CHAPTER 14

1. $E_{x0} = -\frac{\partial \varphi}{\partial x} = kA \sin kx\, e^{kz}$, and at the boundary this is equal to E_{xi}. The normal component of **D** at the boundary, but outside the medium, is ε(ω)kA cos kx, where for a plasma $\varepsilon(\omega) = 1 - \omega_p^2/\omega^2$. The boundary condition is –kA cos kx = ε(ω)kA cos kx, or ε(ω) = –1, or $\omega_p^2 = 2\omega^2$. This frequency $\omega = \omega_p \big/ \sqrt{2}$ is that of a surface plasmon.

2. A solution below the interface is of the form $\varphi(-) = A \cos kx\, e^{kz}$, and above the interface $\varphi(+) = A \cos kx\, e^{-kz}$, just as for Prob. (1). The condition that the normal component of **D** be continuous across the interface reduces to $\varepsilon_1(\omega) = -\varepsilon_2(\omega)$, or

$$1 - \frac{\omega_{p1}^2}{\omega^2} = -1 + \frac{\omega_{p2}^2}{\omega^2}, \text{ so that } \omega^2 = \frac{1}{2}(\omega_{p1}^2 + \omega_{p2}^2) .$$

3. (a) The equation of motion of the electrons is
$-\omega^2 x_e = -(e/m_e)E_x + i\omega\omega_e y_e;\ -\omega^2 y_e = -(e/m_e)E_y - i\omega\omega_e x_e$. For the holes,

$$-\omega^2 x_h = (e/m_h)E_x + i\omega\omega_h y_h;\ -\omega^2 y_h = (e/m_h)E_y - i\omega\omega_h x_h .$$

The result follows on forming $\xi_e = x_e + iy_e$ and $\xi_h = x_h + iy_h$. (b) Expand
$(\omega_e + \omega)^{-1} \simeq \omega_e^{-1}(1 - \omega/\omega_e)$ and $(\omega_h - \omega) \simeq \omega_h^{-1}(1 + \omega/\omega_h)$. In this approximation
$(\xi_h - \xi_e)/E^+ \simeq (c/B)(\omega_h^{-1} + \omega_e^{-1}) = (c^2/eB^2)(m_h + m_e)$.

4. From the solution to Problem 3 we have $P^+ = pe^2E^+/m_h\omega_h\omega$, where we have dropped a term in ω^2 in comparison with $\omega_h\omega$. The dielectric constant
$\varepsilon(\omega) = 1 + 4\pi P^+/E^+ \simeq 4\pi pe^2/m_h\omega_h\omega$, and the dispersion relation $\varepsilon(\omega)\omega^2 = c^2k^2$ becomes $4\pi pe^2\omega/(eB/c) = c^2k^2$. Numerically, $\omega \approx [(10^3)(3\times10^{10})/(10)(3\times10^{22})(5\times10^{-10})] \approx 0.2\ s^{-1}$.
It is true that ωτ will be <<1 for any reasonable relaxation time, but $\omega_c \tau > 1$ can be shown to be the applicable criterion for helicon resonance.

5. $md^2\mathbf{r}/dt^2 = -m\omega^2\mathbf{r} = -e\mathbf{E} = 4\pi e\mathbf{P}/3 = -4\pi ne^2\mathbf{r}/3$. Thus $\omega_o^2 = 4\pi ne^2\big/3m$.

6. $md^2\mathbf{r}\big/dt^2 = -m\omega^2\mathbf{r} = -(e/c)(\mathbf{v}\times B\hat{\mathbf{z}}) - m\omega_o^2\mathbf{r}$, where $\omega_o^2 = 4\pi ne^2/3m$, from the solution to A. Thus, with $\omega_c \equiv eB/mc$,

$$-\omega^2 x = i\omega\omega_c y - \omega_o^2 x ;$$
$$-\omega^2 y = -i\omega\omega_c x - \omega_o^2 y .$$

Form $\xi \equiv x + iy$; then $-\omega^2\xi - \omega\omega_c\xi + \omega_o{}^2\xi = 0$, or $\omega^2 + \omega\omega_c - \omega_o{}^2 = 0$, a quadratic equation for ω.

