Instructor's Solutions Manual

SEVENTH EDITION

ANALYTICAL MECHANICS

FOWLES & CASSIDAY

Instructor's Solutions Manual

To Accompany

Analytical Mechanics 7th Edition

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CHAPTER 1
FUNDAMENTAL CONCEPTS: VECTORS

1.1 (a) $\vec{A}+\vec{B}=(\hat{i}+\hat{j})+(\hat{j}+\hat{k})=\hat{i}+2\hat{j}+\hat{k}$

$$\left|\vec{A}+\vec{B}\right|=(1+4+1)^{\frac{1}{2}}=\sqrt{6}$$

(b) $3\vec{A}-2\vec{B}=3(\hat{i}+\hat{j})-2(\hat{j}+\hat{k})=3\hat{i}+\hat{j}-2\hat{k}$

(c) $\vec{A}\cdot\vec{B}=(1)(0)+(1)(1)+(0)(1)=1$

(d) $$\vec{A}\times\vec{B}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 0\\ 0 & 1 & 1\end{vmatrix}=\hat{i}(1-0)+\hat{j}(0-1)+\hat{k}(1-0)=\hat{i}-\hat{j}+\hat{k}$$

$$\left|\vec{A}\times\vec{B}\right|=(1+1+1)^{\frac{1}{2}}=\sqrt{3}$$

1.2 (a) $\vec{A}\cdot\left(\vec{B}+\vec{C}\right)=\left(2\hat{i}+\hat{j}\right)\cdot\left(\hat{i}+4\hat{j}+\hat{k}\right)=(2)(1)+(1)(4)+(0)(1)=6$

$$\left(\vec{A}+\vec{B}\right)\cdot\vec{C}=\left(3\hat{i}+\hat{j}+\hat{k}\right)\cdot 4\hat{j}=(3)(0)+(1)(4)+(1)(0)=4$$

(b) $$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}2 & 1 & 0\\ 1 & 0 & 1\\ 0 & 4 & 0\end{vmatrix}=-8$$

$$\left(\vec{A}\times\vec{B}\right)\cdot\vec{C}=\vec{A}\cdot\left(\vec{B}\times\vec{C}\right)=-8$$

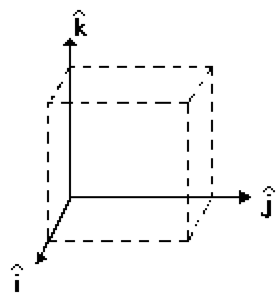
(c) $\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\left(\vec{A}\cdot\vec{C}\right)\vec{B}-\left(\vec{A}\cdot\vec{B}\right)\vec{C}=4\left(\hat{i}+\hat{k}\right)-2\left(4\hat{j}\right)=4\hat{i}-8\hat{j}+4\hat{k}$

$$\left(\vec{A}\times\vec{B}\right)\times\vec{C}=-\vec{C}\times\left(\vec{A}\times\vec{B}\right)=-\left[\left(\vec{C}\cdot\vec{B}\right)\vec{A}-\left(\vec{C}\cdot\vec{A}\right)\vec{B}\right]$$
$$=-\left[0\left(2\hat{i}+\hat{j}\right)-4\left(\hat{i}+\hat{k}\right)\right]=4\hat{i}+4\hat{k}$$

1.3 $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB} = \dfrac{(a)(a)+(2a)(2a)+(0)(3a)}{\sqrt{5a^2}\sqrt{14a^2}} = \dfrac{5a^2}{a^2\sqrt{5}\sqrt{14}}$

$$\theta = \cos^{-1}\sqrt{\frac{5}{14}} \approx 53°$$

1.4



(a) $\vec{A} = \hat{i} + \hat{j} + \hat{k}$: *body diagonal*

$$A = \left|\vec{A}\cdot\vec{A}\right| = \sqrt{\hat{i}\cdot\hat{i} + \hat{j}\cdot\hat{j} + \hat{k}\cdot\hat{k}} = \sqrt{3}$$

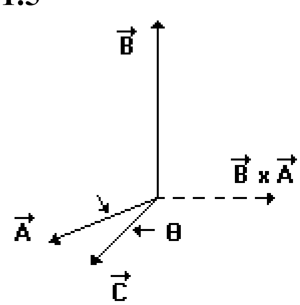
(b) $\vec{B} = \hat{i} + \hat{j}$: *face diagonal*

$$B = \left|\vec{B}\cdot\vec{B}\right| = \sqrt{2}$$

(c) $\vec{C} = \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

(d) $\cos\theta = \dfrac{\vec{A}\cdot\vec{B}}{AB} = \dfrac{1-1}{\sqrt{3}\sqrt{2}} = 0 \qquad \therefore \theta = 90^{\circ}$

1.5



$B = \left|\vec{B}\right| = \left|\vec{A}\times\vec{C}\right| = AC\sin\theta \qquad \therefore C_y = C\sin\theta = \dfrac{B}{A}$

$\vec{A}\cdot\vec{C} = AC\cos\theta = u \qquad \therefore C_x = C\cos\theta = \dfrac{u}{A}$

$$\vec{C} = \frac{\vec{A}}{A}C_x + \frac{\vec{B}\times\vec{A}}{\left|\vec{B}\times\vec{A}\right|}C_y = \frac{u}{A^2}\vec{A} + \frac{\vec{B}\times\vec{A}}{AB}\left(\frac{B}{A}\right)$$

$$= \frac{u}{A^2}\vec{A} + \frac{1}{A^2}\vec{B}\times\vec{A}$$

1.6 $\dfrac{d\vec{A}}{dt} = \hat{i}\dfrac{d}{dt}(\alpha t) + \hat{j}\dfrac{d}{dt}\left(\beta t^2\right) + \hat{k}\dfrac{d}{dt}\left(\gamma t^3\right) = \hat{i}\alpha + \hat{j}2\beta t + \hat{k}3\gamma t^2$

$$\frac{d^2\vec{A}}{dt^2} = \hat{j}2\beta + \hat{k}6\gamma t$$

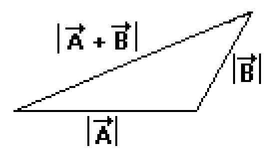
1.7 $0 = \vec{A}\cdot\vec{B} = (q)(q) + (3)(-q) + (1)(2) = q^2 - 3q + 2$

$(q-2)(q-1) = 0$, $q = 1$ or 2

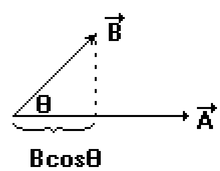
1.8 $\left|\vec{A}+\vec{B}\right|^2 = \left(\vec{A}+\vec{B}\right)\cdot\left(\vec{A}+\vec{B}\right) = A^2 + B^2 + 2\vec{A}\cdot\vec{B}$

$\left[\left|\vec{A}\right| + \left|\vec{B}\right|\right]^2 = A^2 + B^2 + 2AB$

Since $\vec{A}\cdot\vec{B} = AB\cos\theta \le AB$, $\left|\vec{A}+\vec{B}\right| \le \left|\vec{A}\right| + \left|\vec{B}\right|$



$\left|\vec{A}\cdot\vec{B}\right| = \left|AB\cos\theta\right| = \left|\vec{A}\right|\left|\vec{B}\right|\left|\cos\theta\right| \le \left|\vec{A}\right|\left|\vec{B}\right|$



$B\cos\theta \le B$

1.9 Show $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}\cdot\vec{C}\right)\vec{B} - \left(\vec{A}\cdot\vec{B}\right)\vec{C}$

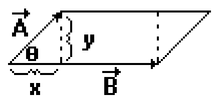
or $$\vec{A}\times\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ B_x & B_y & B_z\\ C_x & C_y & C_z\end{vmatrix} = \left(A_xC_x + A_yC_y + A_zC_z\right)\vec{B} - \left(A_xB_x + A_yB_y + A_zB_z\right)\vec{C}$$

$$= \left(A_xB_xC_x + A_yB_xC_y + A_zB_xC_z - A_xB_xC_x - A_yB_yC_x - A_zB_zC_x\right)\hat{i}$$
$$+\left(A_xB_yC_x + A_yB_yC_y + A_zB_yC_z - A_xB_xC_y - A_yB_yC_y - A_zB_zC_y\right)\hat{j}$$
$$+\left(A_xB_zC_x + A_yB_zC_y + A_zB_zC_z - A_xB_xC_z - A_yB_yC_z - A_zB_zC_z\right)\hat{k}$$

$$\left(A_yB_xC_y + A_zB_xC_z - A_yB_yC_x - A_zB_zC_x\right)\hat{i}$$
$$= +\left(A_xB_yC_x + A_zB_yC_z - A_xB_xC_y - A_zB_zC_y\right)\hat{j}$$
$$+\left(A_xB_zC_x + A_yB_zC_y - A_xB_xC_z - A_yB_yC_z\right)\hat{k}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_yC_z - B_zC_y & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix} \begin{matrix} \hat{i}\left(A_yB_xC_y - A_yB_yC_x - A_zB_zC_x + A_zB_xC_z\right) \\ = +\hat{j}\left(A_zB_yC_z - A_zB_zC_y - A_xB_xC_y + A_xB_yC_x\right) \\ +\hat{k}\left(A_xB_zC_x - A_xB_xC_z - A_yB_yC_z + A_yB_zC_y\right) \end{matrix}$$

1.10



$$y = A\sin\theta$$

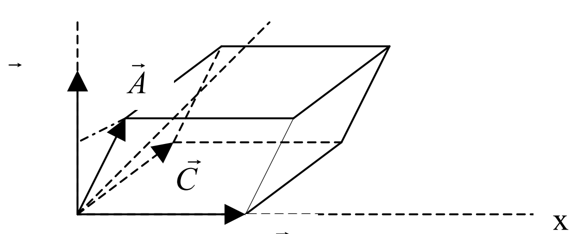
$$\text{A} = 2\left(\frac{1}{2}xy\right) + y\left(B - x\right) = xy + yB - xy = AB\sin\theta$$

$$\text{A} = \left|\vec{A}\times\vec{B}\right|$$

1.11

$$\vec{A}\cdot\left(\vec{B}\times\vec{C}\right) = \vec{A}\cdot\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = -\begin{vmatrix} B_x & B_y & B_z \\ A_x & A_y & A_z \\ C_x & C_y & C_z \end{vmatrix} = -\vec{B}\cdot\left(\vec{A}\times\vec{C}\right)$$

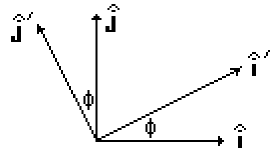
1.12



Let $\vec{A}$ = (Ax,Ay,Az), $\vec{B}$ = (0,By,0) and $\vec{C}$ = (0,Cy,Cz)
C_z is the perpendicular distance between the plane $\vec{A}, \vec{B}$ and its opposite. $\vec{u} = \vec{B}\,x\,\vec{C}$ is directed along the x-axis since the vectors $\vec{B}$, $\vec{C}$ are in the y,z plane. $u_x = \left|\vec{B}\,x\,\vec{C}\right|$ = B_yC_z is the area of the parallelogram formed by the vectors $\vec{B}$, $\vec{C}$. Multiply that area times the height of plane $\vec{A}, \vec{B}$ = A_x to get the volume of the parallopiped

$$V = A_xu_x = A_x\,B_y\,C_z = \vec{A}\bullet\left(\vec{B}\,x\,\vec{C}\right)$$

1.13 For rotation about the z axis:

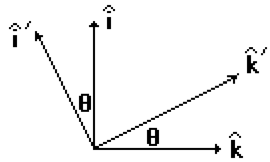


$\hat{i}\cdot\hat{i}'=\cos\phi=\hat{j}\cdot\hat{j}',\quad \hat{k}\cdot\hat{k}'=1$

$\hat{i}\cdot\hat{j}'=-\sin\phi$

$\hat{j}\cdot\hat{i}'=\sin\phi$

For rotation about the y' axis:



$\hat{i}\cdot\hat{i}'=\cos\theta=\hat{k}\cdot\hat{k}',\quad \hat{j}\cdot\hat{j}'=1$

$\hat{i}\cdot\hat{k}'=\sin\theta$

$\hat{k}\cdot\hat{i}'=-\sin\theta$

$$\vec{T}=\begin{pmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{pmatrix}\begin{pmatrix}\cos\phi & \sin\phi & 0\\ -\sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}=\begin{pmatrix}\cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta\\ -\sin\phi & \cos\phi & 0\\ \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta\end{pmatrix}$$

1.14

$\hat{i}\cdot\hat{i}'=\cos 30^\circ=\frac{\sqrt{3}}{2}\qquad \hat{j}\cdot\hat{i}'=\sin 30^\circ=\frac{1}{2}\qquad \hat{k}\cdot\hat{i}'=0$

$\hat{i}\cdot\hat{j}'=-\sin 30^\circ=-\frac{1}{2}\qquad \hat{j}\cdot\hat{j}'=\cos 30^\circ=\frac{\sqrt{3}}{2}\qquad \hat{k}\cdot\hat{j}'=0$

$\hat{i}\cdot\hat{k}'=0\qquad \hat{j}\cdot\hat{k}'=0\qquad \hat{k}\cdot\hat{k}'=1$

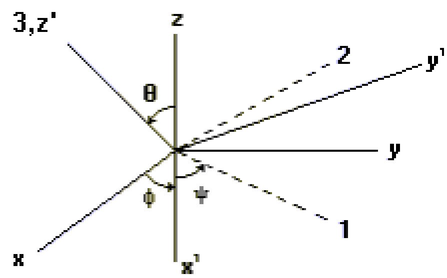
$$\begin{bmatrix}A_{x'}\\ A_{y'}\\ A_{z'}\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2} & \frac{1}{2} & 0\\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}2\\ 3\\ -1\end{bmatrix}=\begin{bmatrix}\sqrt{3}+\frac{3}{2}\\ \frac{3}{2}\sqrt{3}-1\\ -1\end{bmatrix}$$

$\vec{A}=3.232\hat{i}'+1.598\hat{j}'-\hat{k}'$

1.15 1. Rotate thru ϕ about z-axis $\phi = 45^\circ$ R_ϕ

2. Rotate thru θ about x'-axis $\theta = 45^\circ$ R_θ

3. Rotate thru ψ about z'-axis $\psi = 45^\circ$ R_ψ



$$R_\phi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \qquad R_\psi = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$R(\psi,\theta,\phi) = R_\psi R_\theta R_\phi = \begin{pmatrix} \frac{1}{2}-\frac{1}{2\sqrt{2}} & \frac{1}{2}+\frac{1}{\sqrt{2}} & \frac{1}{2} \\ -\frac{1}{2}-\frac{1}{2\sqrt{2}} & -\frac{1}{2}+\frac{1}{2\sqrt{2}} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{\sqrt{2}} \end{pmatrix} \text{ or } \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = R(\psi,\theta,\phi)\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Condition is: $\vec{x}' = R\vec{x}$ where $\vec{x}' = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$

Since $\vec{x} \cdot \vec{x} = 1$ we have: $\psi^2 + \beta^2 + \alpha^2 = 1$

After a lot of algebra: $\alpha = \frac{1}{2} - \frac{\sqrt{2}}{4},\ \beta = \frac{1}{2} + \frac{\sqrt{2}}{4},\ \gamma = \frac{1}{2}$

1.16 $\vec{v} = v\hat{\tau} = ct\hat{\tau}$

$$\vec{a} = \dot{v}\hat{\tau} + \frac{v^2}{\rho}\hat{n} = c\hat{\tau} + \frac{c^2t^2}{b}\hat{n}$$

at $t=\sqrt{\frac{b}{c}}$, $\vec{v}=\hat{\tau}\sqrt{bc}$ and $\vec{a}=c\hat{\tau}+c\hat{n}$

$$\cos\theta=\frac{\vec{v}\cdot\vec{a}}{va}=\frac{c\sqrt{bc}}{\sqrt{bc}\sqrt{2c^2}}=\frac{1}{\sqrt{2}}$$

$$\theta=45^\circ$$

1.17 $\vec{v}(t)=-\hat{i}b\omega\sin(\omega t)+\hat{j}2b\omega\cos(\omega t)$

$$|\vec{v}|=\left(b^2\omega^2\sin^2\omega t+4b^2\omega^2\cos^2\omega t\right)^{\frac{1}{2}}=b\omega\left(1+3\cos^2\omega t\right)^{\frac{1}{2}}$$

$$\vec{a}(t)=-\hat{i}b\omega^2\cos\omega t-\hat{j}2b\omega^2\sin\omega t$$

$$|\vec{a}|=b\omega^2\left(1+3\sin^2\omega t\right)^{\frac{1}{2}}$$

at $t=0$, $|\vec{v}|=2b\omega$; at $t=\frac{\pi}{2\omega}$, $|\vec{v}|=b\omega$

1.18 $\vec{v}(t)=\hat{i}b\omega\cos\omega t-\hat{j}b\omega\sin\omega t+\hat{k}2ct$

$$\vec{a}(t)=-\hat{i}b\omega^2\sin\omega t-\hat{j}b\omega^2\cos\omega t+\hat{k}2c$$

$$|\vec{a}|=\left(b^2\omega^4\sin^2\omega t+b^2\omega^4\cos^2\omega t+4c^2\right)^{\frac{1}{2}}=\left(b^2\omega^4+4c^2\right)^{\frac{1}{2}}$$

1.19 $\vec{v}=\dot{r}\hat{e}_r+r\dot{\theta}\hat{e}_\theta=bke^{kt}\hat{e}_r+bce^{kt}\hat{e}_\theta$

$$\vec{a}=\left(\ddot{r}-r\dot{\theta}^2\right)\hat{e}_r+\left(r\ddot{\theta}+2\dot{r}\dot{\theta}\right)\hat{e}_\theta=b\left(k^2-c^2\right)e^{kt}\hat{e}_r+2bcke^{kt}\hat{e}_\theta$$

$$\cos\phi=\frac{\vec{v}\cdot\vec{a}}{va}=\frac{b^2k\left(k^2-c^2\right)e^{2kt}+2b^2c^2ke^{2kt}}{be^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}be^{kt}\left[\left(k^2-c^2\right)^2+4c^2k^2\right]^{\frac{1}{2}}}$$

$$\cos\phi=\frac{k\left(k^2+c^2\right)}{\left(k^2+c^2\right)^{\frac{1}{2}}\left(k^2+c^2\right)}=\frac{k}{\left(k^2+c^2\right)^{\frac{1}{2}}},\quad\text{a constant}$$

1.20 $\vec{a}=\left(\ddot{R}-R\phi\right)\hat{e}_R+\left(2\dot{R}\phi+R\phi\right)\hat{e}_\phi+\ddot{z}\hat{e}_z$

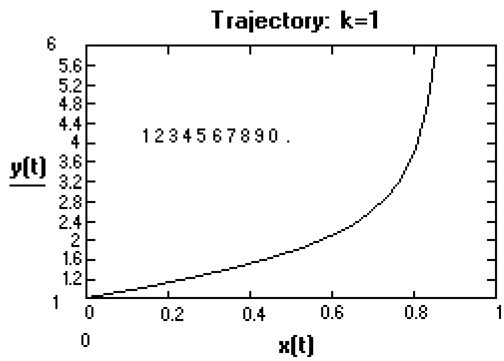
$$\vec{a}=-b\omega^2\hat{e}_R+2c\hat{e}_z$$

$$|\vec{a}|=\left(b^2\omega^4+4c^2\right)^{\frac{1}{2}}$$

1.21 $\vec{r}(t) = \hat{i}\left(1 - e^{-kt}\right) + \hat{j}e^{kt}$

$\vec{r}(t) = \hat{i}ke^{-kt} + \hat{j}ke^{kt}$

$\vec{r}(t) = -\hat{i}k^2e^{-kt} + \hat{j}k^2e^{kt}$



1.22 $\bar{v} = \hat{e}_r\dot{r} + \hat{e}_\phi r\dot{\phi}\sin\theta + \hat{e}_\theta r\dot{\theta}$

$$\bar{v} = \hat{e}_\phi b\omega\sin\left\{\frac{\pi}{2}\left[1 + \frac{1}{4}\cos(4\omega t)\right]\right\} - \hat{e}_\theta b\frac{\pi}{2}\omega\sin(4\omega t)$$

$$\bar{v} = \hat{e}_\phi b\omega\cos\left[\frac{\pi}{8}\cos(4\omega t)\right] - \hat{e}_\theta b\omega\frac{\pi}{2}\sin(4\omega t)$$

$$|\bar{v}| = b\omega\left[\cos^2\left(\frac{\pi}{8}\cos 4\omega t\right) + \frac{\pi^2}{4}\sin^2 4\omega t\right]^{\frac{1}{2}}$$

Path is sinusoidal oscillation about the equator.

