CHAPTER 15

1a. The displacement under this force is

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \alpha(\omega)\, e^{-i\omega t}\, d\omega.$$

With $\omega = \omega_R + i\omega_I$, the integral is $\int \alpha(\omega) e^{-i\omega_R t} e^{\omega_I t}\, d\omega$. This integral is zero for $t < 0$ because we may then complete a contour with a semicircle in the upper half-plane, over which semicircle the integral vanishes. The integral over the entire contour is zero because $\alpha(\omega)$ is analytic in the upper half-plane. Therefore $x(t) = 0$ for $t < 0$.

1b. We want

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}\, d\omega}{\omega_0{}^2 - \omega^2 - i\omega\rho}\,, \tag{A}$$

which is called the retarded Green's function of the problem. We can complete a contour integral by adding to $x(t)$ the integral around an infinite semicircle in the upper half-plane. The complete contour integral vanishes because the integrand is analytic everywhere within the contour. But the integral over the infinite semicircle vanishes at $t < 0$, for then

$$\exp\left[-i(\omega_R + i\omega_I)(-|t|)\right] = \exp(-\omega_I |t|)\ \exp(i\omega_R |t|)\,,$$

which $\to 0$ as $|\omega| \to \infty$. Thus the integral in (A) must also vanish. For $t > 0$ we can evaluate $x(t)$ by carrying out a Cauchy integral in the lower half-plane. The residues at the poles are

$$\pm\tfrac{1}{2}\left(\omega_0{}^2 - \tfrac{1}{4}\rho^2\right)^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\rho t\right)\ \exp\left[\mp i\left(\omega_0{}^2 - \tfrac{1}{4}\rho^2\right)^{\frac{1}{2}} t\right],$$

so that

$$x(t) = \left(\omega_0{}^2 - \tfrac{1}{4}\rho^2\right)^{\frac{1}{2}} \exp\left(-\tfrac{1}{2}\rho t\right) \sin\left(\omega_0{}^2 - \tfrac{1}{4}\rho^2\right)^{\frac{1}{2}} t\,.$$

2. In the limit $\omega \to \infty$ we have

$$\alpha'(\omega) \to -\sum f_j / \omega^2$$

from (9), while from (11a)

$$\alpha'(\omega) \to -\frac{2}{\pi\omega^2}\int_0^\infty s\alpha''(s)\,ds.$$

3. The reflected wave in vacuum may be written as

$$-E_y(\text{refl}) = B_z(\text{refl}) = A'\,e^{-i(kx+\omega t)},$$

where the sign of E_y has been reversed relative to B_z in order that the direction of energy flux (Poynting vector) be reversed in the reflected wave from that in the incident wave. For the transmitted wave in the dielectric medium we find

$$\begin{aligned} E_y(\text{trans}) &= ck\ B_z(\text{trans})/\varepsilon\omega \\ &= \varepsilon^{-1/2}\,B_z(\text{trans}) = A''e^{i(kx-\omega t)}, \end{aligned}$$

by use of the Maxwell equation c curl H = $\varepsilon\partial E/\partial t$ and the dispersion relation $\varepsilon\omega^2 = c^2k^2$ for electromagnetic waves.

The boundary conditions at the interface at x = 0 are that E_y should be continuous: E_y (inc) + E_y (refl) = E_y (trans), or A – A' = A". Also B_z should be continuous, so that A + A' = $\varepsilon^{1/2}$ A". We solve for the ratio A'/A to obtain $\varepsilon^{1/2}$ (A – A') = A + A', whence

$$\frac{A'}{A} = \frac{1-\varepsilon^{1/2}}{\varepsilon^{1/2}+1},$$

and

$$r \equiv \frac{E(\text{refl})}{E(\text{inc})} = -\frac{A'}{A} = \frac{\varepsilon^{1/2}-1}{\varepsilon^{1/2}+1} = \frac{n+ik-1}{n+ik+1}.$$

The power reflectance is

$$R(\omega) = r*r = \left(\frac{n-ik-1}{n-ik+1}\right)\left(\frac{n+ik-1}{n+ik+1}\right) = \frac{(n-1)^2+K^2}{(n+1)^2+K^2}.$$

