is more than half-filled. The ground state symbol is $^2F_{7/2}$.

c. Tb^{+++} has 8 f electrons, or one more than Eu^{++}. Thus L = 3; S = 7/2 – 1/2 = 3; and J = 6. The ground state is $^7F_6$.

3a. The relative occupancy probabilities are

______ $e^{-(\Delta+\mu B)/kT}$ (Here Δ stands for $k_B\Delta$)

______ $e^{-\Delta/kT}$

______ $e^{-(\Delta-\mu B)/kT}$

______ 1

The average magnetic moment is

$$< \mu > = \mu \frac{e^{-(\Delta-\mu B)/kT} - e^{-(\Delta+\mu B)/kT}}{Z}$$

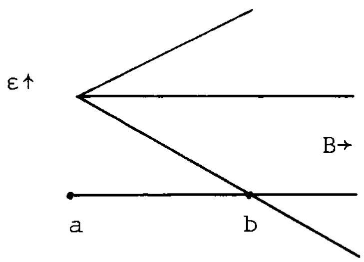
$$\text{where } Z = 1 + e^{-(\Delta-\mu B)/kT} + e^{-\Delta/kT} + e^{-(\Delta+\mu B)/kT}.$$

b. At high temperatures $e^{-\Delta/kT} \to 1$ and

$$< \mu > \to \mu \cdot \frac{\left(1+\frac{\mu B}{kT}+\dots\right)-\left(1-\frac{\mu B}{kT}+\dots\right)}{4}$$

$$= \frac{\mu^2 B}{2kT}\ ; \quad \chi \to \frac{N\mu^2}{2kT}.$$

c. The energy levels as a function of field are:



If the field is applied to take the system from a to b we increase the entropy of the spin system from ≈ 0 to ≈ N log 2. If the magnetization is carried out constant total entropy, it is necessary that the lattice entropy be reduced, which means the temperature ↓.

4a. $Z = 1 + e^{-\Delta/T}$;

$$E = \frac{k_B \Delta e^{-\Delta/T}}{1+e^{-\Delta/T}} = \frac{k_B \Delta}{e^{\Delta/T}+1}$$

$$C = \left(\frac{\partial E}{\partial T}\right)_\Delta = k_B \Delta \frac{\frac{\Delta}{T^2} e^{\Delta/T}}{\left(e^{\Delta/T}+1\right)^2}.$$

b. For $\Delta/T << 1$, $e^{\Delta/T} \to 1$ and $C \to \frac{1}{4} k_B \left(\frac{\Delta}{T}\right)^2$.

5a. If the concentration in the spin-up band is $N^+ = 1/2\ N\ (1 + \zeta)$, the kinetic energy of all the electrons in that band is

$$\frac{3}{5} N^+ \frac{\hbar^2}{2m}\left(3\pi^2 N^+\right)^{2/3} = E_0 \left(1+\zeta\right)^{5/3},$$

and the magnetic energy is $-N^+ \mu B = -1/2\ N(1+\zeta)\ \mu B$.

b. Now $E_{tot} = E_0 \left\{\left(1+\zeta\right)^{5/3} + \left(1-\zeta\right)^{5/3}\right\} - N\zeta\mu B$;

$$\frac{\partial E_{tot}}{\partial \zeta} = \frac{5}{3} E_0 \underbrace{\left\{(1+\zeta)^{2/3} - (1-\zeta)^{2/3}\right\}}_{\simeq \frac{4}{3}\zeta} - N\mu B = 0$$

$$\therefore \zeta = \frac{9N\mu B}{20E_0} = \frac{3\mu B}{2\varepsilon_F}$$

$$M = N\mu\zeta = \frac{3N\mu^2}{2\varepsilon_F} B \,. \quad \text{Q.E.D.}$$

6a. The number of pairs of electrons with parallel spin up is

$$\frac{1}{2}\left(N^+\right)^2 = \frac{1}{8} N^2 (1+\zeta)^2 \ ,$$

so that the exchange energy among the up spins is

$$-\frac{1}{8} VN^2 (1+\zeta)^2 \ ;$$

and among the down spins the exchange energy is

$$-\frac{1}{8} VN^2 (1-\zeta)^2 \ .$$

b. Using these results and those from Prob. 5 we have $E_{tot} = E_0 \left\{(1+\zeta)^{5/3} + (1-\zeta)^{5/3}\right\}$ $-\frac{1}{8} VN^2 \left(1+\zeta^2\right) 2 - N\zeta\mu B.$ Thus (for $\zeta << 1$)

$$\frac{\partial E_{tot}}{\partial \zeta} \simeq \frac{20}{9} E_0 \zeta - \frac{1}{2} VN^2 \zeta - N\mu B = 0 \ ;$$

$$\zeta = \frac{N\mu B}{\frac{20}{9} E_0 - \frac{1}{2} VN^2} = \frac{N\mu B}{\frac{2N\varepsilon_F}{3} - \frac{1}{2} VN^2}$$

and

$$M = N\mu\zeta = \frac{3N\mu^2}{2\varepsilon_F - \frac{3}{2} VN} B$$

c. For B = 0 and ζ = 0.

$$\frac{\partial^2 E_{tot}}{\partial\zeta^2} \simeq \frac{20}{9}E_0 - \frac{1}{2}VN^2 < 0 \text{ if } V > \frac{40}{9}\frac{E_0}{N^2} = \frac{4}{3}\frac{{}^{\varepsilon}F}{N}$$

7a. The Boltzmann factor gives directly, with $\tau = k_B T$

$$U = -\Delta\frac{e^{\Delta/\tau} - e^{-\Delta/\tau}}{e^{\Delta/\tau} + e^{-\Delta/\tau}} = -\Delta \tanh \Delta/\tau;$$

$$C = k_B \; dU/d\tau = k_B \left(\Delta/k_B T\right)^2 \operatorname{sech}^2 \left(\Delta/k_B T\right),$$

because d tanh x/dx = $\operatorname{sech}^2 x$.

b. The probability $P(\Delta)\, d\Delta$ that the upper energy level lies between Δ and $\Delta + d\Delta$, referred to the midpoint as the zero of energy, is $P(\Delta)\, d\Delta = (d\Delta) / \Delta_0$. Thus, from (a),

$$< U >= -\int_0^{\Delta_0} d\Delta \left(\Delta/\Delta_0\right) \tanh \Delta/\tau,$$

$$< C > = k_B \int_0^{\Delta_0} d\Delta \left(\Delta^2/\Delta_0 \tau^2\right) \operatorname{sech}^2 \left(\Delta/\tau\right)$$

$$= \left(k_B\tau/\Delta_0\right) \int_0^{x_0} dx \; x^2 \; \operatorname{sech}^2 x \; ,$$

where $x \equiv \Delta/\tau$. The integrand is dominated by contributions from $0 < \Delta < \tau$, because sech x decreases exponentially for large values of x. Thus

$$< C > \simeq \left(k_B\tau/\Delta_0\right) \int_0^{\infty} dx \; x^2 \; \operatorname{sech}^2 x \; .$$

8. $\dfrac{<\mu>}{\mu} = \dfrac{e^{\mu B/\tau} - e^{-\mu B/\tau}}{1 + 2\cosh x} = \dfrac{2 \sinh x}{1 + 2 \cosh x}$