1.23 $\bar{v}\cdot\bar{v} = v^2$

$$\frac{d\bar{v}}{dt}\cdot\bar{v} + \bar{v}\cdot\frac{d\bar{v}}{dt} = 2v\dot{v}$$

$$2\bar{v}\cdot\bar{a} = 2v\dot{v}$$

$$\bar{v}\cdot\bar{a} = v\dot{v}$$

1.24 $$\frac{d}{dt}\left[\vec{r}\cdot\left(\vec{v}\times\vec{a}\right)\right]=\frac{d\vec{r}}{dt}\cdot\left(\vec{v}\times\vec{a}\right)+\vec{r}\cdot\frac{d}{dt}\left(\vec{v}\times\vec{a}\right)$$

$$=\vec{v}\cdot\left(\vec{v}\times\vec{a}\right)+\vec{r}\cdot\left[\left(\frac{d\vec{v}}{dt}\times\vec{a}\right)+\left(\vec{v}\times\frac{d\vec{a}}{dt}\right)\right]$$

$$=0+\vec{r}\cdot\left[0+\left(\vec{v}\times\dot{\vec{a}}\right)\right]$$

$$\frac{d}{dt}\left[\vec{r}\cdot\left(\vec{v}\times\vec{a}\right)\right]=\vec{r}\cdot\left(\vec{v}\times\dot{\vec{a}}\right)$$

1.25 $\vec{v}=v\hat{\tau}$ and $\vec{a}=a_\tau\hat{\tau}+a_n\hat{n}$

$\vec{v}\cdot\vec{a}=va_\tau$, so $a_\tau=\dfrac{\vec{v}\cdot\vec{a}}{v}$

$a^2=a_\tau^2+a_n^2$, so $a_n=\left(a^2-a_\tau^2\right)^{\frac{1}{2}}$

1.26 For 1.14, $$a_\tau=\frac{-b^2\omega^3\cos\omega t\cdot\sin\omega t+b^2\omega^3\sin\omega t\cdot\cos\omega t+4c^2t}{\left(b^2\omega^2\cos^2\omega t+b^2\omega^2\sin^2\omega t+4c^2t^2\right)^{\frac{1}{2}}}$$

$$a_\tau=\frac{4c^2t}{\left(b^2\omega^2+4c^2t^2\right)^{\frac{1}{2}}}$$

$$a_n=\left(b^2\omega^2+4c^2-\frac{16c^4t^2}{b^2\omega^2+4c^2t^2}\right)^{\frac{1}{2}}$$

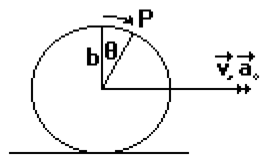
For 1.15, $$a_\tau=\frac{b^2k\left(k^2-c^2\right)e^{2kt}+2b^2c^2ke^{2kt}}{be^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}}=bke^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}$$

$$a_n=\left[b^2e^{2kt}\left(k^2+c^2\right)^2-b^2k^2e^{2kt}\left(k^2+c^2\right)\right]^{\frac{1}{2}}=bce^{kt}\left(k^2+c^2\right)^{\frac{1}{2}}$$

1.27 $\vec{v}=v\hat{\tau}$, $\vec{a}=\dot{v}\hat{\tau}+\dfrac{v^2}{\rho}\hat{n}$

$$\left|\vec{v}\times\vec{a}\right|=v\cdot a_n=v\frac{v^2}{\rho}=\frac{v^3}{\rho}$$

1.28



$$\vec{r}_{\circ P} = \hat{i} b \sin\theta + \hat{j} b \cos\theta$$
$$\vec{v}_{rel} = \hat{i} b\dot{\theta}\cos\theta - \hat{j} b\dot{\theta}\sin\theta$$
$$\vec{a}_{rel} = \hat{i} b\left(\ddot{\theta}\cos\theta - \dot{\theta}^2 \sin\theta\right) - \hat{j} b\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)$$

at the point $\theta = \dfrac{\pi}{2}$, $\vec{v}_{rel} = -\vec{v}$

So, $\left|\vec{v}_{rel}\right| = b\dot{\theta} = v$

$$\dot{\theta} = \frac{v}{b} \qquad \ddot{\theta} = \frac{\dot{v}}{b} = \frac{a_\circ}{b}$$

Now, $\vec{a}_{rel} = \dot{v}_{rel}\hat{\tau} + \dfrac{v_{rel}^2}{\rho}\hat{n} = a_\circ\hat{\tau} + \dfrac{v^2}{b}\hat{n}$

$$\left|\vec{a}_{rel}\right| = \left(a_\circ^2 + \frac{v^4}{b^2}\right)^{\frac{1}{2}}$$

$\vec{v}_P = \vec{v} + \vec{v}_{rel}$ and $\vec{a}_P = \vec{a}_\circ + \vec{a}_{rel}$

$$\vec{a}_P = \hat{i}\left[a_\circ + b\left(\frac{a_\circ}{b}\cos\theta - \frac{v^2}{b^2}\sin\theta\right)\right] - \hat{j} b\left(\frac{a_\circ}{b}\sin\theta + \frac{v^2}{b^2}\cos\theta\right)$$

$$\left|\vec{a}_P\right| = a_\circ\left(2 + 2\cos\theta + \frac{v^4}{a_\circ^2 b^2} - \frac{2v^2}{a_\circ b}\sin\theta\right)^{\frac{1}{2}}$$

$\vec{a}_P$ is a maximum at $\theta = 0$, i.e., at the top of the wheel.

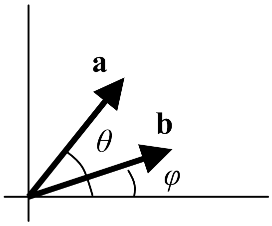
$$-2\sin\theta - \frac{2v^2}{a_\circ b}\cos\theta = 0$$

$$\theta = \tan^{-1}\left(-\frac{v^2}{a_\circ b}\right)$$

1.29 $\tilde{\boldsymbol{R}}\boldsymbol{R} = \begin{pmatrix} x & -x & 0 \\ x & x & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x & x & 0 \\ -x & x & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 2x^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Therefore, $x = \dfrac{1}{\sqrt{2}}$

The transformation represents a rotation of 45° about the z-axis (see Example 1.8.2)

1.30



(a)

$$\boldsymbol{a} = \hat{\boldsymbol{i}}\cos\theta + \hat{\boldsymbol{j}}\sin\theta$$

$$\boldsymbol{b} = \hat{\boldsymbol{i}}\cos\varphi + \hat{\boldsymbol{j}}\sin\varphi$$

$$\boldsymbol{a}\cdot\boldsymbol{b} = \cos(\theta-\varphi) = \left(\hat{\boldsymbol{i}}\cos\theta + \hat{\boldsymbol{j}}\sin\theta\right)\cdot\left(\hat{\boldsymbol{i}}\cos\varphi + \hat{\boldsymbol{j}}\sin\varphi\right)$$

$$\cos(\theta-\varphi) = \cos\theta\cos\varphi + \sin\theta\sin\varphi$$

(b) $$\boldsymbol{b}\times\boldsymbol{a} = \left|\hat{\boldsymbol{k}}\right|\sin(\theta-\varphi) = \left|\left(\hat{\boldsymbol{i}}\cos\theta + \hat{\boldsymbol{j}}\sin\theta\right)\times\left(\hat{\boldsymbol{i}}\cos\varphi + \hat{\boldsymbol{j}}\sin\varphi\right)\right|$$

$$\sin(\theta-\varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi$$

CHAPTER 2
NEWTONIAN MECHANICS:
RECTILINEAR MOTION OF A PARTICLE

2.1 (a) $\ddot{x} = \frac{1}{m}(F_\circ + ct)$

$$\dot{x} = \int_0^t \frac{1}{m}(F_\circ + ct)\,dt = \frac{F_\circ}{m}t + \frac{c}{2m}t^2$$

$$x = \int_0^t \left(\frac{F_\circ}{m}t + \frac{c}{2m}t^2\right)dt = \frac{F_\circ}{m}t^2 + \frac{c}{6m}t^3$$

(b) $\ddot{x} = \frac{F_\circ}{m}\sin ct$

$$\dot{x} = \int_0^t \frac{F_\circ}{m}\sin ct\,dt = -\frac{F_\circ}{cm}\cos ct\Big|_0^t = \frac{F_\circ}{cm}(1-\cos ct)$$

$$x = \int_0^t \frac{F_\circ}{cm}(1-\cos ct)\,dt = \frac{F_\circ}{cm}\left(1 - \frac{1}{c}\sin ct\right)$$

(c) $\ddot{x} = \frac{F_\circ}{m}e^{ct}$

$$\dot{x} = \frac{F_\circ}{cm}e^{ct}\Big|_0^t = \frac{F_\circ}{cm}\left(e^{ct}-1\right)$$

$$x = \frac{F_\circ}{cm}\left(\frac{1}{c}e^{ct} - \frac{1}{c} - t\right) = \frac{F_\circ}{c^2m}\left(e^{ct}-1-ct\right)$$

2.2 (a) $\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx}\cdot\frac{dx}{dt} = \dot{x}\frac{d\dot{x}}{dx}$

$$\dot{x}\frac{d\dot{x}}{dx} = \frac{1}{m}(F_\circ + cx)$$

$$\dot{x}d\dot{x} = \frac{1}{m}(F_\circ + cx)\,dx$$

$$\frac{1}{2}\dot{x}^2 = \frac{1}{m}\left(F_\circ x + \frac{cx^2}{2}\right)$$

$$\dot{x} = \left[\frac{x}{m}(2F_\circ + cx)\right]^{\frac{1}{2}}$$

(b) $\ddot{x} = \dot{x}\frac{d\dot{x}}{dx} = \frac{1}{m}F_\circ e^{-cx}$

$$\dot{x}d\dot{x} = \frac{1}{m}F_{\circ}e^{-cx}dx$$

$$\frac{1}{2}\dot{x}^2 = -\frac{F_{\circ}}{cm}\left(e^{-cx}-1\right) = \frac{F_{\circ}}{cm}\left(1-e^{-cx}\right)$$

$$\dot{x} = \left[\frac{2F_{\circ}}{cm}\left(1-e^{-cx}\right)\right]^{\frac{1}{2}}$$

(c) $\ddot{x} = \dot{x}\frac{d\dot{x}}{dx} = \frac{1}{m}\left(F_{\circ}\cos cx\right)$

$$\dot{x}d\dot{x} = \frac{F_{\circ}}{m}\cos cx\, dx$$

$$\frac{1}{2}\dot{x}^2 = \frac{F_{\circ}}{cm}\sin cx$$

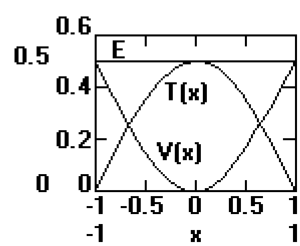
$$\dot{x} = \left(\frac{2F_{\circ}}{cm}\sin cx\right)^{\frac{1}{2}}$$

2.3 (a) $V\left(x\right) = -\int_{x_{\circ}}^{x}\left(F_{\circ}+cx\right)dx = -F_{\circ}x - \frac{cx^2}{2} + C$

(b) $V\left(x\right) = -\int_{x_{\circ}}^{x}F_{\circ}e^{-cx}dx = \frac{F_{\circ}}{c}e^{-cx} + C$

(c) $V\left(x\right) = -\int_{x_{\circ}}^{x}F_{\circ}\cos cx\, dx = -\frac{F_{\circ}}{c}\sin cx + C$

2.4 (a) $F\left(x\right) = -\frac{dV\left(x\right)}{dx} = -kx$



$$V\left(x\right) = \int_0^x kx\, dx = \frac{1}{2}kx^2$$

(b) $T_{\circ} = T\left(x\right) + V\left(x\right)$

$$T\left(x\right) = T_{\circ} - V\left(x\right) = \frac{1}{2}k\left(A - x^2\right)$$

(c) $E = T_{\circ} = \frac{1}{2}kA^2$

(d) *turning points* @ $T\left(x_1\right) \to 0 \quad \therefore x_1 = \pm A$

2.5 (a) $F\left(x\right) - kx + \frac{kx^3}{A^2}$ so $V\left(x\right) = \int_0^x\left(kx - \frac{kx^3}{A^2}\right)dx = \frac{1}{2}kx^2 - \frac{1}{4}\frac{kx^4}{A^2}$

(b) $T\left(x\right) = T_{\circ} - V\left(x\right) = T_{\circ} - \frac{1}{2}kx^2 + \frac{1}{4}\frac{kx^4}{A^2}$

(c) $E = T_{\circ}$

(d) $V(x)$ has maximum at $|F(x_m)| \to 0$

$$kx_m - \frac{kx_m^{\ 3}}{A^2} = 0 \qquad x_m = \pm A$$

$$V(x_m) = \frac{1}{2}kA^2 - \frac{1}{4}\frac{kA^4}{A^2} = \frac{1}{4}kA^2$$

If $E < V(x_m)$ turning points exist.

Turning points @ $T(x_1) \to 0$ let $u = x_1^{\ 2}$

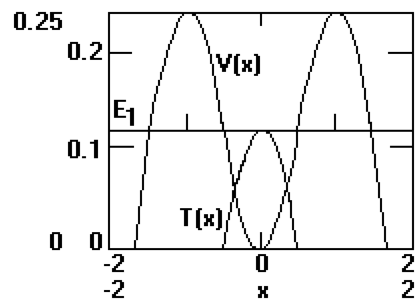
$$E - \frac{1}{2}ku + \frac{1}{4}\frac{ku^2}{A^2} = 0$$

solving for u, we obtain

$$u = A^2\left[1 \pm \left(1 - \frac{4E}{kA^2}\right)^{\frac{1}{2}}\right]$$

or

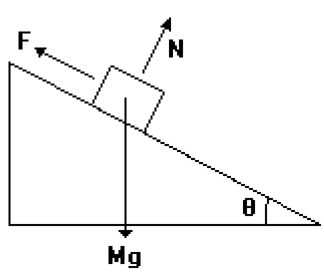
$$x_1 = \pm A\left[1 - \sqrt{\left(1 - \frac{4E}{kA^2}\right)}\right]^{\frac{1}{2}}$$



2.6 $\dot{x} = v(x) = \dfrac{\alpha}{x} \qquad \ddot{x} = -\dfrac{\alpha}{x^2}\dot{x} = -\dfrac{\alpha^2}{x^3}$

$$F(x) = m\ddot{x} = -\frac{m\alpha^2}{x^3}$$

2.7



$F \geq Mg\sin\theta$

2.8 $F = m\ddot{x} = m\dot{x}\dfrac{d\dot{x}}{dx}$

$$\dot{x} = bx^{-3}$$

$$\frac{d\dot{x}}{dx} = -3bx^{-4}$$

$$F = m\left(bx^{-3}\right)\left(-3bx^{-4}\right)$$

$$F = -3mb^2x^{-7}$$

2.9 (a) $V = mgx = (.145kg)\left(9.8\dfrac{m}{s^2}\right)(1250\,ft)\left(.3048\dfrac{m}{ft}\right) = 541J$

(b) $T = \frac{1}{2}mv^2 = \frac{1}{2}mv_t^{\ 2} = \frac{1}{2}m\left(\frac{mg}{c_2}\right) = \frac{1}{2}\frac{m^2 g}{.22D^2}$

$$T = \frac{(.145kg)^2\left(9.8\frac{m}{s^2}\right)}{(2)(.22)\left[(2)(.0366)\right]^2\frac{kg}{m}} = 87J$$

$$\int F dx = \int -cv^2 dx = -c\int v^3 dt = -c\int\left(-v_t \tanh\left(\frac{t}{\tau}\right)\right)^3 dt$$

$$= cv_t^{\ 3}\tau\left[-\frac{1}{2}\tanh^2\left(\frac{t}{\tau}\right) + \int \tanh\left(\frac{t}{\tau}\right) d\left(\frac{t}{\tau}\right)\right]$$

$$= cv_t^{\ 3}\tau\left[-\frac{1}{2}\tanh^2\left(\frac{t}{\tau}\right) + \ln\cosh\left(\frac{t}{\tau}\right)\right]$$

Now $\tanh^2\left(\frac{t}{\tau}\right) \cong 1$ for $t \ll \tau$

Meanwhile $x = \int v dt = \int\left(-v_t \tanh\left(\frac{t}{\tau}\right)\right) dt = v_t \tau \ln\cosh\left(\frac{t}{\tau}\right)$

$$\ln\cosh\left(\frac{t}{\tau}\right) = \frac{x}{v_t \tau}$$

$$x = (1250 ft)\left(.3048\frac{m}{ft}\right) = 381m$$

$$v_t = \left(\frac{mg}{c_2}\right)^{\frac{1}{2}} = \left[\frac{(.145kg)\left(9.8\frac{m}{s^2}\right)}{(.22)(.0732)^2\frac{kg}{m}}\right]^{\frac{1}{2}} = 34.72\frac{m}{s}$$

$$\tau = \left(\frac{m}{c_2 g}\right)^{\frac{1}{2}} = \left[\frac{(.145kg)}{(.22)(.0732)^2\frac{kg}{m}\left(9.8\frac{m}{s^2}\right)}\right]^{\frac{1}{2}} = 3.543s$$

$$\int F dx = (.22)(.0732)^2(34.72)^3(3.543)\left[-.5 + \frac{3.81}{(34.72)(3.54)}\right] = 454J$$

$$V - T = 541J - 87J = 454J$$

2.10 For $0 \le t \le t_1$: $v = \frac{F_\circ}{m}t$, $x = \frac{1}{2}\frac{F_\circ}{m}t^2$

For $t_1 \le t \le 2t_1$: $v_\circ = \frac{F_\circ}{m}t_1$, $x_\circ = \frac{F_\circ}{2m}t_1^{\ 2}$, $t_\circ = t_1$

$$x = \frac{F_\circ}{2m} t_1^{\,2} + \frac{F_\circ}{m} t_1 \left(t - t_1 \right) + \frac{1}{2} \frac{2F_\circ}{m} \left(t - t_1 \right)^2$$

At $\quad t = 2t_1: \quad x = \frac{F_\circ}{2m} t_1^{\,2} + \frac{F_\circ}{m} t_1^{\,2} + \frac{F_\circ}{m} t_1^{\,2} = \frac{5F_\circ}{2m} t_1^{\,2}$

2.11 $\quad a = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v \cdot \frac{dv}{dx} = -\frac{c}{m} v^{\frac{3}{2}}$

$$v^{-\frac{1}{2}} dv = -\frac{c}{m} dx$$

$$\int_{v_\circ}^{v} v^{-\frac{1}{2}} dv = \int_0^{x_{\max}} -\frac{c}{m} dx$$

$$-2v_\circ^{\frac{1}{2}} = -\frac{c}{m} x_{\max}$$

$$x_{\max} = \frac{2mv_\circ^{\frac{1}{2}}}{c}$$

2.12 $\quad$ Going up: $\ F_x = -mg \sin 30^\circ - \mu mg \cos 30^\circ$

$$\ddot{x} = -g \left(\sin 30^\circ + 0.1 \cos 30^\circ \right) = -5.749 \frac{m}{s^2}$$

$$v = v_\circ + at$$

at the highest point $v = 0$ so $t_{up} = -\frac{v_\circ}{a} = 0.174 v_\circ s$

$$x_{up} = v_\circ t_{up} + \frac{1}{2} a t_{up}^2 = 0.174 v_\circ^{\,2} - .087 v_\circ^{\,2} = 0.087 v_\circ^{\,2} m$$

Going down: $\ x_\circ' = 0.087 v_\circ^{\,2}$, $\ v_\circ' = 0$, $\ a' = -9.8 \left(0.5 - 0.0866 \right)$

$$x_{down} = 0 = 0.087 v_\circ^{\,2} - \frac{1}{2} 4.0513 t_{down}^2$$

$$t_{down} = 0.207 v_\circ s$$

$$t_{total} = t_{up} + t_{down} = 0.381 v_\circ s$$

2.13 $\quad$ At the top $v = 0$ so $\ e^{-2kx_{\max}} = \dfrac{\frac{g}{k}}{\frac{g}{k} + v_\circ^{\,2}}$

Coming down $x_\circ = x_{\max}$ and at the bottom $x = 0$

$$v^2 = \frac{g}{k} - \left(\frac{g}{k} \right)^2 \frac{1}{\left(\frac{g}{k} + v_\circ^{\,2} \right)} (1) = \frac{\left(\frac{g}{k} \right) v_\circ^{\,2}}{\frac{g}{k} + v_\circ^{\,2}}$$

$$v = \frac{v_t v_\circ}{\left(v_t^2 + v_\circ^2\right)^{\frac{1}{2}}}, \quad v_t = \sqrt{\frac{g}{k}} = \sqrt{\frac{mg}{c_2}}$$