4. (a) From (11) we have

$$\sigma''(\omega) = -\frac{2\omega}{\pi}P\int_0^\infty \frac{\sigma'(s)}{s^2-\omega^2}\,ds.$$

In the limit $\omega \to \infty$ the denominator comes out of the integrand and we have

$$\lim_{\omega \to \infty} \sigma''(\omega) = \frac{2}{\pi\omega} \int_0^\infty \sigma'(s)\, ds.$$

(b) A superconductor has infinite conductivity at zero frequency and zero conductivity at frequencies up to ω_g at the energy gap. We can replace the lost portion of the integral (approximately $\sigma'_n\omega_g$) by a delta function $\sigma'_n\omega_g\, \delta(\omega)$ in $\sigma'_s(\omega)$ at the origin. Then the KK relation above gives

$$\sigma''_s(\omega) = \frac{2}{\pi\omega} \sigma'_n\, \omega_g.$$

(c) At very high frequencies the drift velocity of the conduction electrons satisfies the free electron equation of motion

$$m dv/dt = -eE; \ -i\omega mv = -\, eE,$$

so that the current density is

$$j = n(-e)v = -ine^2\, E/m\omega$$

and $\omega\sigma''(\omega) = ne^2/m$ in this limit. Then use (a) to obtain the desired result.

5. From (11a) we have

$$\varepsilon'(\omega) - 1 = \frac{4\pi ne^2}{m} P \int_0^\infty \frac{\delta(s-\omega_g)}{s^2-\omega^2}\, ds = \frac{\omega_p{}^2}{\omega_g{}^2 - \omega^2}.$$

6. $n^2 - K^2 + 2inK = 1 + 4\pi i\sigma_0/\omega$. For normal metals at room temperature $\sigma_0 \sim 10^{17} - 10^{18}$ $\sec^{-1}$, so that in the infrared $\omega \ll \sigma_0$. Thus $n^2 \simeq K^2$, so that $R \simeq 1 - 2/n$ and $n \simeq \sqrt{(2\pi\sigma_0/\omega)}$, whence $R \simeq 1 - \sqrt{(2\omega/\pi\sigma_0)}$. (The units of σ_0 are $\sec^{-1}$ in CGS.)

7. The ground state of the line may be written $\psi_g = A_1B_1A_2B_2 \ldots A_NB_N$. Let the asterisk denote excited state; then if specific single atoms are excited the states are $\varphi_j = A_1B_1A_2B_2 \ldots A_j^*B_j \ldots A_NB_N$; $\theta_j = A_1B_1A_2B_2 \ldots A_jB_j^* \ldots A_NB_N$. The hamiltonian acts thusly:

$$H\varphi_j = \varepsilon_A\varphi_j + T_1\theta_j + T_2\theta_{j-1};$$
$$H\theta_j = \varepsilon_B\theta_j + T_1\varphi_j + T_2\varphi_j + 1.$$

An eigenstate for a single excitation will be of the form $\psi_k = \sum_j e^{ijka}\left(\alpha\varphi_j + \beta\theta_j\right)$. We form

$$H\psi_k = \sum_j e^{ijka}[\alpha\varepsilon_A\varphi_j + \alpha T_1\theta_j + \alpha T_2\theta_{j-1}$$

$$+\beta\varepsilon_B{}^{\theta}j + \beta T_1\varphi_j + \beta T_2\varphi_j + 1].$$

$$= \sum_j e^{ijka}[\left(\alpha\varepsilon_A + \beta T_1 + e^{-ika}\beta T_2\right)\varphi_j$$

$$+\left(\alpha T_1 + \beta\varepsilon_B + e^{ika}\alpha T_2\right)\theta_j]$$

$$= E\psi_k = \sum e^{ijka}[\alpha E\varphi_j + \beta E\theta_j].$$

This is satisfied if

$$\left(\varepsilon_A - E\right)\alpha + \left(T_1 + e^{-ika}T_2\right)\beta = 0;$$

$$\left(T_1 + e^{ika}T_2\right)\alpha + \left(\varepsilon_B - E\right)\beta = 0.$$

The eigenvalues are the roots of

$$\begin{vmatrix} \varepsilon_A - E & T_1 + e^{-ika}T_2 \\ T_1 + e^{ika}T_2 & \varepsilon_B - E \end{vmatrix} = 0.$$