7. Eq. (53) becomes $c^2K^2E = \omega^2[\varepsilon(\infty)E + 4\pi P]$, where P is the ionic contribution to the polarization. Then (55) becomes

$$\begin{vmatrix} \omega^2\varepsilon(\infty) - c^2K^2 & 4\pi\omega^2 & \\ Nq^2/M & \omega^2 & -\omega_T{}^2 \end{vmatrix} = 0,$$

or

$$\omega^4\varepsilon^2(\infty) - \omega^2\left[c^2K^2 + \varepsilon(\omega)\omega_T{}^2 + 4\pi\,Nq^2/M\right] + c^2K^2\omega_T{}^2 = 0.$$

One root at K = 0 is $\omega^2 = \omega_T{}^2 + 4\pi Nq^2/\varepsilon(\infty)M.$ For the root at low ω and K we neglect terms in ω^4 and in ω^2K^2. Then

$$\begin{aligned}\omega^2 &= c^2K^2\ \omega_T{}^2/[\varepsilon(\infty)\omega_T{}^2 + 4\pi Nq^2/M] \\ &= c^2K^2/[\varepsilon(\infty) + 4\pi Nq^2/M\omega_T{}^2] = c^2K^2/\varepsilon(0)\ ,\end{aligned}$$

where ε (0) is given by (58) with ω = 0.

8(a). $\sigma = ne^2\tau/m = (\omega_p{}^2/4\pi)\tau = 0.73 \times 10^{15}\ s^{-1} = 800\,(\Omega\ cm)^{-1}$

(b) $\omega_p{}^2 = 4\pi ne^2/m^*;\ \ m^* = 4\pi ne^2/\omega_p{}^2 = 4.2 \times 10^{-27}\,g;\ \ m^*/m = 4.7.$

9. The kinetic energy of a Fermi gas of N electrons in volume V is $U = N(3/5)(\hbar^2/2m)(3\pi^2N/V)^{2/3}$. Then dU/dV = – (2/3)U/V and d^2U/dV^2 = (10/9)U/V^2. The bulk modulus $B = Vd^2U/dV^2 = (10/9)\ U/V = (10/9)\ (3/5)\,n\,(mv_F{}^2/2) = nmv_F{}^2/3.$ The velocity of sound $v = (B/\rho)^{1/2}$, where the density $\rho = n\,(m + M) \simeq nM$, whence $v \simeq (m/3M)^{1/2}\ v_F$.

10. The response is given, with ρ = 1/τ, by

$$m\left(d^2\,x/dt^2 + \rho dx/dt + \omega_p{}^2x\right) = F(t)\ .$$

The conductivity σ does not enter this equation directly, although it may be written as $\sigma = \omega_p{}^2\tau/4\pi$. For order of magnitude,

$\sigma = \left(1/10^{-6}\right)\left(9\times10^{11}\right) \simeq 10^{18}\ \mathrm{s}^{-1}$;

$\rho = 1/\tau = v_F/\ell \simeq \left(1.6\times10^{8}\right)/\left(4\times10^{-6}\right) \simeq 0.4\times10^{14}\ \mathrm{s}^{-1}$;

$\omega_p = \left(4\pi ne^2/m\right)^{1/2} \simeq \left(10\times10^{23}\times23\times10^{-20}/10^{-27}\right)^{1/2}$

$\simeq 1.5\times10^{16}\ \mathrm{s}^{-1}$.

The homogeneous equation has a solution of the form $x\left(t>0\right) = Ae^{-\lambda t}\sin\left(\omega t+\phi\right)$, where $\omega = \left[\omega_p{}^2 + \left(\rho/2\right)^2\right]^{1/2}$ and $\lambda = \rho/2$. To this we add the particular solution $x = -e/m\omega$ and find A and ϕ to satisfy the initial conditions $x(0) = 0$ and $\dot{x}\left(0\right) = 0$.

11. The Laplacian $\nabla^2\varphi = 0,$ whence

$$\frac{d^2f}{dz^2} - K^2 f = 0\ .$$

This has solutions

$$f = Ae^{Kz}\ \text{for}\ z < 0$$
$$f = Ae^{-K(z-d)}\ \text{for}\ z > d$$
$$f = B\cosh K\left(z - d/2\right)\ \text{for}\ 0 < z < d\ .$$

This solution assures that φ will be continuous across the boundaries if $B = A/\cosh(Kd/2)$. To arrange that the normal component of D is continuous, we need $\varepsilon(\omega)\ \partial\varphi/\partial z$ continuous, or $\varepsilon(\omega) = -\tanh(Kd/2)$.