2.14 Going up: $F_x = -mg - c_2 v^2$

$$a = v\frac{dv}{dx} = -g - kv^2, \quad k = \frac{c^2}{m}$$

$$\int_{v_\circ}^{v} \frac{vdv}{-g - kv^2} = \int_0^x dx$$

$$-\frac{1}{2k}\ln\left(-g - kv^2\right)\Big|_{v_\circ}^{v} = x$$

$$\frac{g + kv^2}{g + kv_\circ^2} = e^{-2kx}$$

$$v^2 = \left(\frac{g}{k} + v_\circ^2\right)e^{-2kx} - \frac{g}{k}$$

Going down: $F_x = -mg + c_2 v^2$

$$v\frac{dv}{dx} = -g + kv^2$$

$$\int_0^{v} \frac{vdv}{-g + kv^2} = \int_0^x dx$$

$$\frac{1}{2k}\ln\left(-g + kv^2\right)\Big|_0^{v} = x - x_\circ$$

$$1 - \frac{k}{g}v^2 = e^{2kx}e^{-2kx_\circ}$$

$$v^2 = \frac{g}{k} - \left(\frac{g}{k}e^{-2kx_\circ}\right)e^{2kx}$$

2.15 $m\dfrac{dv}{dt} = mg - c_1 v - c_2 v^2$

$$\int_0^t \frac{dt}{m} = \int_0^v \frac{dv}{mg - c_1 v - c_2 v^2}$$

Using $\displaystyle\int \frac{dx}{a + bx + cx^2} = \frac{1}{\sqrt{b^2 - 4ac}}\ln\frac{2cx + b - \sqrt{b^2 - 4ac}}{2cx + b + \sqrt{b^2 - 4ac}}$,

$$\frac{t}{m} = \frac{1}{\sqrt{c_1^{\,2} + 4mgc_2}} \ln \frac{-2c_2 v - c_1 - \sqrt{c_1^{\,2} + 4mgc_2}}{-2c_2 v - c_1 + \sqrt{c_1^{\,2} + 4mgc_2}} \Bigg|_0^v$$

$$\frac{t}{m}\left(c_1^{\,2} + 4mgc_2\right)^{\frac{1}{2}} = \ln \frac{\left(2c_2 v + c_1 + \sqrt{c_1^{\,2} + 4mgc_2}\right)\left(c_1 - \sqrt{c_1^{\,2} + 4mgc_2}\right)}{\left(2c_2 v + c_1 - \sqrt{c_1^{\,2} + 4mgc_2}\right)\left(c_1 + \sqrt{c_1^{\,2} + 4mgc_2}\right)}$$

as $t \to \infty$, $2c_2 v_t + c_1 - \sqrt{c_1^{\,2} + 4mgc_2} = 0$

$$v_t = -\frac{c_1}{2c_2} + \left[\left(\frac{c_1}{2c_2}\right)^2 + \frac{mg}{c_2}\right]^{\frac{1}{2}}$$

Alternatively, when $v = v_t$,

$$m\frac{dv}{dt} = 0 = mg - c_1 v_t - c_2 v_t^{\,2}$$

$$v_t = -\frac{c_1}{2c_2} + \left[\left(\frac{c_1}{2c_2}\right)^2 + \frac{mg}{c_2}\right]^{\frac{1}{2}}$$

2.16 $\quad a = v\frac{dv}{dx} = -\frac{k}{m}x^{-2}$

$$\int_0^v v\,dv = \int_b^x -\frac{k\,dx}{mx^2}$$

$$\frac{1}{2}v^2 = \frac{k}{m}\left(\frac{1}{x} - \frac{1}{b}\right)$$

$$v = \frac{dx}{dt} = \left[\frac{2k}{m}\left(\frac{1}{x} - \frac{1}{b}\right)\right]^{\frac{1}{2}} = \left[\frac{2k}{mb}\left(\frac{b-x}{x}\right)\right]^{\frac{1}{2}}$$

$$\int_0^t dt = \int_b^0 \left[\frac{mb}{2k}\left(\frac{x}{b-x}\right)\right]^{\frac{1}{2}} dx = \left(\frac{mb^3}{2k}\right)^{\frac{1}{2}} \int_1^0 \left(\frac{\frac{x}{b}}{1 - \frac{x}{b}}\right)^{\frac{1}{2}} d\left(\frac{x}{b}\right)$$

Since $x \le b$, say $\frac{x}{b} = \sin^2\theta$

$$t = \left(\frac{mb^3}{2k}\right)^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^0 \frac{\sin\theta\left(2\sin\theta\cos\theta\,d\theta\right)}{\cos\theta} = \left(\frac{2mb^3}{k}\right)^{\frac{1}{2}} \int_{-\frac{\pi}{2}}^0 \sin^2\theta\,d\theta$$

$$t = \left(\frac{mb^3}{8k}\right)^{\frac{1}{2}} \pi$$

2.17 $m\frac{dv}{dt} = mv\frac{dv}{dx} = f(x)\bullet g(v)$

$$\frac{mvdv}{g(v)} = f(x)dx$$

By integration, get $v = v(x) = \frac{dx}{dt}$

If $F(x,t) = f(x)\bullet g(t)$:

$$m\frac{d^2x}{dt^2} = m\frac{d}{dt}\left(\frac{dx}{dt}\right) = f(x)\bullet g(t)$$

This cannot, in general, be solved by integration.

If $F(v,t) = f(v)\bullet g(t)$:

$$m\frac{dv}{dt} = f(v)\bullet g(t)$$

$$\frac{mdv}{f(v)} = g(t)dt$$

Integration gives $v = v(t)$

$$\frac{dx}{dt} = v(t)$$

$$dx = v(t)dt$$

A second integration gives $x = x(t)$

2.18

$$c_1 = \left(1.55\times10^{-4}\right)\left(10^{-2}\right) = 1.55\times10^{-6}\frac{kg}{s}$$

$$c_2 = (0.22)\left(10^{-2}\right)^2 = 2.2\times10^{-5}\frac{kg}{s}$$

$$v_t = -\frac{1.55\times10^{-6}}{2\times2.2\times10^{-5}} + \left[\left(\frac{1.55\times10^{-6}}{2\times2.2\times10^{-5}}\right)^2 + \frac{\left(10^{-7}\right)(9.8)}{2.2\times10^{-5}}\right]^{\frac{1}{2}}$$

$$v_t = 0.179\frac{m}{s}$$

Using equation 2.29, $v_t = \sqrt{\frac{\left(10^{-7}\right)(9.8)}{2.2\times10^{-5}}} = 0.211\frac{m}{s}$

2.19

$F(x) = -Ae^{\alpha\dot{x}} = m\ddot{x}$ or $F(v) = -Ae^{\alpha v} = m\dot{v}$ $\quad \frac{dv}{e^{\alpha v}} = -\frac{A}{m}dt$

Let $u = e^{\alpha v}$ $\quad du = \alpha e^{\alpha v}dv$ $\quad dv = \frac{du}{\alpha e^{\alpha v}} = \frac{du}{\alpha u}$ $\quad \therefore \frac{du}{u^2} = -\frac{\alpha A}{m}dt$

Integrating

$$\frac{1}{u}-\frac{1}{u_\circ}=\frac{A}{m}\alpha t \qquad \text{and substituting } e^{\alpha v}=u$$

(a) $v=v_\circ-\frac{1}{\alpha}\ln\left[1+\frac{A}{m}e^{\alpha v_\circ}\alpha t\right]$

(b) $t=T\,@\,v=0$

$$\alpha v_\circ=\ln\left[1+\frac{A}{m}e^{\alpha v_\circ}\alpha T\right]$$

$$e^{\alpha v_\circ}=1+\frac{A}{m}e^{\alpha v_\circ}\alpha T \qquad T=\frac{m}{\alpha A}\left[1-e^{-\alpha v_\circ}\right]$$

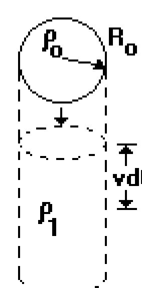
(c) $v\frac{dv}{dx}=v=-\frac{A}{m}e^{\alpha v} \qquad \frac{vdv}{e^{\alpha v}}=-\frac{A}{m}dx$

again, let $u=e^{\alpha v} \quad du=\alpha u dv$ or $dv=\frac{du}{\alpha u} \qquad v=\frac{1}{\alpha}\ln u$

$$\frac{\left[\frac{1}{\alpha}\ln u\right]\frac{du}{\alpha u}}{u}=-\frac{A}{m}dx \qquad \text{Integrating and solving}$$

$$x=\frac{m}{\alpha^2 A}\left[1-\left(1+\alpha v_\circ\right)e^{-\alpha v_\circ}\right]$$

2.20



$$F=\frac{d(mv)}{dt}=mv+vm=mg$$

but $m=\rho_\circ\frac{4}{3}\pi r^3 \qquad m=\rho_1\pi r^2 v$

so (1) $\frac{4}{3}\pi\rho_\circ r^3 v+\pi\rho_1 r^2 v^2=\frac{4}{3}\pi^2=\frac{4}{3}\pi\rho_\circ r^3 g$

Now $\frac{\rho_1}{\rho_\circ}\approx 10^{-3}$ so, second term is negligible-small

hence $v\approx g$ and $\boxed{v\approx gt}$ speed $\propto t$ but

$\dot{m}=\rho_\circ 4\pi r^2\dot{r}=\rho_1\pi r^2 v$ or $\dot{r}\cong\frac{1}{4}\frac{\rho_1}{\rho_\circ}v$ Hence $\boxed{r\approx\frac{1}{4}\frac{\rho_1}{\rho_\circ}gt}$ and rate of growth $\propto t$

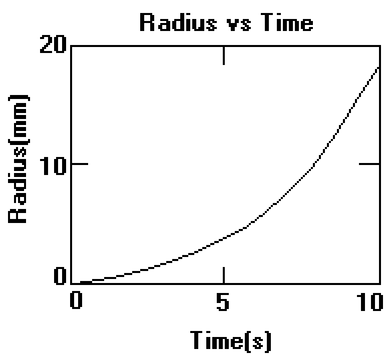
The exact differential equation from (1) above is:

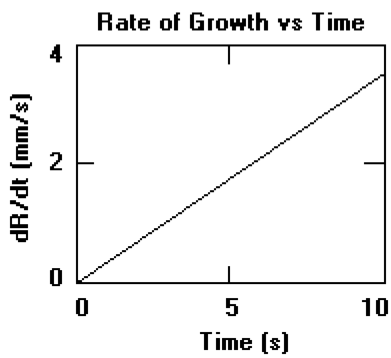
$$\frac{4}{3}\pi\rho_\circ r\left|\frac{4\rho_\circ}{\rho_1}\ddot{r}\right|+\pi\rho_1\left|\frac{4\rho_\circ\dot{r}}{\rho_1}\right|^2=\frac{4}{3}\pi\rho_\circ rg$$

which reduces to: $\ddot{r} + \frac{3\dot{r}^2}{r} = \frac{\rho_1}{4\rho_\circ} g$

Using Mathcad, solve the above non-linear d.e. letting $\frac{\rho_1}{\rho_\circ} \approx 10^{-3}$ and $R_\circ \approx 0.01mm$ (small raindrop). Graphs show that

$$v \propto \dot{r} \propto t \quad \text{and} \quad r \propto t^2$$





CHAPTER 3
OSCILLATIONS

3.1 $x = 0.002\sin\left[2\pi\left(512\,s^{-1}\right)t\right]\,[m]$

$$\dot{x}_{\max} = (0.002)(2\pi)(512)\left[\frac{m}{s}\right] = 6.43\left[\frac{m}{s}\right]$$

$$\ddot{x}_{\max} = (0.002)(2\pi)^2(512)^2\left[\frac{m}{s^2}\right] = 2.07\times10^4\left[\frac{m}{s^2}\right]$$

3.2 $x = 0.1\sin\omega_\circ t\ \ [m]$ $\qquad$ $\dot{x} = 0.1\,\omega_\circ\cos\omega_\circ t\left[\frac{m}{s}\right]$

When $t = 0$, $x = 0$ and $\qquad$ $\dot{x} = 0.5\left[\frac{m}{s}\right] = 0.1\,\omega_\circ$

$\omega_\circ = 5\,s^{-1}$ $\qquad$ $T = \dfrac{2\pi}{\omega_\circ} = 1.26\,s$

3.3 $x(t) = x_\circ\cos\omega_\circ t + \dfrac{\dot{x}_\circ}{\omega_\circ}\sin\omega_\circ t$ and $\omega_\circ = 2\pi f$

$$x = 0.25\cos(20\pi t) + 0.00159\sin(20\pi t)\,[m]$$

3.4 $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

$$x = A\cos(\omega_\circ t - \phi) = A\cos\phi\cos\omega_\circ t + A\sin\phi\sin\omega_\circ t$$

$$x = \mathrm{A}\cos\omega_\circ t + \mathrm{B}\sin\omega_\circ t\,,\ \mathrm{A} = A\cos\phi\,,\ \mathrm{B} = A\sin\phi$$

3.5 $\frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}kx_1^2 = \frac{1}{2}m\dot{x}_2^2 + \frac{1}{2}kx_2^2$

$$k\left(x_1^2 - x_2^2\right) = m\left(\dot{x}_2^2 - \dot{x}_1^2\right)$$

$$\omega_\circ = \sqrt{\frac{k}{m}} = \left(\frac{\dot{x}_2^2 - \dot{x}_1^2}{x_1^2 - x_2^2}\right)^{\frac{1}{2}}$$

$$\frac{1}{2}kA^2 = \frac{1}{2}m\dot{x}_1^2 + \frac{1}{2}kx_1^2$$

$$A^2 = \frac{m}{k}\dot{x}_1^2 + x_1^2 = \frac{x_1^2\dot{x}_1^2 - x_2^2\dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2} + x_1^2$$

$$A = \left(\frac{x_1^2\dot{x}_2^2 - x_2^2\dot{x}_1^2}{\dot{x}_2^2 - \dot{x}_1^2}\right)^{\frac{1}{2}}$$

3.6 $\frac{1}{2}T_{\circ} = \pi\sqrt{\frac{l}{g}} = \pi\sqrt{\frac{1}{\frac{9.8}{6}}}\ s \approx 2.5\,s$

3.7 For springs tied in parallel:

$F_s(x) = -k_1 x - k_2 x = -(k_1 + k_2)x$

$$\omega = \left[\frac{(k_1 + k_2)}{m}\right]^{\frac{1}{2}}$$

For springs tied in series:

The upward force m is $k_{eq}\ x$.

Therefore, the downward force on spring k_2 is $k_{eq}\ x$.

The upward force on the spring k_2 is $k_1 x'$ where x' is the displacement of P, the point at which the springs are tied.

Since the spring k_2 is in equilibrium, $k_1 x' = k_{eq}\ x$.

Meanwhile,
The upward force at P is $k_1 x'$.
The downward force at P is $k_2(x - x')$.

Therefore, $k_1 x' = k_2(x - x')$

$$x' = \frac{k_2 x}{k_1 + k_2}$$

And $k_{eq}\ x = k_1\left(\frac{k_2 x}{k_1 + k_2}\right)$

$$\omega = \sqrt{\frac{k_{eq}}{m}} = \left[\frac{k_1 k_2}{(k_1 + k_2)m}\right]^{\frac{1}{2}}$$

3.8 For the system $(M + m)$, $-kX = (M + m)\ddot{X}$

The position and acceleration of *m* are the same as for $(M + m)$:

$$\ddot{x}_m = -\frac{k}{M + m}x_m$$

$$x_m = A\cos\left(\sqrt{\frac{k}{M + m}}t + \delta\right) = d\cos\sqrt{\frac{k}{M + m}}t$$

The total force on *m*, $F_m = m\ddot{x}_m = mg - F_r$

$$F_r = mg + \frac{mk}{M+m} x_m = mg + \frac{mkd}{M+m} \cos \sqrt{\frac{k}{M+m}} t$$

For the block to just begin to leave the bottom of the box at the top of the vertical oscillations, $F_r = 0$ at $x_m = -d$:

$$0 = mg - \frac{mkd}{M+m}$$

$$d = \frac{g(M+m)}{k}$$

3.9 $x = e^{-\gamma t} A \cos(\omega_d t - \phi)$

$$\frac{dx}{dt} = -e^{-\gamma t} A\omega_d \sin(\omega_d t - \phi) - \gamma e^{-\gamma t} A \cos(\omega_d t - \phi)$$

maxima at $\frac{dx}{dt} = 0 = \omega_d \sin(\omega_d t - \phi) + \gamma \cos(\omega_d t - \phi)$

$$\tan(\omega_d t - \phi) = -\frac{\gamma}{\omega_d}$$

thus the condition of relative maximum occurs every time that *t* increases by $\frac{2\pi}{\omega_d}$:

$$t_{i+1} = t_i + \frac{2\pi}{\omega_d}$$

For the *i* th maximum: $x_i = e^{-\gamma t_i} A \cos(\omega_d t_i - \phi)$

$$x_{i+1} = e^{-\gamma t_{i+1}} A \cos(\omega_d t_{i+1} - \phi) = e^{-\gamma \frac{2\pi}{\omega_d}} x_i$$

$$\frac{x_i}{x_{i+1}} = e^{-\gamma \frac{2\pi}{\omega_d}} = e^{\gamma T_d}$$

3.10 (a) $\gamma = \frac{c}{2m} = 3\,s^{-1}$ $\qquad \omega_\circ^2 = \frac{k}{m} = 25\,s^{-2}$

$\omega_d{}^2 = \omega_\circ^2 - \gamma^2 = 16\,s^{-2}$ $\qquad \omega_r{}^2 = \omega_d{}^2 - \gamma^2 = 7\,s^{-2}$

$\therefore \omega_r = \sqrt{7}\,s^{-1}$

(b) $A_{\max} = \frac{F_\circ}{C\omega_d} = \frac{48}{60.4} m = 0.2\ m$

(c) $\tan\phi = \frac{2\gamma\omega_r}{(\omega_\circ^2 - \omega_r^2)} = \frac{2\gamma\omega_r}{2\gamma^2} = \frac{\omega_r}{\gamma} = \frac{\sqrt{7}}{3}$ $\qquad \therefore \phi \approx 41.4^\circ$

3.11 (a) $$m\ddot{x}+3\beta m\dot{x}+\frac{17}{2}\beta^2 mx=0$$

$$\gamma=\frac{3}{2}\beta \quad \text{and} \quad \omega_\circ^2=\frac{17}{2}\beta^2$$

$$\omega_r{}^2=\omega_\circ^2-2\gamma^2=4\beta^2 \qquad \therefore \omega_r=2\beta$$

(b) $$A_{\max}=\frac{F_\circ}{2m\gamma\omega_d} \qquad \omega_d^2=\omega_\circ^2-\gamma^2=\frac{25}{4}\beta^2 \quad \therefore \omega_d=\frac{5}{2}\beta$$

$$=\frac{2A}{15\beta^2}$$

3.12 $$e^{-\gamma T_d}=\frac{1}{2}$$

$$\gamma=\frac{1}{T_d}\ln 2=f_d\ln 2$$

(a) $$\omega_d=(\omega_\circ^2-\gamma^2)^{\frac{1}{2}}$$

So, $$\omega_\circ=(\omega_d^2+\gamma^2)^{\frac{1}{2}}$$

$$f_\circ=\left[f_d^2+\left(\frac{\gamma}{2\pi}\right)^2\right]^{\frac{1}{2}}=f_d\left[1+\left(\frac{\ln 2}{2\pi}\right)^2\right]^{\frac{1}{2}}$$

$$f_\circ=100.6Hz$$

(b) $$\omega_r=\left(\omega_d^2-\gamma^2\right)^{\frac{1}{2}}$$

$$f_r=\left[f_d^2-\left(\frac{\gamma}{2\pi}\right)^2\right]^{\frac{1}{2}}=f_d\left[1-\left(\frac{\ln 2}{2\pi}\right)^2\right]^{\frac{1}{2}}$$

$$f_r=99.4Hz$$

3.13 Since the amplitude diminishes by $e^{-\gamma T_d}$ in each complete period,

$$\left(e^{-\gamma T_d}\right)^n=\frac{1}{e}=e^{-1}$$

$$\gamma T_d n=1$$

$$\gamma=\frac{1}{T_d n}=\frac{\omega_d}{2\pi n}$$

Now $$\omega_d=\left(\omega_\circ{}^2-\gamma^2\right)^{\frac{1}{2}}$$

So $$\omega_\circ=\left(\omega_d^2+\gamma^2\right)^{\frac{1}{2}}=\omega_d\left(1+\frac{1}{4\pi^2n^2}\right)^{\frac{1}{2}}$$

$$\frac{T_d}{T_\circ} = \frac{\dfrac{2\pi}{\omega_d}}{\dfrac{2\pi}{\omega_\circ}} = \frac{\omega_\circ}{\omega_d} = \left(1 + \frac{1}{4\pi^2 n^2}\right)^{\frac{1}{2}}$$

For large n, $\dfrac{T_d}{T_\circ} \approx 1 + \dfrac{1}{8\pi^2 n^2}$

3.14 (a) $\omega_r = \left(\omega_\circ^{\,2} - 2\gamma^2\right)^{\frac{1}{2}} = \left[\omega_\circ^2 - 2\left(\dfrac{\omega_\circ}{2}\right)^2\right]^{\frac{1}{2}} = 0.707\omega_\circ$

(b) $Q = \dfrac{\omega_d}{2\gamma} = \dfrac{\left(\omega_\circ^{\,2} - \gamma^2\right)^{\frac{1}{2}}}{2\gamma} = \dfrac{\omega_\circ\left(1 - \dfrac{1}{4}\right)^{\frac{1}{2}}}{2\left(\dfrac{\omega_\circ}{2}\right)} = 0.866$

(c) $\tan\phi = \dfrac{2\gamma\omega}{\omega_\circ^2 - \omega^2} = \dfrac{2\left(\dfrac{\omega_\circ}{2}\right)\left(2\omega_\circ\right)}{\omega_\circ^2 - 4\omega_\circ^2} = -\dfrac{2}{3}$

$\phi = \tan^{-1}\left(-\dfrac{2}{3}\right) = 146.3^\circ$

(d) $D(\omega) = \left[\left(\omega_\circ^2 - 4\omega_\circ^2\right)^2 + 4\left(\dfrac{\omega_\circ}{2}\right)^2\left(4\omega_\circ^{\,2}\right)\right]^{\frac{1}{2}} = 3.606\,\omega_\circ^2$

$A(\omega) = \dfrac{\dfrac{F_\circ}{m}}{D(\omega)} = 0.277\dfrac{F_\circ}{m\omega_\circ^2}$

3.15 $A(\omega) \approx \dfrac{A_{\max}\gamma}{\left[\left(\omega_\circ - \omega\right)^2 + \gamma^2\right]^{\frac{1}{2}}}$

for $A(\omega) = \dfrac{1}{2}A_{\max}$, $\quad \dfrac{1}{2} = \dfrac{\gamma}{\left[\left(\omega_\circ - \omega\right)^2 + \gamma^2\right]^{\frac{1}{2}}}$

$$\left(\omega_\circ - \omega\right)^2 + \gamma^2 = 4\gamma^2$$

$$\omega_\circ - \omega = \pm\gamma\sqrt{3}$$

$$\omega = \omega_\circ \pm \gamma\sqrt{3}$$

3.16 (b) $Q = \dfrac{\omega_d}{2\gamma} = \dfrac{\sqrt{\omega_\circ^2 - \gamma^2}}{2\gamma}$

$$\omega_\circ^2 = \frac{1}{LC}, \qquad \gamma = \frac{R}{2L}$$

$$Q = \frac{\sqrt{\left(\frac{1}{LC}\right) - \left(\frac{R^2}{4L^2}\right)}}{2\left(\frac{R}{2L}\right)} = \sqrt{\left(\frac{L}{R^2C}\right) - \left(\frac{1}{4}\right)}$$

(c) $Q = \left(\dfrac{\omega_\circ}{2\gamma}\right) = \dfrac{\sqrt{\frac{L}{C}}}{R} = \dfrac{R_\circ}{R}$

3.17 $F_{ext} = F_\circ \sin\omega t = \mathrm{Im}\left[F_\circ e^{i\omega t}\right]$

and $x(t)$ is the imaginary part of the solution to:

$$m\ddot{x} + c\dot{x} + kx = F_\circ e^{i\omega t}$$

i.e. $\quad x(t) = \mathrm{Im}\left[Ae^{i(\omega t - \phi)}\right] = A\sin(\omega t - \phi)$

where, as derived in the text,

$$A = \frac{F_\circ}{\left[\left(k - m\omega^2\right)^2 + c^2\omega^2\right]^{\frac{1}{2}}}$$

and

$$\tan\phi = \frac{2\gamma\omega}{\omega_\circ^2 - \omega^2}$$

3.18 Using the hint, $F_{ext} = \mathrm{Re}\left(F_\circ e^{\beta t}\right)$, where $\beta = -\alpha + i\omega$,

and $x(t)$ is the real part of the solution to:

$$m\ddot{x} + c\dot{x} + kx = F_\circ e^{\beta t}.$$

Assuming a solution of the form: $\quad x = Ae^{\beta t - i\phi}$

$$\left(m\beta^2 + c\beta + k\right)x = \left(\frac{F_\circ}{A}\right)xe^{i\phi}$$

$$m\alpha^2 - 2im\alpha\omega - m\omega^2 - c\alpha + ic\omega + k = \frac{F_\circ}{A}(\cos\phi + i\sin\phi)$$

$$m\left(\alpha^2 - \omega^2\right) - c\alpha + k = \frac{F_\circ}{A}\cos\phi$$

$$\omega(-2m\alpha + c) = \frac{F_\circ}{A}\sin\phi$$

$$\phi = \tan^{-1}\frac{\omega(c-2m\alpha)}{m(\alpha^2-\omega^2)-c\alpha+k}$$

Using $\sin^2\phi+\cos^2\phi=1$,

$$\frac{F_\circ^2}{A^2}=\left[m(\alpha^2-\omega^2)-c\alpha+k\right]^2+\omega^2(c-2m\alpha)^2$$

$$A=\frac{F_\circ}{\left\{\left[m(\alpha^2-\omega^2)-c\alpha+k\right]^2+\omega^2(c-2m\alpha)^2\right\}^{\frac{1}{2}}}$$

and $x(t)=Ae^{-\alpha t}\cos(\omega t-\phi)+$ the transient term.

3.19 (a) $$T\approx 2\pi\sqrt{\frac{l}{g}}\left(1-\frac{A^2}{8}\right)^{-\frac{1}{2}}$$

for $A=\frac{\pi}{4}$, $T\approx 2\pi\sqrt{\frac{l}{g}}\times 1.041$

(b) $$g=\frac{4\pi^2 l}{T^2}\times 1.084$$

Using $T_\circ=2\pi\sqrt{\frac{l}{g}}$ gives $g=\frac{4\pi^2 l}{T_\circ^2}$, approximately 8% too small.

(c) $$B=-\frac{\lambda A^3}{32\omega_\circ^2} \text{ and } \lambda=\frac{\omega_\circ^2}{6}$$

$$\left|\frac{B}{A}\right|=\frac{A^2}{192}$$

for $A=\frac{\pi}{4}$, $\left|\frac{B}{A}\right|=0.0032$

3.20 $$f(t)=\sum_n c_n e^{in\omega t} \qquad n=0,\pm1,\pm2,\ \ldots$$

$$f(t)=\sum_n c_n\cos n\omega t+\sum_n c_n i\sin n\omega t\,,\quad n=0,\pm1,\pm2,\ \ldots$$

and $$c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)e^{-in\omega t}dt\,,\quad n=0,\pm1,\pm2,\ \ldots$$

$$c_n=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\cos(n\omega t)\,dt-\frac{i}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(n\omega t)\,dt$$

The first term on c_n is the same for n and $-n$; the second term changes sign for n vs. $-n$. The same holds true for the trigonometric terms in $f(t)$. Therefore, when terms that cancel in the summations are discarded:

$$f(t) = c_\circ + \sum_n \left(\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(n\omega t)\, dt \right) \cos n\omega t$$

$$+ \sum_n \left(\frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(n\omega t)\, dt \right) \sin n\omega t \,,$$

$$n = \pm 1, \pm 2, \ \ldots, \text{ and } c_\circ = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)\, dt$$

Now, due to the equality of terms in $\pm n$:

$$f(t) = c_\circ + \sum_n \left(\frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \cos(n\omega t)\, dt \right) \cos n\omega t$$

$$+ \sum_n \left(\frac{2}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) \sin(n\omega t)\, dt \right) \sin n\omega t \,,$$

$$n = 1, 2, 3, \ \ldots$$

Equations 3.9.9 and 3.9.10 follow directly.

3.21 $f(t) = \sum_n c_n e^{in\omega t}$, $\quad c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f(t) e^{-in\omega t} dt$, $\quad$ and $n = 0, \pm 1, \pm 2, \ \ldots$

$T = \frac{2\pi}{\omega}$ $\quad$ so $\quad c_n = \frac{\omega}{2\pi} \int_{-\frac{\pi}{\omega}}^{\frac{\pi}{\omega}} f(t) e^{-in\omega t} dt$

$$= \frac{\omega}{2\pi} \left[\int_{-\frac{\pi}{\omega}}^{0} \left(-e^{-in\omega t} \right) dt + \int_{0}^{\frac{\pi}{\omega}} e^{-in\omega t} dt \right]$$

$$= \frac{\omega}{2\pi} \left[\frac{1}{in\omega} e^{-in\omega t} \Big|_{-\frac{\pi}{\omega}}^{0} - \frac{1}{in\omega} e^{-in\omega t} \Big|_{0}^{\frac{\pi}{\omega}} \right]$$

$$= \frac{1}{2\pi i n} \left[1 - e^{+in\pi} - e^{-in\pi} + 1 \right]$$

For n even, $e^{in\pi} = e^{-in\pi} = 1$ and the term in brackets is zero.
For n odd, $e^{in\pi} = e^{-in\pi} = -1$

$$c_n = \frac{4}{2\pi i n}, \qquad n = \pm 1, \pm 3, \ \ldots$$

$$f(t) = \sum_n \frac{4}{2\pi i n} e^{in\omega t}, \quad n = \pm 1, \pm 3, \ \ldots$$

$$= \sum_n \frac{4}{\pi} \frac{1}{n} \frac{1}{2i} \left(e^{in\omega t} - e^{-in\omega t} \right), \qquad n = 1, 3, 5, \ \ldots$$

$$= \sum_n \frac{4}{\pi} \frac{1}{n} \sin(n\omega t), \quad n = 1, 3, 5, \ \ldots$$

$$f(t) = \frac{4}{\pi} \left[\sin \omega t + \frac{1}{3} \sin 3\omega t + \frac{1}{5} \sin 5\omega t + \ldots \right]$$

3.22 In steady state, $x(t)=\sum_n A_n e^{i(n\omega t-\phi_n)}$

$$A_n = \frac{\frac{F_n}{m}}{\left[\left(\omega_\circ^2 - n^2\omega^2\right)^2 + 4\gamma^2 n^2\omega^2\right]^{\frac{1}{2}}}$$

Now $F_n = \dfrac{4F_\circ}{n\pi}$, $n = 1,3,5,\ . \ . \ .$ and $\omega_\circ = 3\omega$

$Q = 100 \approx \dfrac{\omega_\circ}{2\gamma}$ so $\gamma^2 \approx \dfrac{9\omega^2}{40,000}$

$$A_1 = \frac{4F_\circ}{m\pi}\cdot\frac{1}{\left[\left(9\omega^2-\omega^2\right)^2 + 4\dfrac{9\omega^2\cdot\omega^2}{40000}\right]^{\frac{1}{2}}}$$

$$A_1 \approx \frac{F_\circ}{2m\pi\omega^2}$$

$$A_3 = \frac{4F_\circ}{3m\pi}\cdot\frac{1}{\left[\left(9\omega^2-9\omega^2\right)^2 + 4\left(\dfrac{9\omega^2}{200}\right)^2\right]^{\frac{1}{2}}}$$

$$A_3 \approx \frac{400F_\circ}{27m\pi\omega^2}$$

$$A_5 = \frac{4F_\circ}{5m\pi}\cdot\frac{1}{\left[\left(9\omega^2-25\omega^2\right)^2 + 4\left(\dfrac{3\omega}{200}\right)^2(5\omega)^2\right]^{\frac{1}{2}}}$$

$$A_5 \approx \frac{F_\circ}{20m\pi\omega^2}$$

i.e., $A_1 : A_3 : A_5 = 1 : 29.6 : 0.1$

3.23 (a) $\ddot{x}+\omega_\circ^2 x = 0$ $y=\dot{x}$ Thus $\dot{y} = -\omega_\circ^2 x$ $\dot{x} = y$

divide these two equations: $\dfrac{\dot{y}}{\dot{x}} = \dfrac{dy}{dx} = -\dfrac{\omega_\circ^2 x}{y}$

(b) Solving $\dfrac{ydy}{\omega_\circ^2} + xdx = 0$ and Integrating $\dfrac{y^2}{2\omega_\circ^2} + \dfrac{x^2}{2} = C$

Let $2C = A^2$

$$\frac{y^2}{\omega_\circ^2 A^2} + \frac{x^2}{A^2} = 1 \qquad \rightarrow \quad \textit{an ellipse}$$

3.24 The equation of motion is $F(x) = x - x^3 = m\ddot{x}$. For simplicity, let m=1. Then

$\ddot{x} = x - x^3$. This is equivalent to the two first order equations …

$\dot{x} = y$ and $\dot{y} = x - x^3$

(a) The equilibrium points are defined by

$$x - x^3 = x(1-x)(1+x) = 0$$

Thus, the points are: (-1,0), (0,0) and (+1,0). We can tell whether or not the points represent stable or unstable points of equilibrium by examining the phase space plots in the neighborhood of the equilibrium points. We'll do this in part (c).

(b) The energy can be found by integrating $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{x - x^3}{y}$ or

$$\int y\,dy = \int \left(x - x^3\right) dx + C \text{ or}$$

$$\frac{y^2}{2} = \frac{x^2}{2} - \frac{x^4}{4} + C$$

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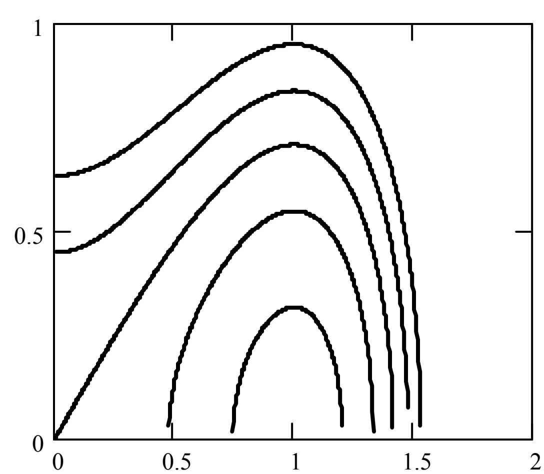
In other words … $E = T + V = \frac{y^2}{2} + \left[\frac{x^4}{4} - \frac{x^2}{2}\right] = C$. The total energy C is constant.

(c) The phase space trajectories are given by solutions to the above equation

$$y = \pm\left(x^2 - \frac{x^4}{2} + 2C\right)^{\frac{1}{2}}.$$

The upper right quadrant of the trajectories is shown in the figure below. The trajectories are symmetrically disposed about the x and y axes. They form closed paths for energies C<0 about the two points (-1,0) and (+1,0). Thus, these are points of stable equilibrium for small excursions away from these points. The trajectory passes thru the point (0,0) for C=0 and is a saddle point. Trajectories never pass thru the point (0,0) for positive energies C>0. Thus, (0,0) is a point of unstable equilibrium.



3.25 $\ddot{\theta}+\sin\theta=0 \quad \Rightarrow \quad \frac{d}{dt}\left\{\frac{\theta^2}{2}-\cos\theta\right\}=0$

Integrating: $\left.\frac{\dot{\theta}^2}{2}\right|_0^{\dot{\theta}}=\cos\theta\Big|_{\theta_\circ}^{\theta}$ or $\dot{\theta}^2=2\left(\cos\theta-\cos\theta_\circ\right)$

$$\therefore T=4\int_0^{\theta_\circ}\frac{d\theta}{\left[2\left(\cos\theta-\cos\theta_\circ\right)\right]^{\frac{1}{2}}}$$

Time for pendulum to swing from $\theta=0$ to $\theta=\theta_\circ$ is $\frac{T}{4}$

Now—substitute $\sin\phi=\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_\circ}{2}}$ so $\phi=\frac{\pi}{2}$ at $\theta=\theta_\circ$

and use the identity $\cos\theta=1-2\sin^2\frac{\theta}{2}$ $\quad\therefore T=4\int_0^{\theta_\circ}\frac{d\theta}{\left[4\left(\sin^2\frac{\theta_\circ}{2}-\sin^2\frac{\theta}{2}\right)\right]^{\frac{1}{2}}}$

and after some algebra … $\frac{d\phi}{\left[1-\sin^2\frac{\theta}{2}\right]^{\frac{1}{2}}}=\frac{d\theta}{\left[4\left(\sin^2\frac{\theta_\circ}{2}-\sin^2\frac{\theta}{2}\right)\right]^{\frac{1}{2}}}$ or

(a) $T=4\int_0^{\frac{\pi}{2}}\frac{d\phi}{\left[1-\alpha\sin^2\phi\right]^{\frac{1}{2}}}$ where $\alpha=\sin^2\frac{\theta_\circ}{2}$

(b) $\left(1-\alpha\sin^2\phi\right)^{-\frac{1}{2}}\approx 1+\frac{1}{2}\alpha\sin^2\phi+\frac{3}{8}\alpha^2\sin^4\phi+\ldots$

$T=4\int_0^{\frac{\pi}{2}}d\phi\left[1+\frac{1}{2}\alpha\sin^2\phi+\frac{3}{8}\alpha^2\sin^4\phi+\ldots\right]$ $\qquad T=2\pi\left[1+\frac{\alpha}{4}+\frac{9}{64}\alpha^2+\ldots\right]$

(c) $\alpha=\sin^2\frac{\theta_\circ}{2}\approx\left[\frac{\theta_\circ}{2}-\frac{\theta_\circ^3}{48}+\ldots\right]^2\sim\frac{\theta_\circ^2}{4}\ \ldots$

$$T=2\pi\left[1+\frac{\theta_\circ^2}{16}+\ldots\right]$$

CHAPTER 4
GENERAL MOTION OF A PARTICLE IN THREE DIMENSIONS

Note to instructors … there is a typo in equation 4.3.14. The range of the projectile is …

$$R = x = \frac{v_0^{\,2}\sin 2\alpha}{g} \quad \ldots \text{ NOT } \ldots \frac{v_0^{\,2}\sin^2 2\alpha}{g}$$

4.1 (a) $\vec{F} = -\vec{\nabla}V = -\hat{i}\frac{\partial V}{\partial x} - \hat{j}\frac{\partial V}{\partial y} - \hat{k}\frac{\partial V}{\partial z}$

$\vec{F} = -c\left(\hat{i}yz + \hat{j}xz + \hat{k}xy\right)$

(b) $\vec{F} = -\vec{\nabla}V = -\hat{i}2\alpha x - \hat{j}2\beta y - \hat{k}2\gamma z$

(c) $\vec{F} = -\vec{\nabla}V = ce^{-(\alpha x+\beta y+\gamma z)}\left(\hat{i}\alpha + \hat{j}\beta + \hat{k}\gamma\right)$

(d) $\vec{F} = -\vec{\nabla}V = -\hat{e}_r\frac{\partial V}{\partial r} - \hat{e}_\theta\frac{1}{r}\frac{\partial V}{\partial \theta} - \hat{e}_\phi\frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$

$\vec{F} = -\hat{e}_r cnr^{n-1}$

4.2 (a)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

conservative

(b)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z^2 \end{vmatrix} = \hat{k}\left(-1-1\right) \neq 0$$

~~conservative~~

(c)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & z^3 \end{vmatrix} = \hat{k}\left(1-1\right) = 0$$

conservative

(d)

$$\vec{\nabla}\times\vec{F}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{e}_r & \hat{e}_\theta r & \hat{e}_\phi r\sin\theta\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi}\\ -kr^{-n} & 0 & 0\end{vmatrix}=0 \quad \text{conservative}$$

4.3 (a)

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & cx^2 & z^3\end{vmatrix}=k\left(2cx-x\right)$$

$$2cx-x=0$$

$$c=\frac{1}{2}$$

(b)

$$\vec{\nabla}\times\vec{F}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{z}{y} & \frac{cxz}{y^2} & \frac{x}{y}\end{vmatrix}$$

$$=\hat{i}\left(-\frac{x}{y^2}-\frac{cx}{y^2}\right)+\hat{j}\left(\frac{1}{y}-\frac{1}{y}\right)+\hat{k}\left(\frac{cz}{y^2}+\frac{z}{y^2}\right)$$

$$-\frac{x}{y^2}-\frac{cx}{y^2}=0 \qquad\qquad c=-1$$

also $\frac{cz}{y^2}+\frac{z}{y^2}=0$ implies that $c=-1$ as it must

4.4 (a) $E=\text{constant}=V\left(x,y,z\right)+\frac{1}{2}mv^2$

at the origin $E=0+\frac{1}{2}mv_\circ^2$

at $\left(1,1,1\right)$ $E=\alpha+\beta+\gamma+\frac{1}{2}mv^2=\frac{1}{2}mv_\circ^2$

$$v^2 = v_\circ^2 - \frac{2}{m}(\alpha + \beta + \gamma)$$

$$v = \left[v_\circ^2 - \frac{2}{m}(\alpha + \beta + \gamma) \right]^{\frac{1}{2}}$$

(b) $$v_\circ^2 - \frac{2}{m}(\alpha + \beta + \gamma) = 0$$

$$v_\circ = \left[\frac{2}{m}(\alpha + \beta + \gamma) \right]^{\frac{1}{2}}$$

(c) $$m\ddot{x} = F_x = -\frac{\partial V}{\partial x}$$

$$m\ddot{x} = -\alpha$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} = -2\beta y$$

$$m\ddot{z} = -\frac{\partial V}{\partial z} = -3\gamma z^2$$

4.5 (a) $\vec{F} = \hat{i}x + \hat{j}y$

on the path $x = y$: $d\vec{r} = \hat{i}dx + \hat{j}dy$

$$\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r} = \int_0^1 F_x dx + \int_0^1 F_y dy = \int_0^1 x dx + \int_0^1 y dy = 1$$

on the path along the x-axis: $d\vec{r} = \hat{i}dx$

and on the line $x = 1$: $d\vec{r} = \hat{j}dy$

$$\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r} = \int_0^1 F_x dx + \int_0^1 F_y dy = 1$$

$\vec{F}$ is conservative.

(b) $\vec{F} = \hat{i}y - \hat{j}x$

on the path $x = y$:

$$\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r} = \int_0^1 F_x dx + \int_0^1 F_y dy = \int_0^1 y dx - \int_0^1 x dy$$

and, with $x = y$ $\quad \int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r} = \int_0^1 x dx - \int_0^1 y dy = 0$

on the x-axis: $\quad \int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r} = \int_0^1 F_x dx = \int_0^1 y dx$

and, with $y = 0$ on the x-axis $\quad \int_{(0,0)}^{(1,0)} \vec{F} \cdot d\vec{r} = 0$

on the line $x = 1$: $\quad \int_{(1,0)}^{(1,1)} \vec{F} \cdot d\vec{r} = \int_0^1 F_y dy = \int_0^1 x dy$

and, with $x=1$ $$\int_{(1,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = \int_0^1 dy = 1$$

on this path $$\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = 0+1=1$$

$\vec{F}$ is *not* conservative.

4.6 From Example 2.3.2, $V(z) = -mg\dfrac{r_e^2}{(r_e+z)}$

$$V(z) = -mgr_e\left(1+\frac{z}{r_e}\right)^{-1}$$

From Appendix D, $(1+x)^{-1} = 1-x+x^2+\ldots$

$$V(z) = -mgr_e\left(1-\frac{z}{r_e}+\frac{z^2}{r_e^2}+\ldots\right)$$

$$V(z) = -mgr_e + mgz - \frac{mgz^2}{r_e}+\ldots$$

With $-mgr_e$ an additive constant,

$$V(z) \approx mgz\left(1-\frac{z}{r_e}\right)$$

$$\vec{F} = -\vec{\nabla}V = -\hat{k}\frac{\partial}{\partial z}V(z)$$

$$= -\hat{k}mg\left[1-\frac{z}{r_e}+z\left(-\frac{1}{r_e}\right)\right]$$

$$\vec{F} = -\hat{k}mg\left(1-\frac{2z}{r_e}\right)$$

$$m\ddot{x} = F_x = 0, \qquad m\ddot{y} = F_y = 0 \qquad m\ddot{z} = -mg\left(1-\frac{2z}{r_e}\right)$$

$$m\dot{z}\frac{d\dot{z}}{dz} = -mg\left(1-\frac{2z}{r_e}\right)$$

$$\int_{v_{\circ z}}^{0} \dot{z}dz = -g\int_0^h\left(1-\frac{2z}{r_e}\right)dz$$

$$-\frac{1}{2}v_{\circ z}^2 = -g\left(h-\frac{h^2}{r_e}\right)$$

$$h^2 - r_e h + \frac{r_e v_{\circ z}^2}{2g} = 0$$

$$h = \frac{r_e}{2} - \frac{1}{2}\sqrt{r_e^2 - \frac{2r_e v_{\circ z}^2}{g}} \qquad (h, z \ll r_e)$$

$$h = \frac{r_e}{2} - \frac{r_e}{2}\sqrt{1 - \frac{2v_{\circ z}^2}{gr_e}}$$

From Appendix D, $(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} - \frac{x^2}{8} + \ldots$

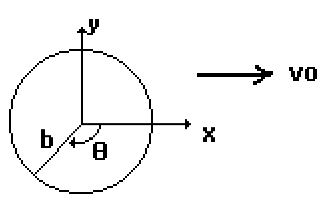
$$h = \frac{r_e}{2} - \frac{r_e}{2} + \frac{v_{\circ z}^2}{2g} + \frac{v_{\circ z}^4}{4gr_e} + \ldots$$

$$h \approx \frac{v_{\circ z}^2}{2g}\left(1 + \frac{v_{\circ z}^2}{2gr_e}\right)$$

From Example 2.3.2, $h = \frac{v_\circ^2}{2g}\left(1 - \frac{v_\circ^2}{2gr_e}\right)^{-1}$

And with $(1-x)^{-1} \approx 1 + x$, $\quad h \approx \frac{v_\circ^2}{2g}\left(1 + \frac{v_\circ^2}{2gr_e}\right)$

4.7



For a point on the rim measured from the center of the wheel:

$$\vec{r} = \hat{i}b\cos\theta - \hat{j}b\sin\theta$$

$$\theta = \omega t = \frac{v_\circ t}{b}, \qquad \text{so } \dot{\vec{r}} = -\hat{i}v_\circ \sin\theta - \hat{j}v_\circ \cos\theta$$

Relative to the ground, $\vec{v} = \hat{i}v_\circ(1 - \sin\theta) - \hat{j}v_\circ\cos\theta$

For a particle of mud leaving the rim:

$y_\circ = -b\sin\theta$ and $v_{\circ y} = -v_\circ\cos\theta$

So $\quad v_y = v_{\circ y} - gt = -v_\circ\cos\theta - gt$

and $\quad y = -b\sin\theta - v_\circ t\cos\theta - \frac{1}{2}gt^2$

At maximum height, $v_y = 0$:

$$t = -\frac{v_\circ\cos\theta}{g}$$

$$h = -b\sin\theta - v_\circ\left(-\frac{v_\circ\cos\theta}{g}\right)\cos\theta - \frac{1}{2}g\left(-\frac{v_\circ\cos\theta}{g}\right)^2$$

$$h = -b\sin\theta + \frac{v_\circ^2\cos^2\theta}{2g}$$

Maximum h occurs for $\quad \frac{dh}{d\theta} = 0 = -b\cos\theta - \frac{2v_\circ^2\cos\theta\sin\theta}{2g}$

$$\sin\theta = -\frac{gb}{v_\circ^2}$$

$$\cos^2\theta = 1-\sin^2\theta = \frac{v_\circ^4 - g^2b^2}{v_\circ^4}$$

$$h_{max} = \frac{gb^2}{v_\circ^2} + \frac{v_\circ^4 - g^2b^2}{2gv_\circ^2} = \frac{gb^2}{2v_\circ^2} + \frac{v_\circ^2}{2g}$$

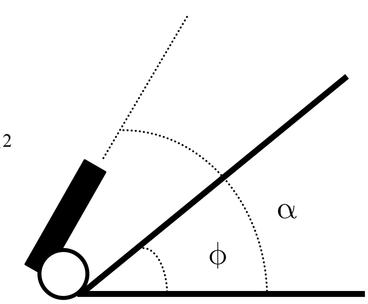
Measured from the ground,

$$h'_{max} = b + \frac{gb^2}{2v_\circ^2} + \frac{v_\circ^2}{2g}$$

The mud leaves the wheel at $\theta = \sin^{-1}\left(-\frac{gb}{v_\circ^2}\right)$

4.8 $x = R\cos\phi$ and $x = v_{\circ x}t = (v_\circ \cos\alpha)t$

so $t = \dfrac{R\cos\phi}{v_\circ \cos\alpha}$



$y = R\sin\phi$ and $y = v_{\circ y}t - \frac{1}{2}gt^2 = (v_\circ \sin\alpha)t - \frac{1}{2}gt^2$

$$R\sin\phi = (v_\circ \sin\alpha)\frac{R\cos\phi}{v_\circ\cos\alpha} - \frac{1}{2}g\left(\frac{R\cos\phi}{v_\circ\cos\alpha}\right)^2$$

$$\sin\phi = \tan\alpha\cos\phi - \frac{gR\cos^2\phi}{2v_\circ^2\cos^2\alpha}$$

$$R = \frac{2v_\circ^2\cos^2\alpha}{g\cos^2\phi}(\tan\alpha\cos\phi - \sin\phi) = \frac{2v_\circ^2\cos\alpha}{g\cos^2\phi}(\sin\alpha\cos\phi - \cos\alpha\sin\phi)$$

From Appendix B, $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$R = \frac{2v_\circ^2\cos\alpha}{g\cos^2\phi}\sin(\alpha-\phi)$$

R is a maximum for $\dfrac{dR}{d\alpha} = 0 = \dfrac{2v_\circ^2}{g\cos^2\phi}\left[-\sin\alpha\sin(\alpha-\phi) + \cos\alpha\cos(\alpha-\phi)\right]$

Implies that $\cos\alpha\cos(\alpha-\phi) - \sin\alpha\sin(\alpha-\phi) = 0$

From appendix B, $\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

so $\cos(2\alpha-\phi) = 0$

$$2\alpha - \phi = \frac{\pi}{2} \qquad \alpha = \frac{\pi}{4} + \frac{\phi}{2}$$

$$R_{max} = \frac{2v_\circ^2}{g\cos^2\phi}\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)$$

Now $\sin\left(\frac{\pi}{4}-\frac{\phi}{2}\right)=\cos\left[\frac{\pi}{2}-\left(\frac{\pi}{4}-\frac{\phi}{2}\right)\right]=\cos\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$

$$R_{\max}=\frac{2v_\circ^2}{g\cos^2\phi}\cos^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$$

Again using Appendix B, $\cos 2\theta=\cos^2\theta-\sin^2\theta=2\cos^2\theta-1$

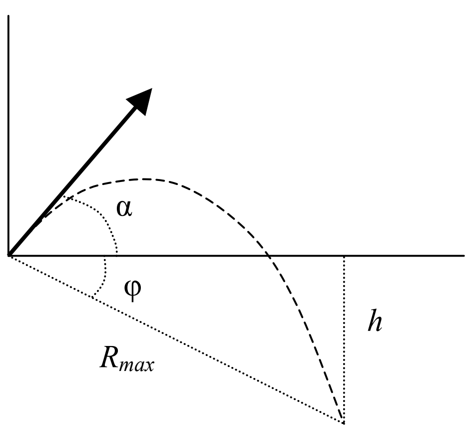
$$R_{\max}=\frac{2v_\circ^2}{g\cos^2\phi}\left[\frac{1}{2}\cos\left(\frac{\pi}{2}+\phi\right)+\frac{1}{2}\right]=\frac{v_\circ^2}{g\cos^2\phi}\left[\cos\left(\frac{\pi}{2}+\phi\right)+1\right]$$

Using $\cos\left(\frac{\pi}{2}+\theta\right)=-\sin\theta$,

$$R_{\max}=\frac{v_\circ^2}{g\left(1-\sin^2\phi\right)}\left(1-\sin\phi\right)$$

$$R_{\max}=\frac{v_\circ^{\ 2}}{g\left(1+\sin\phi\right)}$$

4.9



(a) Here we note that the projectile is launched "downhill" towards the target, which is located a distance h below the cannon along a line at an angle φ below the horizon. α is the angle of projection that yields maximum range, R_{max}. We can use the results from problem 4.8 for this problem. We simply have to replace the angle φ in the above result with the angle -φ, to account for the downhill slope. Thus, we get for the downhill range …

$$R=\frac{2v_0^{\ 2}}{g}\frac{\cos\alpha\sin\left(\alpha+\varphi\right)}{\cos^2\varphi}$$

The maximum range and the angle is α are obtained from the problem above again by replacing φ with the angle -φ … $R_{\max}=\frac{v_0^{\ 2}}{g}\frac{\left(1+\sin\varphi\right)}{\cos^2\varphi}$ and … $2\alpha=\frac{\pi}{2}-\varphi$.

We can now calculate α … $R_{\max}=\frac{h}{\sin\varphi}=\frac{v_0^{\ 2}}{g}\frac{\left(1+\sin\varphi\right)}{\cos^2\varphi}=\frac{v_0^{\ 2}}{g\left(1-\sin\varphi\right)}$

Solving for $\sin\varphi$ … $\sin\varphi=\frac{gh}{v_0^{\ 2}}\Big/\left(1+\frac{gh}{v_0^{\ 2}}\right)$

But, from the above … $\sin\varphi=\sin\left(\frac{\pi}{2}-2\alpha\right)=\cos 2\alpha=1-2\sin^2\alpha$

Thus … $1-2\sin^2\alpha=\frac{gh}{v_0^{\ 2}}\Big/\left(1+\frac{gh}{v_0^{\ 2}}\right)$

$$2\sin^2\alpha = \frac{2}{\csc^2\alpha} = 1 - \frac{gh}{v_0^2}\Bigg/\left(1 + \frac{gh}{v_0^2}\right) = \frac{1}{1 + \frac{gh}{v_0^2}}$$

Finally … $\csc^2\alpha = 2\left(1 + \frac{gh}{v_0^2}\right)$

(b)

Solving for R_{max} … $R_{\max} = \frac{h}{\sin\varphi} = \frac{h}{1 - 2\sin^2\alpha} = \frac{h}{1 - 2/\csc^2\alpha}$

Substituting for $\csc^2\alpha$ and solving …

$$R_{\max} = \frac{v_0^2}{g}\left(1 + \frac{gh}{v_0^2}\right)$$

4.10 We can again use the results of problem 4.8. The maximum slope range from problem 4.8 is given by …

$$R_{\max} = \frac{v_0^2}{g(1 + \sin\varphi)} = \frac{h}{\sin\varphi}$$

Solving for $\sin\varphi$ …

$$\sin\varphi = \frac{gh}{v_0^2}\Bigg/\left(1 - \frac{gh}{v_0^2}\right)$$

Thus …

$$x_{\max} = R_{\max}\cos\varphi = h\frac{\cos\varphi}{\sin\varphi}$$

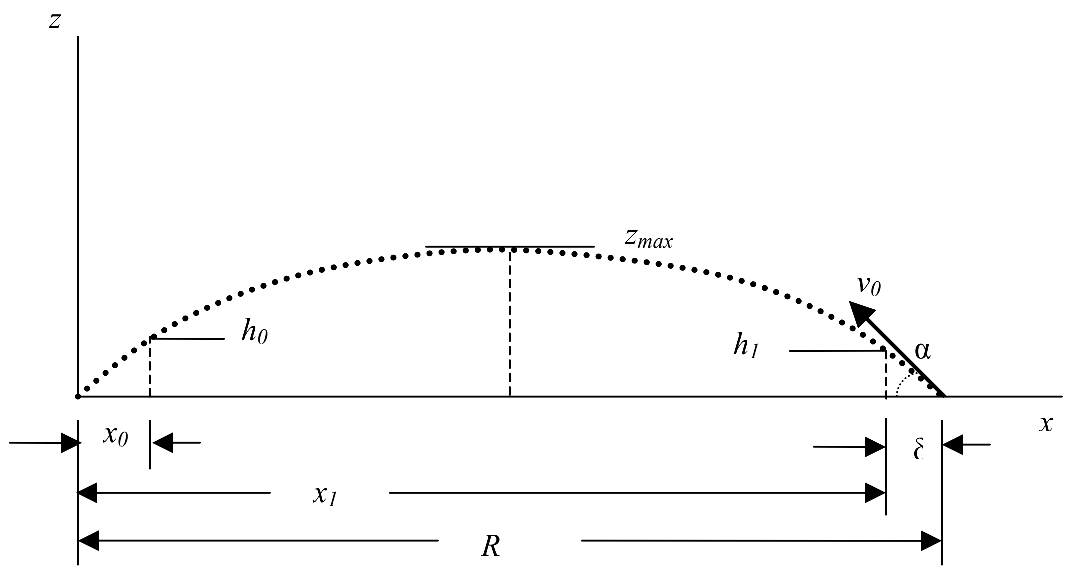
We can calculate $\cos\varphi$ from the above relation for $\sin\varphi$ …

$$\cos\varphi = \left(1 - \sin^2\varphi\right)^{\frac{1}{2}} = \left(1 - 2\frac{gh}{v_0^2}\right)^{\frac{1}{2}}\Bigg/\left(1 - \frac{gh}{v_0^2}\right)$$

Inserting the results for $\sin\varphi$ and $\cos\varphi$ into the above …

$$x_{\max} = h\frac{\cos\varphi}{\sin\varphi} = \frac{v_0^2}{g}\left(1 - 2\frac{gh}{v_0^2}\right)^{\frac{1}{2}}$$

4.11 We can simplify this problems somewhat by noting that the trajectory is symmetric about a vertical line that passes through the highest point of the trajectory. Thus we have the following picture …



We have “reversed the trajectory so that h_0 (= 9.8 ft), and x_0 , the height and range within which Mickey can catch the ball represent the starting point of the trajectory. h_1 (=3.28 ft) is the height of the ball when Mickey strikes it at home plate. δ is the distance behind home plate where the ball would be hypothetically launched at some angle α to achieve the total range R. x_1 (=328 ft) is the distance the ball actually would travel from home plate if not caught by Mickey. (Note, because of the symmetry, v_0 is the speed of the ball when it strikes the ground … also at the same angle α at which was launched. We will calculate the value of x_0 assuming a time-reversed trajectory!)

(1) The range of the ball … $R = \frac{v_0^2 \sin 2\alpha}{g} = \frac{2v_0^2 \sin\alpha\cos\alpha}{g}$

(2) The maximum height … $z_{\max} = \frac{R}{2}\tan\alpha - \frac{g}{2v_0^2\cos^2\alpha}\left(\frac{R}{2}\right)^2$

(3) The height at x_1 … $h_1 = x_1\tan\alpha - \frac{g}{2v_0^2\cos^2\alpha}(x_1)^2$

From (1) … $\frac{g}{2v_0^2\cos^2\alpha} = \frac{\tan\alpha}{R}$ and inserting this into (2) gives …

$$z_{\max} = \frac{R}{2}\tan\alpha - \frac{R}{4}\tan\alpha = \frac{R}{4}\tan\alpha$$

Thus, $R = \frac{4z_{\max}}{\tan\alpha}$ and inserting this expression and the first previously derived into (3)

(4) $h_1 = x_1\tan\alpha - \frac{(x_1\tan\alpha)^2}{4z_{\max}}$

Let $u = x_1\tan\alpha$ and we obtain the following quadratic …

$u^2 - 4z_{\max}u + 4z_{\max}h_1 = 0$ and solving for u …

$u = 2z_{\max}\left[1 \pm (1 - h_1/z_{\max})^{\frac{1}{2}}\right]$ and letting $\varepsilon = \frac{h_1}{z_{\max}}$, we get …

$u \approx z_{\max}\varepsilon = h_1$

or $\quad u \approx 2z_{\max}\left(2-\varepsilon\right) = 2z_{\max}\left(2-.0475\right) = 3.9z_{\max}$. This result is the correct one …

Thus, $\quad \tan\alpha = \dfrac{3.9z_{\max}}{x_1} = 0.821 \quad \therefore\ \alpha = 39.4^0$

Now solve for x_0 using a relation identical to (4) …

$$h_0 = x_0 \tan\alpha - \frac{\left(x_0 \tan\alpha\right)^2}{4z_{\max}}$$

Again we obtain a quadratic expression for $u = x_0 \tan\alpha$ which we solve as before. This time, though, the first result for u is the correct one to use …

$u = z_{\max}\varepsilon \approx h_0$ and we obtain …

$$x_0 = \frac{h_0}{\tan\alpha} = 11.9\, ft$$

4.12 The x and z positions of the ball vs. time are

$$x = v_\circ t \cos\frac{1}{2}\theta_\circ \qquad z = v_\circ t \cos\frac{1}{2}\theta_\circ \sin\theta_\circ - \frac{1}{2}gt^2$$

Since $v_x = v_\circ \cos\frac{1}{2}\theta_\circ$

The horizontal range is $\quad R = \dfrac{v_\circ^{\,2}}{g}\cos^2\dfrac{1}{2}\theta_\circ \sin 2\theta_\circ$

The maximum range occurs @ $\dfrac{dR}{d\theta_\circ} = 0$

$$\frac{dR}{d\theta_\circ} = \frac{v_\circ^{\,2}}{g}\left(2\cos^2\frac{1}{2}\theta_\circ \cos 2\theta_\circ - \cos\frac{1}{2}\theta_\circ \sin\frac{1}{2}\theta_\circ \sin 2\theta_\circ\right) = 0$$

Thus, $2\cos^2\frac{1}{2}\theta_\circ \cos 2\theta_\circ = \cos\frac{1}{2}\theta_\circ \sin\frac{1}{2}\theta_\circ \sin 2\theta_\circ$

Using the identities: $\quad 2\cos^2\theta_\circ = 1+\cos 2\theta_\circ \quad$ and $\quad \sin 2\theta_\circ = 2\sin\theta_\circ \cos\theta_\circ$

We get:

$$\left(1+\cos\theta_\circ\right)\left(2\cos^2\theta_\circ - 1\right) = \sin\theta_\circ \sin\theta_\circ \cos\theta_\circ = \left(1-\cos^2\theta_\circ\right)\cos\theta_\circ$$

or $\quad \left(1+\cos\theta_\circ\right)\left(3\cos^2\theta_\circ - \cos\theta_\circ - 1\right) = 0$

Thus $\quad \cos\theta_\circ = -1, \quad \cos\theta_\circ = \dfrac{1}{6}\left(1 \pm \sqrt{13}\right)$

Only the positive root applies for the $\theta_\circ$-range: $\qquad 0 \le \theta_\circ \le \dfrac{\pi}{2}$

$$\cos\theta_\circ = \frac{1}{6}\left(1+\sqrt{13}\right) = 0.7676 \qquad \underline{\theta_\circ = 39^\circ 51'}$$

Thus (b) for $v_\circ = 25\, m\, s^{-1}$ $\underline{R_{\max} = 55.4\, m \quad @\theta_\circ = 39^\circ 51'}$

(a) The maximum height occurs at $\frac{dz}{dt} = 0$

$$v_\circ \cos\frac{1}{2}\theta_\circ \sin\theta_\circ = gT \qquad \text{or at} \qquad T = \frac{v_\circ \cos\frac{1}{2}\theta_\circ \sin\theta_\circ}{g}$$

or $\quad H = \frac{v_\circ^2}{2g}\cos^2\frac{1}{2}\theta_\circ \sin^2\theta_\circ \qquad$ maximum at fixed $\theta_\circ$

The maximum possible height occurs @ $\frac{dH}{d\alpha} = 0$

$$\frac{dH}{d\alpha} = \frac{v_\circ^2}{2g}\left(2\cos^2\frac{1}{2}\theta_\circ \sin\theta_\circ \cos\theta_\circ - \cos\frac{1}{2}\theta_\circ \sin\frac{1}{2}\theta_\circ \sin^2\theta_\circ\right) = 0$$

Using the above trigonometric identities, we get

$$(1+\cos\theta_\circ)\sin\theta_\circ \cos\theta_\circ = \frac{1}{2}\sin\theta_\circ \sin^2\theta_\circ = \frac{1}{2}\sin\theta_\circ\left(1-\cos^2\theta_\circ\right)$$

or $\quad \sin\theta_\circ(1+\cos\theta_\circ)(3\cos\theta_\circ - 1) = 0$

There are 3-roots: $\quad \sin\theta_\circ = 0, \quad \cos\theta_\circ = -1, \quad \cos\theta_\circ = \frac{1}{3}$

The first two roots give minimum heights; the last gives the maximum

Thus, $\quad H_{max} = 18.9m \quad @\theta_\circ = \cos^{-1}\frac{1}{3} = 70^\circ 32'$

4.13 The trajectory of the shell is given by Eq. 4.3.11 with r replacing x

$$z = \frac{\dot{z}_\circ}{\dot{r}_\circ}r - \frac{g}{2\dot{r}_\circ^2}r^2 \qquad \text{where} \quad \dot{r}_\circ = v_\circ\cos\theta_\circ \quad \dot{z}_\circ = v_\circ\sin\theta_\circ$$

Thus, $\quad z = r\tan\theta_\circ - \frac{g r^2}{2v_\circ^2}\sec^2\theta_\circ$

Since $\quad \sec^2\theta_\circ = 1 + \tan^2\theta_\circ$

We have:

$$\frac{g r^2}{2v_\circ^2}\tan^2\theta_\circ - r\tan\theta_\circ + z + \frac{g r^2}{2v_\circ^2} = 0$$

(r,z) are target coordinates.

The above equation yields two possible roots:

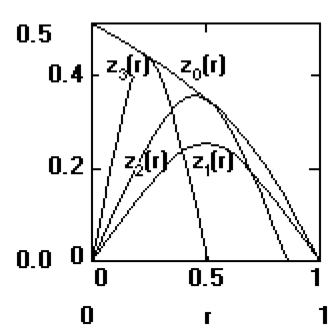
$$\tan\theta_\circ = \frac{1}{gr}\left[v_\circ^2 \pm \left(v_\circ^4 - 2gzv_\circ^2 - g^2r^2\right)^{\frac{1}{2}}\right]$$

The roots are only real if

$v_\circ^4 - 2gzv_\circ^2 - g^2r^2 \ge 0$

The critical surface is therefore:

$v_\circ^4 - 2gzv_\circ^2 - g^2r^2 = 0$



4.14 If the velocity vector, of magnitude $\dot{s}$, makes an angle θ with the z-axis, and its projection on the xy-plane make an angle ϕ with the x-axis:

$$\dot{x} = \dot{s}\sin\theta\cos\phi \ , \qquad \text{and} \quad F_x = F_r \sin\theta\cos\phi = m\ddot{x}$$

$$\dot{y} = \dot{s}\sin\theta\sin\phi \ , \qquad \text{and} \quad F_y = F_r \sin\theta\sin\phi = m\ddot{y}$$

$$\dot{z} = \dot{s}\cos\theta \ , \qquad \text{and} \quad F_z = -mg + F_r\cos\theta = m\ddot{z}$$

Since $F_r = -c_2\dot{s}^2 = -c_2\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$, the differential equations of motion are not separable.

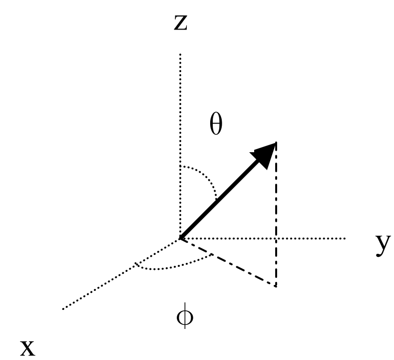
$$m\ddot{x} = -c_2\dot{s}^2\sin\theta\cos\phi = -c_2\dot{s}\dot{x}$$

$$m\frac{d\dot{x}}{dt} = m\frac{d\dot{x}}{ds}\cdot\frac{ds}{dt} = m\dot{s}\frac{d\dot{x}}{ds} = -c_2\dot{s}\dot{x}$$

$$\frac{d\dot{x}}{\dot{x}} = -\frac{c_2}{m}ds = -\gamma ds \text{, where } \gamma = \frac{c_2}{m}$$

$$\ln\dot{x} - \ln\dot{x}_\circ = \ln\frac{\dot{x}}{\dot{x}_\circ} = -\gamma s$$

$$\dot{x} = \dot{x}_\circ e^{-\gamma s}$$



Similarly $\qquad \dot{y} = \dot{y}_\circ e^{-\gamma s}$

4.15 From eqn 4.3.16, $\left(\frac{\dot{z}_\circ}{\gamma} + \frac{g}{\gamma^2}\right)\frac{\gamma x_{max}}{\dot{x}_\circ} + \frac{g}{\gamma^2}\ln\left(1 - \frac{\gamma x_{max}}{\dot{x}_\circ}\right) = 0$

From Appendix D: $\ln(1-u) = -u - \frac{u^2}{2} - \frac{u^3}{3} - \ldots \qquad \text{for} \quad |u| < 1$

$$\ln\left(1 - \frac{\gamma x_{max}}{\dot{x}_\circ}\right) = -\frac{\gamma x_{max}}{\dot{x}_\circ} - \frac{\gamma^2 x_{max}{}^2}{2\dot{x}_\circ^2} - \frac{\gamma^3 x_{max}^3}{3\dot{x}_\circ^3} + \text{ terms in } \gamma^4$$

$$\frac{\dot{z}_\circ x_{max}}{\dot{x}_\circ} + \frac{g x_{max}}{\gamma\dot{x}_\circ} - \frac{g x_{max}}{\gamma\dot{x}_\circ} - \frac{g x_{max}^2}{2\dot{x}_\circ^2} - \frac{g\gamma x_{max}^3}{3\dot{x}_\circ^3} + \text{ terms in } \gamma^2 = 0$$

$$x_{max}^2 + \frac{3\dot{x}_\circ}{2\gamma}x_{max} - \frac{3\dot{x}_\circ^2\dot{z}_\circ}{g\gamma} \approx 0$$

$$x_{max} \approx -\frac{3\dot{x}_\circ}{4\gamma} \pm \left(\frac{9\dot{x}_\circ^2}{16\gamma^2} + \frac{3\dot{x}_\circ^2\dot{z}_\circ}{g\gamma}\right)^{\frac{1}{2}}$$

$$x_{max} \approx -\frac{3\dot{x}_\circ}{4\gamma} \pm \frac{3\dot{x}_\circ}{4\gamma}\left(1 + \frac{16\gamma\dot{z}_\circ}{3g}\right)^{\frac{1}{2}}$$

Since $x_{max} > 0$, the + sign is used.

From Appendix D:

$$\left(1 + \frac{16\gamma\dot{z}_\circ}{3g}\right)^{\frac{1}{2}} = 1 + \frac{8\gamma\dot{z}_\circ}{3g} - \frac{1}{8}\left(\frac{16\gamma\dot{z}_\circ}{3g}\right)^2 + \text{ terms in } \gamma^3$$

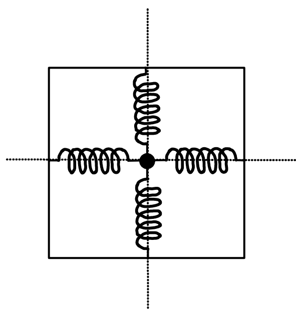
$$x_{max} = -\frac{3\dot{x}_\circ}{4\gamma} + \frac{3\dot{x}_\circ}{4\gamma} + \frac{2\dot{x}_\circ\dot{z}_\circ}{g} - \frac{8\dot{x}_\circ\gamma\dot{z}_\circ^2}{3g^2} + \text{ terms in } \gamma^2$$

$$x_{max} = \frac{2\dot{x}_\circ \dot{z}_\circ}{g} - \frac{8\dot{x}_\circ \dot{z}_\circ^2}{3g^2}\gamma + \ldots$$

For $\quad \dot{z}_\circ = v_\circ \sin\alpha$ and $2\dot{x}_\circ \dot{z}_\circ = v_\circ^2 \sin 2\alpha$:

$$x_{max} = \frac{v_\circ^2 \sin 2\alpha}{g} - \frac{4v_\circ^3 \sin 2\alpha \sin\alpha}{3g^2}\gamma + \ldots$$

4.16



$$x = A\cos(\omega t + \alpha), \qquad \dot{x} = -A\omega \sin(\omega t + \alpha)$$

from $\quad \dot{x}_\circ = 0, \qquad \alpha = 0$

from $\quad x_\circ = A, \qquad x = A\cos\omega t$

$$y = B\cos(\omega t + \beta), \qquad \dot{y} = -\omega B \sin(\omega t + \beta)$$

$$\frac{1}{2}kB^2 = \frac{1}{2}ky_\circ^2 + \frac{1}{2}m\dot{y}_\circ^2$$

with $\quad y_\circ = 4A,\ \dot{y}_\circ = 3\omega A \quad$ and $\omega = \sqrt{\frac{k}{m}}$:

$$B^2 = 16A^2 + \frac{1}{\omega^2}\left(9\omega^2 A^2\right) = 25A^2$$

$$B = 5A$$

Then $\quad 4A = 5A\cos\beta \quad$ and $\quad 3\omega A = -5\omega A \sin\beta$

$$\beta = \cos^{-1}\left(\frac{4}{5}\right) = \sin^{-1}\left(-\frac{3}{5}\right) = -36.9^\circ$$

$$y = 5A\cos\left(\omega t - 36.9^\circ\right)$$

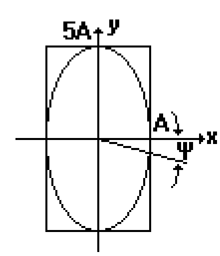
Since maximum x and y displacements are $\pm A$ and $\pm 5A$, respectively, the motion takes place entirely within a rectangle of dimension $2A$ and $10A$.

$$\Delta = \beta - \alpha = -36.9^\circ - 0 = -36.9^\circ$$

From eqn 4.4.15, $\tan 2\psi = \dfrac{2AB\cos\Delta}{A^2 - B^2}$

$$\tan 2\psi = \frac{(2A)(5A)\cos\left(-36.9^\circ\right)}{A^2 - (5A)^2} = \frac{10\left(\frac{4}{5}\right)}{-24} = -\frac{1}{3}$$

$$\psi = \frac{1}{2}\tan^{-1}\left(-\frac{1}{3}\right) = -9.2^\circ$$



4.17 $\quad m\ddot{x} = F_x = -\dfrac{\partial V}{\partial x} = -kx = -\pi^2 mx$

$$\dot{x} = A\cos\left(\sqrt{\frac{k}{m}}t + \alpha\right) = A\cos(\pi t + \alpha)$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} = -4\pi^2 mx$$

$$y = B\cos\left(2\pi t + \beta\right)$$

$$m\ddot{z} = -\frac{\partial V}{\partial z} = -9\pi^2 mz$$

$$z = C\cos\left(3\pi t + \gamma\right)$$

Since $x = y = z = 0$ at $t = 0$, $\alpha = \beta = \gamma = -\frac{\pi}{2}$

$$x = A\cos\left(\pi t - \frac{\pi}{2}\right) = A\sin \pi t$$

$$\dot{x} = A\pi \cos \pi t$$

Since $v_\circ^2 = \dot{x}_\circ^2 + \dot{y}_\circ^2 + \dot{z}_\circ^2$ and $\dot{x}_\circ = \dot{y}_\circ = \dot{z}_\circ$,

$$\dot{x}_\circ = \frac{v_\circ}{\sqrt{3}} = A\pi$$

$$A = \frac{v_\circ}{\pi\sqrt{3}}$$

$$x = \frac{v_\circ}{\pi\sqrt{3}}\sin \pi t$$

$$y = B\sin 2\pi t, \qquad \dot{y} = 2B\pi\cos 2\pi t$$

$$\dot{y}_\circ = \frac{v_\circ}{\sqrt{3}} = 2\pi B$$

$$B = \frac{v_\circ}{2\pi\sqrt{3}}$$

$$y = \frac{v_\circ}{2\pi\sqrt{3}}\sin 2\pi t$$

$$z = C\sin 3\pi t, \qquad \dot{z} = 3C\pi\cos 3\pi t$$

$$\dot{z}_\circ = \frac{v_\circ}{\sqrt{3}} = 3C\pi$$

$$C = \frac{v_\circ}{3\pi\sqrt{3}}$$

$$z = \frac{v_\circ}{3\pi\sqrt{3}}\sin 3\pi t$$

Since $\omega_x = \pi$, $\omega_y = 2\pi$, and $\omega_z = 3\pi$ the ball does retrace its path.

$$t_{\min} = \frac{2\pi n_1}{\omega_x} = \frac{2\pi n_2}{\omega_y} = \frac{2\pi n_3}{\omega_z}$$

The minimum time occurs at $n_1 = 1$, $n_2 = 2$, $n_3 = 3$.

$$t_{\min} = \frac{2\pi}{\pi} = 2$$

4.18 Equation 4.4.15 is $\tan 2\psi = \dfrac{2AB\cos\Delta}{A^2 - B^2}$

Transforming the coordinate axes xyz to the new axes $x'y'z'$ by a rotation about the z-axis through an angle ψ given, from Section 1.8:

$x' = x\cos\psi + y\sin\psi$, $\quad y' = -x\sin\psi + y\cos\psi$

or, $x = x'\cos\psi - y'\sin\psi$, and $\quad y = x'\sin\psi + y'\cos\psi$

From eqn. 4.4.10: $\dfrac{x^2}{A^2} - xy\dfrac{2\cos\Delta}{AB} + \dfrac{y^2}{B^2} = \sin^2\Delta$

Substituting:

$$\frac{1}{A^2}\left(x'^2\cos^2\psi - 2x'y'\cos\psi\sin\psi + y'^2\sin^2\psi\right)$$
$$-\frac{2\cos\Delta}{AB}\left[x'^2\cos\psi\sin\psi + x'y'\left(\cos^2\psi - \sin^2\psi\right) - y'^2\cos\psi\sin\psi\right]$$
$$+\frac{1}{B^2}\left(x'^2\sin^2\psi + 2x'y'\cos\psi\sin\psi + y'^2\cos^2\psi\right) = \sin^2\Delta$$

For x' to be a major or minor axis of the ellipse, the coefficient of $x'y'$ must vanish.

$$-\frac{2\cos\psi\sin\psi}{A^2} - \frac{2\cos\Delta}{AB}\left(\cos^2\psi - \sin^2\psi\right) + \frac{2\cos\psi\sin\psi}{B^2} = 0$$

From Appendix B, $2\cos\psi\sin\psi = \sin 2\psi$ and $\cos^2\psi - \sin^2\psi = \cos 2\psi$

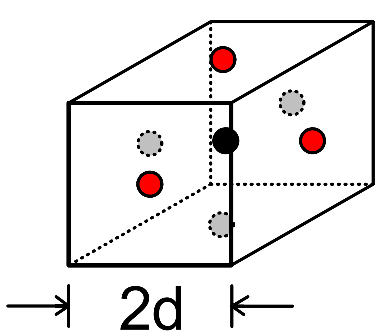
$$-\frac{\sin 2\psi}{A^2} - \frac{2\cos\Delta\cos 2\psi}{AB} + \frac{\sin 2\psi}{B^2} = 0$$

$$\tan 2\psi\left(\frac{1}{B^2} - \frac{1}{A^2}\right) = \frac{2\cos\Delta}{AB}$$

$$\tan 2\psi = \frac{2AB\cos\Delta}{A^2 - B^2}$$

4.19 Shown below is a face-centered cubic lattice. Each atom in the lattice is centered within a cube on whose 6 faces lies another adjacent atom. Thus each atom is surrounded by 6 nearest neighbors at a distance d. We neglect the influence of atoms that lie at further distances. Thus, the potential energy of the central atom can be approximated as

$$V = \sum_{i=1}^{6} c r_i^{-\alpha}$$



$$r_1 = \left[(d-x)^2 + y^2 + z^2\right]^{\frac{1}{2}}$$

$$r_1^{-\alpha} = \left(d^2 - 2dx + x^2 + y^2 + z^2\right)^{-\frac{\alpha}{2}} = d^{-\alpha}\left(1 - \frac{2x}{d} + \frac{x^2+y^2+z^2}{d^2}\right)^{-\frac{\alpha}{2}}$$

From Appendix D, $(1+x)^n = 1 + nx + \frac{1}{2}n(n-1)x^2 + \ldots$

$$r_1^{-\alpha} = d^{-\alpha}\left\{1 - \frac{\alpha}{2}\left(-\frac{2x}{d} + \frac{x^2+y^2+z^2}{d^2}\right) + \frac{1}{2}\left(-\frac{\alpha}{2}\right)\left(-\frac{\alpha}{2}-1\right)\right.$$

$$\left.\left[\left(-\frac{2x}{d}\right)^2 - 2\left(\frac{2x}{d}\right)\left(\frac{x^2+y^2+z^2}{d^2}\right) + \left(\frac{x^2+y^2+z^2}{d^2}\right)^2\right] + \text{terms in } \frac{x^3}{d^3}\right\}$$

$$r_1^{-\alpha} = d^{-\alpha}\left\{1 + \frac{\alpha x}{d} - \frac{\alpha}{2d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{4}\left(\frac{\alpha}{2}+1\right)\left[\frac{4x^2}{d^2} + \text{terms in } \frac{x^3}{d^3}\right]\right\}$$

$$r_1^{-\alpha} \approx d^{-\alpha}\left[1 + \frac{\alpha x}{d} - \frac{\alpha}{2d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{d^2}\left(\frac{\alpha}{2}+1\right)x^2\right]$$

$$r_2 = \left[(-d-x)^2 + y^2 + z^2\right]^{\frac{1}{2}} = \left[d^2 + 2dx + x^2 + y^2 + z^2\right]^{\frac{1}{2}}$$

$$r_2^{-\alpha} = d^{-\alpha}\left(1 + \frac{2x}{d} + \frac{x^2+y^2+z^2}{d^2}\right)^{-\frac{\alpha}{2}}$$

$$r_2^{-\alpha} \approx d^{-\alpha}\left[1 - \frac{\alpha x}{d} - \frac{\alpha}{2d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{d^2}\left(\frac{\alpha}{2}+1\right)x^2\right]$$

$$r_1^{-\alpha} + r_2^{-\alpha} \approx d^{-\alpha}\left[2 - \frac{\alpha}{d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{d^2}(\alpha+2)x^2\right]$$

Similarly:

$$r_3^{-\alpha} + r_4^{-\alpha} \approx d^{-\alpha}\left[2 - \frac{\alpha}{d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{d^2}(\alpha+2)y^2\right]$$

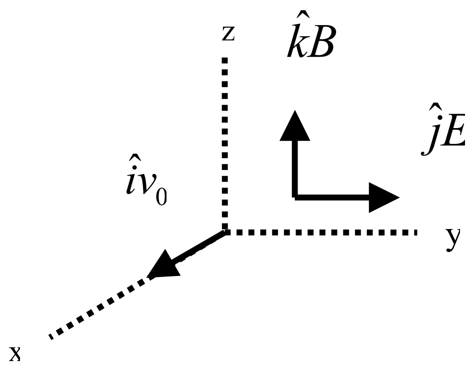
$$r_5^{-\alpha} + r_6^{-\alpha} \approx d^{-\alpha}\left[2 - \frac{\alpha}{d^2}\left(x^2+y^2+z^2\right) + \frac{\alpha}{d^2}(\alpha+2)z^2\right]$$

$$V \approx cd^{-\alpha}\left[6 - \frac{3\alpha}{d^2}\left(x^2+y^2+z^2\right) + \left(\frac{\alpha^2}{d^2} + \frac{2\alpha}{d^2}\right)\left(x^2+y^2+z^2\right)\right]$$

$$\approx 6cd^{-\alpha} + cd^{-\alpha-2}\left(\alpha^2-\alpha\right)\left(x^2+y^2+z^2\right)$$

$$V \approx A + B\left(x^2+y^2+z^2\right)$$

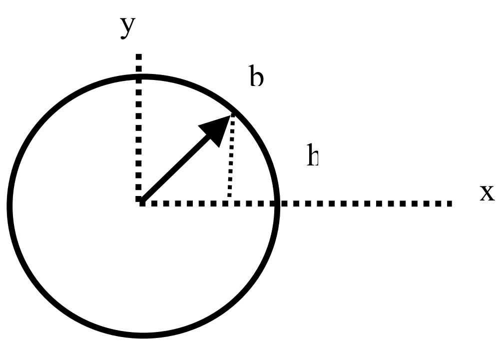
4.20



$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$
$$\vec{v} \times \vec{B} = \left(\hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z}\right) \times \hat{k}B = \hat{i}\dot{y}B - \hat{j}\dot{x}B$$
$$\vec{F} = \hat{i}q\dot{y}B + \hat{j}q\left(E - \dot{x}B\right)$$
$$m\ddot{x} = F_x = q\dot{y}B$$
$$\dot{x} - \dot{x}_\circ = \frac{qB}{m} y$$

$$m\ddot{y} = F_y = qE - q\dot{x}B = qE - qB\left(\dot{x}_\circ + \frac{qB}{m} y\right)$$
$$\ddot{y} = \frac{qE}{m} - \frac{qB\dot{x}_\circ}{m} - \left(\frac{qB}{m}\right)^2 y = -\frac{eE}{m} + \frac{eB\dot{x}_\circ}{m} - \left(\frac{eB}{m}\right)^2 y$$
$$\ddot{y} + \omega^2 y = -\frac{eE}{m} + \omega\dot{x}_\circ, \qquad \omega = \frac{eB}{m}$$
$$y = \frac{1}{\omega^2}\left(-\frac{eE}{m} + \omega x_\circ\right) + A\cos\left(\omega t + \theta_\circ\right)$$
$$\dot{y} = -A\omega\sin\left(\omega t + \theta_\circ\right)$$
$$\dot{y}_\circ = 0, \text{ so } \quad \theta_\circ = 0$$
$$y_\circ = 0, \text{ so } \quad A = -\frac{1}{\omega^2}\left(-\frac{eE}{m} + \omega\dot{x}_\circ\right)$$
$$y = a\left(1 - \cos\omega t\right), \qquad a = \frac{1}{\omega^2}\left(-\frac{eE}{m} + \omega\dot{x}_\circ\right)$$
$$\dot{x} = \dot{x}_\circ + \frac{qB}{m} y = \dot{x}_\circ - \omega y = \dot{x}_\circ - \omega a\left(1 - \cos\omega t\right)$$
$$\dot{x} = \left(\dot{x}_\circ - \omega a\right) + \omega a\cos\omega t$$
$$x = \left(\dot{x}_\circ - \omega a\right)t + a\sin\omega t$$
$$x = a\sin\omega t + bt, \qquad b = \dot{x}_\circ - \omega a$$
$$m\ddot{z} = F_z = 0$$
$$z = \dot{z}_\circ t + z_\circ = 0$$

4.21



$$\frac{1}{2}mv^2 + mqh = mg\frac{b}{2}$$
$$v^2 = g\left(b - 2h\right)$$
$$F_r = -\frac{mv^2}{b} = -mg\cos\theta + R$$

$$\cos\theta = \frac{h}{b}$$

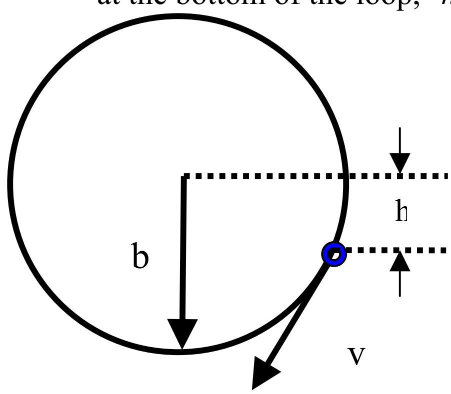
$$R = mg\frac{h}{b} - \frac{mv^2}{b} = \frac{mg}{b}\left[h - (b - 2h)\right] = \frac{mg}{b}(3h - b)$$

the particle leaves the side of the sphere when $R = 0$

$h = \frac{b}{3}$, i.e., $\frac{b}{3}$ above the central plane

4.22 $\frac{1}{2}mv^2 + mgh = 0$

at the bottom of the loop, $h = -b$



so $\frac{1}{2}mv^2 = mgb$,

$$v = \sqrt{2gb}$$

$$F_r = -mg + R = \frac{mv^2}{b}$$

$$R = mg + \frac{mv^2}{b} = mg + 2mg = 3mg$$

4.23 From the equation for the energy as a function of s in Example 4.6.2,

$$E = \frac{1}{2}m\dot{s}^2 + \frac{1}{2}\left(\frac{mg}{4A}\right)s^2,$$

s is undergoing harmonic motion with:

$$\omega = \sqrt{\frac{"k"}{m}} = \sqrt{\frac{g}{4A}} = \frac{1}{2}\sqrt{\frac{g}{A}}$$

Since $s = 4A\sin\phi$, ϕ increases by 2π radians during the time interval:

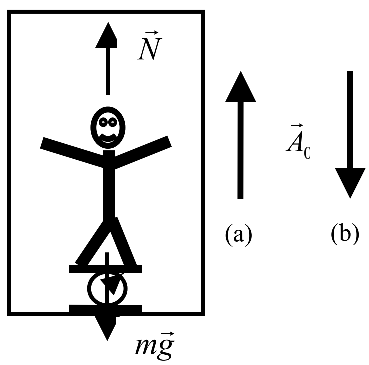
$$T' = \frac{2\pi}{\omega} = 2\pi\left(2\sqrt{\frac{A}{g}}\right)$$

For cycloidal motion, x and z are functions of 2ϕ so they undergo a complete cycle every time ϕ changes by π. Therefore, the period for the cycloidal motion is one-half the period for s.

$$T = \frac{1}{2}T' = 2\pi\sqrt{\frac{A}{g}}$$

CHAPTER 5
NONINERTIAL REFERENCE SYSTEMS

5.1 (a) The non-inertial observer believes that he is in equilibrium and that the net force acting on him is zero. The scale exerts an upward force, $\vec{N}$, whose value is equal to the scale reading --- the "weight," W', of the observer in the accelerated frame. Thus



$$\vec{N} + m\vec{g} - m\vec{A}_0 = 0$$

$$N - mg - mA_0 = N - mg - m\frac{g}{4} = N - \frac{5}{4}mg = 0$$

$$W' = N = \frac{5}{4}mg = \frac{5}{4}W$$

$$W' = 150lb.$$

(b) The acceleration is downward, in the same direction as $\vec{g}$

$$N - mg + m\left(\frac{g}{4}\right) = 0 \qquad W' = W - \frac{W}{4} = \frac{3}{4}W$$

$$W' = 90lb.$$

5.2 (a) $\vec{F}_{cent} = -m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}'\right)$

For $\vec{\omega}\perp\vec{r}'$, $\vec{F}_{cent} = m\omega^2 r'\,\hat{e}_{r'}$

$$\omega = 500\,s^{-1} = 1000\pi\,s^{-1}$$

$$\vec{F}_{cent} = 10^{-6}\times\left(1000\pi\right)^2\times 5\,\hat{e}_r = 5\pi^2 \text{ dynes outward}$$

(b) $$\frac{F_{cent}}{F_g} = \frac{m\omega^2 r'}{mg} = \frac{\left(1000\pi\right)^2 5}{980} = 5.04\times 10^4$$

5.3 $m\vec{g} + \vec{T} - m\vec{A}_\circ = 0$ (See Figure 5.1.2)

$$-mg\,\hat{j} + T\cos\theta\,\hat{j} + T\sin\theta\,\hat{i} - m\left(\frac{g}{10}\right)\hat{i} = 0$$

$T\cos\theta = mg$, and $T\sin\theta = \dfrac{mg}{10}$

$$\tan\theta = \frac{1}{10}, \qquad \theta = 5.71^\circ$$

$$T = \frac{mg}{\cos\theta} = 1.005mg$$

5.4 The non-inertial observer thinks that $\vec{g}'$ points downward in the direction of the hanging plumb bob… Thus

$$\vec{g}' = \vec{g} - \vec{A}_\circ = g\,\hat{j} - \frac{g}{10}\hat{i}$$

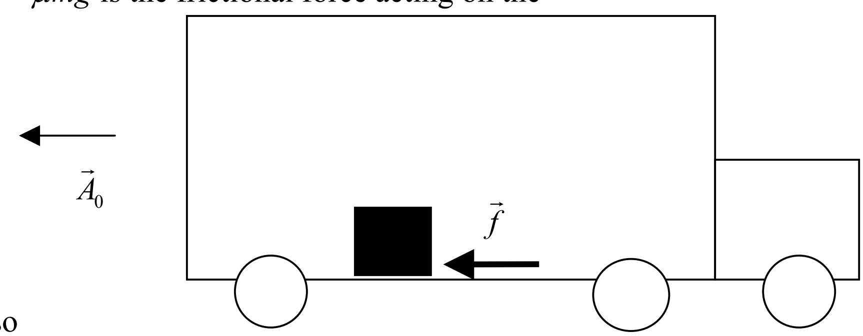
For small oscillations of a simple pendulum:

$$T = 2\pi\sqrt{\frac{1}{g'}}$$

$$g' = \sqrt{g^2 + \left(\frac{g}{10}\right)^2} = 1.005g$$

$$T = 2\pi\sqrt{\frac{1}{1.005g}} = 1.995\pi\sqrt{\frac{1}{g}}$$

5.5 (a) $f = -\mu mg$ is the frictional force acting on the



box, so

$\vec{f} - m\vec{A}_0 = m\vec{a}'$ ($\vec{a}'$ is the acceleration of the box relative to the truck. See Equation 5.1.4b.) Now, $\vec{f}$ the only real force acting horizontally, so the acceration relative to the road is

(b) $a = \dfrac{f}{m} = -\dfrac{\mu mg}{m} = -\mu g = -\dfrac{g}{3}$

(For + in the direction of the moving truck, the – indicates that friction opposes the forward sliding of the box.)

$A_\circ = -\dfrac{g}{2}$ (The truck is decelerating.)

from above, $ma - mA_0 = ma'$ so

(a) $a' = a - A_\circ = -\dfrac{g}{3} + \dfrac{g}{2} = \dfrac{g}{6}$

5.6 (a) $\vec{r} = \hat{i}\left(x_\circ + R\cos\Omega t\right) + \hat{j}R\sin\Omega t$

$\dot{\vec{r}} = -\hat{i}\Omega R\sin\Omega t + \hat{j}\Omega R\cos\Omega t$

$\dot{\vec{r}}\cdot\dot{\vec{r}} = v^2 = \Omega^2 R^2$ $\therefore \underline{v = \Omega R}$ circular motion of radius R

(b) $\dot{\vec{r}}' = \dot{\vec{r}} - \vec{\omega}\times\vec{r}'$ where $\vec{r}' = \hat{i}x' + \hat{j}y'$

$= -\hat{i}\Omega R\sin\Omega t + \hat{j}\Omega R\cos\Omega t - \omega\hat{k}\times\left(\hat{i}x' + \hat{j}y'\right)$

$$= -\hat{i}\Omega R\sin\Omega t + \hat{j}\Omega R\cos\Omega t - \hat{j}\omega x' + \hat{i}\omega y'$$

$$\dot{x}' = \omega y' - \Omega R\sin\Omega t$$

$$\dot{y}' = -\omega x' + \Omega R\cos\Omega t$$

(c) Let $u' = x' + iy'$ here $i = \sqrt{-1}$!

$$\dot{u}' = \dot{x}' + i\dot{y}' = \omega y' - \Omega R\sin\Omega t - i\omega x' + i\Omega R\cos\Omega t$$

$$i\omega u' = \quad = -\omega y' \qquad + i\omega x'$$

$$\therefore \dot{u}' + i\omega u' = -\Omega R\sin\Omega t + i\Omega R\cos\Omega t = i\Omega \mathrm{R}e^{i\Omega t}$$

Try a solution of the form

$$u' = Ae^{-i\omega t} + Be^{i\Omega t}$$

$$\dot{u}' = -i\omega Ae^{-i\omega t} + i\Omega Be^{i\Omega t}$$

$$i\omega u' = i\omega Ae^{-i\omega t} + i\omega Be^{i\Omega t}$$

$$\therefore \dot{u}' + i\omega u' = i\left(\omega + \Omega\right)Be^{i\omega t} \qquad \text{so} \quad B = \frac{\Omega R}{\omega + \Omega}$$

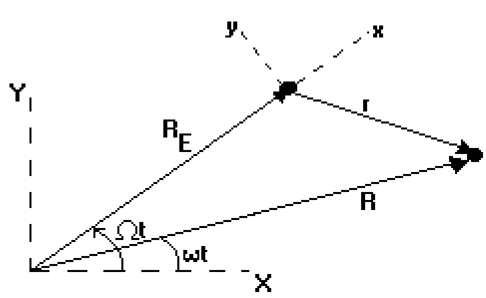
Also at $t = 0$ the coordinate systems coincide so

$$u' = A + B = x'(0) + iy'(0) = x_\circ + R$$

$$\therefore A = x_\circ + R - B = x_\circ + R - \frac{\Omega R}{\omega + \Omega} \qquad \text{so,} \quad A = x_\circ + \frac{\omega R}{\omega + \Omega}$$

Thus, $u' = \left[x_\circ + \dfrac{\omega R}{\omega + \Omega}\right]e^{-i\omega t} + \dfrac{\Omega R}{\omega + \Omega}e^{i\Omega t}$

5.7 The x, y frame of reference is attached to the Earth, but the x-axis always points away from the Sun. Thus, it rotates once every year relative to the fixed stars. The X,Y frame of reference is fixed inertial frame attached to the Sun.



(a) In the x, y rotating frame of reference

$$x(t) = R\cos(\Omega - \omega)t - R_\varepsilon$$

$$y(t) = -R\sin(\Omega - \omega)t$$

where R is the radius of the asteroid's orbit and R_E is the radius of the Earth's orbit. Ω is the angular frequency of the Earth's revolution about the Sun and ω is the angular frequency of the asteroid's orbit.

(b) $\dot{x}(t) = -(\Omega - \omega)R\sin(\Omega - \omega)t \to 0$ at $t = 0$

$\dot{y}(t) = -(\Omega - \omega)R\cos(\Omega - \omega)t \to -(\Omega - \omega)R$ at $t = 0$

(c) $\vec{a} = \vec{A} - \vec{A}_\varepsilon - \dot{\vec{\Omega}} \times \vec{r} - 2\vec{\Omega} \times \dot{\vec{r}} - \vec{\Omega} \times \vec{\Omega} \times \vec{r}$

Where $\vec{a}$ is the acceleration of the asteroid in the x, y frame of reference,

$\vec{A}, \vec{A}_\varepsilon$ are the accelerations of the asteroid and the Earth in the fixed, inertial frame of reference.

1^{st} : examine: $\vec{A} - \vec{A}_\varepsilon - \vec{\Omega}\times\vec{\Omega}\times\vec{r}$

$$= \vec{\omega}\times\vec{\omega}\times\vec{R} - \vec{\Omega}\times\vec{\Omega}\times\vec{R}_\varepsilon - \vec{\Omega}\times\vec{\Omega}\times\left(\vec{R}-\vec{R}_\varepsilon\right)$$

$$= \left(\vec{\omega}\times\vec{\omega} - \vec{\Omega}\times\vec{\Omega}\right)\times\vec{R} = -\left(\omega^2-\Omega^2\right)\vec{R} \qquad \text{note: } \vec{\omega} = \omega\hat{k}\,,\ \vec{\Omega} = \Omega\hat{k}$$

Thus:

$$\vec{a} = \left(\Omega^2-\omega^2\right)\vec{R} - 2\vec{\Omega}\times\vec{v}$$

Therefore:

$$\left(\hat{i}\ddot{x}+\hat{j}\ddot{y}\right) = \left(\Omega^2-\omega^2\right)\left[\hat{i}R\cos\left(\Omega-\omega\right)t - \hat{j}R\sin\left(\Omega-\omega\right)t\right] - 2\hat{j}\Omega\dot{x} + 2\hat{i}\Omega\dot{y}$$

Thus:

$$\ddot{x} = \left(\Omega^2-\omega^2\right)R\cos\left(\Omega-\omega\right)t + 2\Omega\dot{y}$$

$$\ddot{y} = -\left(\Omega^2-\omega^2\right)R\sin\left(\Omega-\omega\right)t - 2\Omega\dot{x}$$

Let $\ddot{x} = \left(\Omega-\omega\right)\dot{y}$ and $\ddot{y} = \left(\Omega-\omega\right)\dot{x}$

Then, we have

$$\dot{y} = \left(\Omega-\omega\right)R\cos\left(\Omega-\omega\right)t + \frac{2\Omega}{\left(\Omega-\omega\right)}\dot{y} \qquad \text{which reduces to}$$

$$\dot{y} = -\left(\Omega-\omega\right)R\cos\left(\Omega-\omega\right)t$$

Integrating … $\boxed{y = -R\sin\left(\Omega-\omega\right)t} \rightarrow 0$ at $t = 0$

Also,

$$-\dot{x}\left(\Omega-\omega\right) = -\left(\Omega^2-\omega^2\right)R\sin\left(\Omega-\omega\right)t - 2\Omega\dot{x}$$

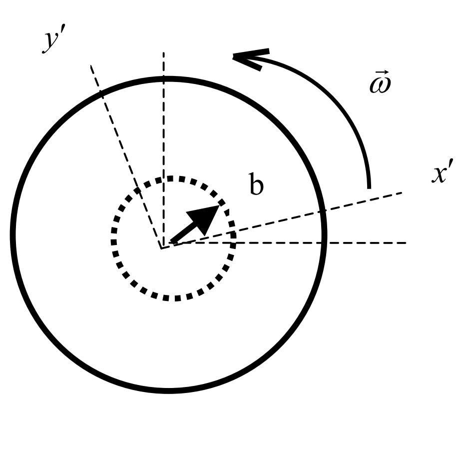
or $\dot{x} = \left(\Omega+\omega\right)R\sin\left(\Omega-\omega\right)t + 2\Omega\dot{x}$

$$\dot{x} = -\left(\Omega-\omega\right)R\sin\left(\Omega-\omega\right)t$$

Integrating … $x = R\cos\left(\Omega-\omega\right)t + const$

$\boxed{x = R\cos\left(\Omega-\omega\right)t - R_\varepsilon} \rightarrow R - R_\varepsilon$ at $t = 0$

5.8 Relative to a reference frame fixed to the turntable the cockroach travels at a constant speed v' in a circle. Thus



$$\vec{a}' = -\frac{v'^2}{b}\hat{e}_{r'}\,.$$

Since the center of the turntable is fixed.

$\vec{A}_\circ = 0$

The angular velocity, ω, of the turntable is constant, so

$$\vec{\omega} = \omega\hat{k}', \text{ with } \dot{\vec{\omega}} = 0$$
$$\vec{r}' = b\hat{e}_{r'}, \text{ so } \vec{\omega}\times(\vec{\omega}\times\vec{r}') = -b\omega^2\hat{e}_{r'}$$
$$\vec{v}' = v'\hat{e}_{\theta'}, \text{ so } \vec{\omega}\times\vec{v}' = -\omega v'\hat{e}_{r'}$$

From eqn 5.2.14, $\vec{a} = \vec{a}' + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times(\vec{\omega}\times\vec{r}')$ and putting in terms from above

$$a_{r'} = -\frac{v'^2}{b} - 2\omega v' - b\omega^2$$

For no slipping $\left|\vec{F}\right| \le \mu_s mg$, so $\left|\vec{a}\right| \le \mu_s g$

$$\frac{v'^2}{b} + 2\omega v' + b\omega^2 \le \mu_s g$$
$$v_m'^2 + 2\omega b v_m' + b^2\omega^2 - b\mu_s g = 0$$
$$v_m' = -\omega b \pm \sqrt{\omega^2 b^2 - b^2\omega^2 + b\mu_s g}$$

Since v' was defined positive, the +square root is used.

$$v_m' = -\omega b + \sqrt{b\mu_s g}$$

(b) $\vec{v}' = -v'\hat{e}_{\theta'}$

$$\vec{\omega}\times\vec{v}' = +\omega v'\hat{e}_{r'}$$
$$a_{r'} = -\frac{v'^2}{b} + 2\omega v' - b\omega^2$$
$$\frac{v'^2}{b} - 2\omega v' + b\omega^2 \le \mu_s g$$
$$v_m' = \omega b + \sqrt{b\mu_s g}$$

5.9 As in Example 5.2.2, $\vec{\omega} = \frac{V_\circ}{\rho}\hat{j}'$ and $\vec{A}_\circ = \frac{V_\circ^2}{\rho}\hat{i}'$

For the point at the front of the wheel:

$$\ddot{\vec{r}}' = \frac{V_\circ^2}{b}\hat{j}' \text{ and } \vec{v}' = -V_\circ\hat{k}'$$
$$\dot{\vec{\omega}} = 0$$
$$\vec{\omega}\times\vec{r}' = \frac{V_\circ}{\rho}\hat{k}'\times\left(-b\hat{j}'\right) = \frac{V_\circ b}{\rho}\hat{i}'$$
$$\vec{\omega}\times\left(\vec{\omega}\times\vec{r}'\right) = \frac{V_\circ}{\rho}\hat{k}'\times\frac{V_\circ b}{\rho}\hat{i}' = \frac{V_\circ^2 b}{\rho}\hat{j}'$$
$$\vec{\omega}\times\vec{v}' = \frac{V_\circ}{\rho}\hat{k}'\times\left(-V_\circ\hat{k}'\right) = 0$$
$$\vec{a} = \ddot{\vec{r}}' + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}'\right) + \vec{A}_\circ = \frac{V_\circ^2}{\rho}\hat{i}' + \left(\frac{V_\circ^2}{b} + \frac{V_\circ^2 b}{\rho^2}\right)\hat{j}'$$

5.10 (See Example 5.3.3)

$$m\omega^2 x' = m\ddot{x}'$$

$$x'(t) = Ae^{\omega t} + Be^{-\omega t}$$

$$\dot{x}'(t) = \omega Ae^{\omega t} - \omega Be^{-\omega t}$$

Boundary Conditions:

$$x^2(0) = \frac{l}{2} = A + B$$

$$\dot{x}'(0) = 0 = \omega(A - B)$$

$$\therefore A = \frac{l}{4} \qquad B = \frac{l}{4}$$

(a) $x'(t) = \frac{l}{2}\cosh\omega t \qquad \dot{x}'(t) = \omega\frac{l}{2}\sinh\omega t$

(b) $x'(T) = \frac{l}{2} + \frac{l}{2} = \frac{l}{2}\cosh\omega T$ when the bead reaches the end of the rod

$\therefore \cosh\omega T = 2$ or $T = \frac{1}{\omega}\cosh^{-1}2 = \frac{1.317}{\omega}$

(c) $\dot{x}'(T) = \omega\frac{l}{2}\sinh\omega T$

$$= \omega\frac{l}{2}\sinh\left[\cosh^{-1}2\right] = \omega\frac{l}{2}(1.732) = 0.866\omega l$$

$$\text{or } \omega\frac{l}{2}\left[\cosh^2\omega T - 1\right]^{\frac{1}{2}} = \sqrt{3}\frac{\omega l}{2} = 0.866\omega l$$

5.11 $\vec{v}' = 400\hat{j}'\text{ mph} = 586.67\hat{j}'\ \text{ft}\cdot\text{s}^{-1}$

$$\vec{\omega} = 7.27\times10^{-5}\left(\cos 41^\circ\,\hat{j}' + \sin 41^\circ\,\hat{k}'\right)\ \text{s}^{-1}$$

$$\vec{\omega}\times\vec{v}' = -\left(7.27\times10^{-5}\right)(586.67)\left(\sin 41^\circ\right)\hat{i}'\ \text{ft}\cdot\text{s}^{-2}$$

$$\frac{F_{cor}}{F_{grav}} = \frac{\left|-2m\vec{\omega}\times\vec{v}'\right|}{mg}$$

$$= \frac{2\left(7.27\times10^{-5}\right)(586.67)\left(\sin 41^\circ\right)}{32}$$

$$\frac{F_{cor}}{F_{grav}} = 0.0017$$

The Coriolis force is in the $-\vec{\omega}\times\vec{v}'$ direction, i.e., $+\hat{i}'$ or east.

5.12 (See Figure 5.4.3)

$$\vec{\omega} = \omega_{y'}\hat{j}' + \omega_{z'}\hat{k}'$$

$$\vec{v}' = v_{x'}\hat{i}' + v_{y'}\hat{j}'$$

$$\vec{\omega}\times\vec{v}' = -\omega_{z'}v_{y'}\hat{i}' + \omega_{z'}v_{x'}\hat{j}' - \omega_{y'}v_{x'}\hat{k}'$$

$$\left(\vec{\omega}\times\vec{v}'\right)_{horiz} = -\omega_{z'}v'_{y'}\hat{i}' + \omega_{z'}v_{x'}\hat{j}'$$

$$\left|\left(\vec{\omega}\times\vec{v}'\right)_{horiz}\right| = \left(\omega_{z'}^2 v_{y'}^2 + \omega_{z'}^2 v_{x'}^2\right)^{\frac{1}{2}} = \omega_{z'}\left(v_{x'}^2 + v_{y'}^2\right)^{\frac{1}{2}} = \omega_{z'}v'$$

$$\vec{F}_{cor} = -2m\vec{\omega}\times\vec{v}'$$

$\left|\left(\vec{F}_{cor}\right)_{horiz}\right| = 2m\left|\left(\vec{\omega}\times\vec{v}'\right)_{horiz}\right| = 2m\omega'_z v'$, independent of the direction of $\vec{v}'$.

5.13 From Example 5.4.1 …

$x'_h = \frac{1}{3}\omega\left(\frac{8h^3}{g}\right)^{\frac{1}{2}}\cos\lambda$ and $y'_h = 0$.

$$x'_h = \frac{1}{3}\left(7.27\times10^{-5}s^{-1}\right)\left(\frac{8\times1250^3 ft^3}{32 ft\cdot s^{-2}}\right)^{\frac{1}{2}}\cos 41^\circ$$

$x'_h = 0.404\ ft$ to the east.

5.14 From Example 5.4.2:

$\Delta \approx \frac{\omega H^2}{v_\circ}\left|\sin\lambda\right|$ is the deflection of the baseball towards the south since it was struck due East at Yankee Stadium at latitude $\lambda = 41^\circ N$ (problem 5.13). v_0 is the initial speed of the baseball whose range is H. From eqn 4.3.18b, without air resistance in an inertial reference frame, the horizontal range is …

$$H = \frac{v_\circ^2\sin 2\alpha}{g}$$

Solving for v_0 … $v_\circ = \left(\frac{gH}{\sin 2\alpha}\right)^{\frac{1}{2}}$

$$v_\circ = \left(\frac{32 ft\cdot s^{-2}\times 200 ft}{\sin 30^\circ}\right)^{\frac{1}{2}} = 113 ft\cdot s^{-1}$$

$$\therefore\quad \Delta \approx \frac{\left(7.27\times10^{-5}s^{-1}\right)\left(200^2 ft^2\right)}{113 ft\cdot s^{-1}}\sin 41^\circ = 0.0169 ft = 0.2\ in$$

A deflection of 0.2 inches should not cause the outfielder any difficulty.

5.15 Equation 5.2.10 gives the relationship between the time derivative of any vector in a fixed and rotating frame of reference. Thus …

$$\dddot{\vec{r}} = \left(\frac{d\vec{a}}{dt}\right)_{fixed} = \left(\frac{d\vec{a}}{dt}\right)_{rot} + \vec{\omega}\times\vec{a}$$

$$\vec{a} = \ddot{\vec{r}}' + \dot{\vec{\omega}}\times\vec{r}' + 2\vec{\omega}\times\dot{\vec{r}}' + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}'\right)$$

$$\left(\frac{d\vec{a}}{dt}\right)_{rot} = \dddot{\vec{r}}' + \ddot{\vec{\omega}}\times\vec{r}' + \dot{\vec{\omega}}\times\dot{\vec{r}}' + 2\dot{\vec{\omega}}\times\dot{\vec{r}}' + 2\vec{\omega}\times\ddot{\vec{r}}'$$

$$+\dot{\vec{\omega}}\times(\vec{\omega}\times\vec{r}') + \vec{\omega}\times\left(\dot{\vec{\omega}}\times\vec{r}'\right) + \vec{\omega}\times\left(\vec{\omega}\times\dot{\vec{r}}'\right)$$

$$\vec{\omega}\times\vec{a} = \vec{\omega}\times\ddot{\vec{r}}' + \vec{\omega}\times\left(\dot{\vec{\omega}}\times\vec{r}'\right) + 2\vec{\omega}\times\left(\vec{\omega}\times\dot{\vec{r}}'\right) + \vec{\omega}\times\left[\vec{\omega}\times(\vec{\omega}\times\vec{r}')\right]$$

Now $(\vec{\omega}\times\vec{r}')$ is $\perp$ to $\vec{\omega}$ and $\vec{r}'$. Let this define a direction $\hat{n}$:

$$\vec{\omega}\times\vec{r}' = |\vec{\omega}\times\vec{r}'|\hat{n}$$

Since $\vec{\omega}\perp\hat{n}$, $\omega\times(\vec{\omega}\times\vec{r}')$ is in the plane defined by $\vec{\omega}$ and $\vec{r}'$ and

$$|\vec{\omega}\times(\vec{\omega}\times\vec{r}')| = |\vec{\omega}\times\hat{n}||\vec{\omega}\times\vec{r}'| = \omega|\vec{\omega}\times\vec{r}'|.$$

Since $\vec{\omega}\perp\vec{\omega}\times(\vec{\omega}\times\vec{r}')$ …

$$\left|\vec{\omega}\times\left[\vec{\omega}\times(\vec{\omega}\times\vec{r}')\right]\right| = \omega^2|\vec{\omega}\times\vec{r}'|$$

And $\vec{\omega}\times\left[\vec{\omega}\times(\vec{\omega}\times\vec{r}')\right]$ is in the direction $-\hat{n}$

Thus $\vec{\omega}\times\left[\vec{\omega}\times(\vec{\omega}\times\vec{r}')\right] = -\omega^2(\vec{\omega}\times\vec{r}')$

$$\vec{\omega}\times\vec{a} = \vec{\omega}\times\ddot{\vec{r}}' + \vec{\omega}\times\left(\dot{\vec{\omega}}\times\vec{r}'\right) + 2\omega\times\left(\vec{\omega}\times\dot{\vec{r}}'\right) - \omega^2(\vec{\omega}\times\vec{r}')$$

$$\dddot{\vec{r}} = \dddot{\vec{r}}' + \ddot{\vec{\omega}}\times\vec{r}' + 3\dot{\vec{\omega}}\times\dot{\vec{r}}' + 3\vec{\omega}\times\ddot{\vec{r}}' + \dot{\vec{\omega}}\times(\vec{\omega}\times\vec{r}')$$

$$+2\vec{\omega}\times\left(\dot{\vec{\omega}}\times\vec{r}'\right) + 3\vec{\omega}\times\left(\vec{\omega}\times\dot{\vec{r}}'\right) - \omega^2(\vec{\omega}\times\vec{r}')$$

5.16 With $x'_\circ = y'_\circ = z'_\circ = \dot{x}'_\circ = \dot{y}'_\circ = 0$, and $\dot{z}'_\circ = v'_\circ$ Equations 5.4.15a – 5.4.15c become:

$$x'(t) = \frac{1}{3}\omega g t^3\cos\lambda - \omega t^2 v'_\circ\cos\lambda$$

$$y'(t) = 0$$

$$z'(t) = -\frac{1}{2}gt^2 + v'_\circ t$$

When the bullet hits the ground $z'(t) = 0$, so

$$t = \frac{2v'_\circ}{g}$$

$$x' = \frac{1}{3}\omega g\left(\frac{8v'^3_\circ}{g^3}\right)\cos\lambda - \omega\left(\frac{4v'^2_\circ}{g^2}\right)v'_\circ\cos\lambda$$

$$x' = -\frac{4\omega v'^3_\circ}{3g^2}\cos\lambda$$

x' is negative and therefore is the distance the bullet lands to the *west*.

5.17 With $x'_\circ = y'_\circ = z'_\circ = 0$ and $\dot{x}'_\circ = v_\circ\cos\alpha$ $\dot{y}'_\circ = 0$ $\dot{z}'_\circ = v_\circ\sin\alpha$
we can solve equation 5.4.15c to find the time it takes the projectile to strike the ground …

$$z'(t) = -\frac{1}{2}gt^2 + v'_\circ t\sin\alpha + \omega v_\circ t^2 \cos\alpha\cos\lambda = 0$$

or $$t = \frac{2v'_\circ \sin\alpha}{g - 2\omega v'_\circ \cos\alpha\cos\lambda} \approx \frac{2v'_\circ \sin\alpha}{g}$$

We have ignored the second term in the denominator—since $v'_\circ$ would have to be impossibly large for the value of that term to approach the magnitude g

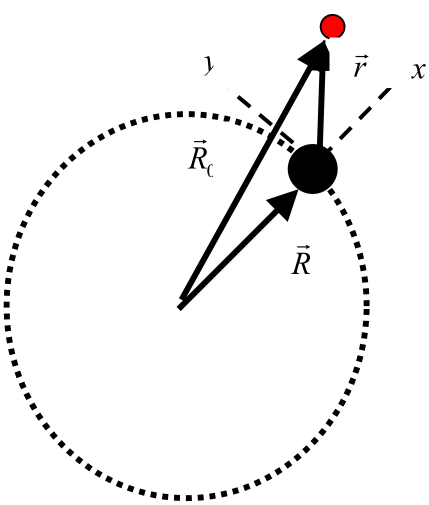
For example, for $\lambda = 41^o$ *and* $\alpha = 45^o$ $\quad g - 2\omega v'_\circ \cos\alpha\cos\lambda \approx g - \omega v'_\circ$

or $$v'_\circ \approx \frac{g}{\omega} \approx 144\frac{km}{s}!$$

Substituting t into equation 5.4.15b to find the lateral deflection gives

$$\dot{y}(t) = -[\omega v_\circ \cos\alpha\sin\lambda]t^2 = -\frac{4\omega v_\circ'^3}{g^2}\sin\lambda\sin^2\alpha\cos\alpha$$

5.18 Let …



$\vec{a}_\circ$ = acceleration of object relative to Earth

$\vec{\omega}_\circ = \omega_\circ\hat{k}$ = its angular speed

$\vec{A}_\circ$ = acceleration of satellite

$\vec{\omega} = \omega\hat{k}$ = its angular speed

$$\vec{a}_\circ = \vec{a} + 2\vec{\omega}\times\vec{v} + \vec{\omega}\times(\vec{\omega}\times\vec{r}) + \vec{A}_\circ \quad \text{(Equation 5.2.14)}$$

$$\vec{a} = \vec{a}_\circ - \vec{A}_\circ - 2\vec{\omega}\times\vec{v} - \vec{\omega}\times\vec{\omega}\times\vec{r}$$

As in problem 5.7 Evaluate the term …

$$\vec{\Delta}_a = \vec{a}_\circ - \vec{A}_\circ - \vec{\omega}\times\vec{\omega}\times\vec{r} = \vec{\omega}_\circ\times\vec{\omega}_\circ\times\vec{R}_\circ - \vec{\omega}\times\vec{\omega}\times\vec{R} - \vec{\omega}\times\vec{\omega}\times(\vec{R}_\circ - \vec{R})$$

$$\vec{\Delta}_a = (\omega_\circ^2 - \omega^2)\vec{R}_\circ$$

… given the condition that $\quad \omega_\circ^2 R_\circ^3 = \omega^2 R^3$

$$\vec{\Delta}_a = -\omega^2\vec{R}_\circ\left(1 - \frac{R^3}{R_\circ^3}\right)$$

but $$\vec{R}_\circ\cdot\vec{R}_\circ = (\vec{R}+\vec{r})\cdot(\vec{R}+\vec{r}) = R^2 + r^2 + 2rR\cos\theta$$

Letting $x = \cos\theta$ $\qquad \vec{R}_\circ\cdot\vec{R}_\circ = R^2 + r^2 + 2Rx \approx R^2\left(1 + \frac{2x}{R}\right)$ for small r

and $$R_\circ^3 \approx R^3\left(1+\frac{2x}{R}\right)^{\frac{3}{2}} \quad \text{or} \quad \frac{R^3}{R_\circ^3} = \left(1+\frac{2x}{R}\right)^{-\frac{3}{2}}$$

$$\vec{\Delta}_a = -\omega^2\vec{R}_\circ\left[1 - \left(1+\frac{2x}{R}\right)^{-\frac{3}{2}}\right] \approx -3\omega^2 x\frac{\vec{R}_\circ}{R} \approx -3\omega^2 x\hat{i} \text{ for small } r$$

Hence: $\vec{a} = \vec{\Delta}_a - 2\vec{\omega}\times\vec{v} = -3\omega^2 x\hat{i} - 2\omega\hat{k}\times\left(\hat{i}\dot{x} + \hat{j}\dot{y}\right)$

$$\vec{a} = \hat{i}\ddot{x} + \hat{j}\ddot{y} = -3\omega^2 x\hat{i} + 2\omega\dot{y}\hat{i} - 2\omega\dot{x}\hat{j}$$

So $\ddot{x} - 2\omega\dot{y} - 3\omega^2 x = 0$

$$\ddot{y} + 2\omega\dot{x} = 0$$

5.19 $m\ddot{\vec{r}} = q\vec{E} + q\left(\vec{v}\times\vec{B}\right)$

Equation 5.2.14 $\quad \ddot{\vec{r}} = \ddot{\vec{r}}' + \dot{\vec{\omega}}\times\vec{r}' + 2\vec{\omega}\times\vec{v}' + \vec{\omega}\times\left(\vec{\omega}\times\vec{r}'\right)$

Equation 5.2.13 $\quad \vec{v} = \vec{v}' + \vec{\omega}\times\vec{r}'$

$\vec{\omega} = -\dfrac{q}{2m}\vec{B}$ so $\dot{\vec{\omega}} = 0$

$$m\ddot{\vec{r}}' - q\left(\vec{B}\times\vec{v}'\right) - \frac{q}{2}\vec{B}\times\left(\vec{\omega}\times\vec{r}'\right) = q\vec{E} + q\left[\left(\vec{v}' + \vec{\omega}\times\vec{r}'\right)\times\vec{B}\right]$$

$$m\ddot{\vec{r}}' + q\left(\vec{v}'\times\vec{B}\right) + \frac{q}{2}\left(\vec{\omega}\times\vec{r}'\right)\times\vec{B} = q\vec{E} + q\left(\vec{v}'\times\vec{B}\right) + q\left(\vec{\omega}\times\vec{r}'\right)\times\vec{B}$$

$$m\ddot{\vec{r}}' = q\vec{E} + \frac{q}{2}\left(\vec{\omega}\times\vec{r}'\right)\times\vec{B}$$

$$\left|\frac{q}{2}\left(\vec{\omega}\times\vec{r}'\right)\times\vec{B}\right| = \frac{q}{2}\left(\frac{qB}{2m}\right)\left(r'\right)\left(\sin\theta\right)\left(B\right) \propto B^2$$

Neglecting terms in B^2, $m\ddot{\vec{r}}' = q\vec{E}$

5.20 For $x' = x\cos\omega' t + y\sin\omega' t$

$$y' = -x\sin\omega' t + y\cos\omega' t$$

$$\dot{x}' = \dot{x}\cos\omega' t - x\omega'\sin\omega' t + \dot{y}\sin\omega' t + y\omega'\cos\omega' t$$

$$\dot{y}' = -\dot{x}\sin\omega' t - x\omega'\cos\omega' t + \dot{y}\cos\omega' t - y\omega'\sin\omega' t$$

$$\dot{x}' = \dot{x}\cos\omega' t + \dot{y}\sin\omega' t + \omega' y'$$

$$\dot{y}' = -\dot{x}\sin\omega' t + \dot{y}\cos\omega' t - \omega' x'$$

$$\ddot{x}' = \ddot{x}\cos\omega' t - \dot{x}\omega'\sin\omega' t + \ddot{y}\sin\omega' t + \dot{y}\omega'\cos\omega' t + \omega'\dot{y}'$$

$$\ddot{y}' = -\ddot{x}\sin\omega' t - \dot{x}\omega'\cos\omega' t + \ddot{y}\cos\omega' t - \dot{y}\omega'\sin\omega' t - \omega'\dot{x}'$$

$$\ddot{x}' = \ddot{x}\cos\omega' t + \ddot{y}\sin\omega' t + 2\omega'\dot{y}' + \omega'^2 x'$$

$$\ddot{y}' = -\ddot{x}\sin\omega' t + \ddot{y}\cos\omega' t - 2\omega'\dot{x}' + \omega'^2 y'$$

Substituting into Eqns 5.6.3:

$$\ddot{x}\cos\omega' t + \ddot{y}\sin\omega' t + 2\omega'\dot{y}' + \omega'^2 x'$$

$$= -\frac{g}{l}x\cos\omega' t - \frac{g}{l}y\sin\omega' t + 2\omega'\dot{y}'$$

$$\ddot{x}\sin\omega' t + \ddot{y}\cos\omega' t - 2\omega'\dot{x}' + \omega'^2 y'$$

$$= +\frac{g}{l}x\sin\omega' t - \frac{g}{l}y\cos\omega' t - 2\omega'\dot{x}'$$

Collecting terms and neglecting terms in ω'^2:

$$\left(\ddot{x}+\frac{g}{l}x\right)\cos\omega't+\left(\ddot{y}+\frac{g}{l}y\right)\sin\omega't=0$$

$$\left(\ddot{x}+\frac{g}{l}x\right)\sin\omega't-\left(\ddot{y}+\frac{g}{l}y\right)\cos\omega't=0$$

5.21 $T=\dfrac{24}{\sin\lambda}$ hours

$$T=\frac{24}{\sin 19^\circ}=73.7 \text{ hours}$$

5.22 Choose a coordinate system with the origin at the center of the wheel, the x' and y' axes pointing toward fixed points on the rim of the wheel, and the z' axis pointing toward the center of curvature of the track. Take the initial position of the x' axis to be horizontal in the $-\vec{V}_\circ$ direction, so the initial position of the y' axis is vertical.

The bicycle wheel is rotating with angular velocity $\dfrac{V_\circ}{b}$ about its axis, so …

$$\vec{\omega}_l=\hat{k}'\frac{V_\circ}{b}$$

A unit vector in the vertical direction is:

$$\hat{n}=\hat{i}'\sin\frac{V_\circ t}{b}+\hat{j}'\cos\frac{V_\circ t}{b}$$

At the instant a point on the rim of the wheel reaches its highest point:

$$\vec{r}'=b\hat{n}=b\left(\hat{i}'\sin\frac{V_\circ t}{b}+\hat{j}'\cos\frac{V_\circ t}{b}\right)$$

Since the coordinate system is moving with the wheel, every point on the rim is fixed in that coordinate system.

$$\dot{\vec{r}}'=0 \quad \text{and} \quad \ddot{\vec{r}}'=0$$

The $x'y'z'$ coordinate system also rotates as the bicycle wheel completes a circle around the track:

$$\vec{\omega}_2=\hat{n}\frac{V_\circ}{\rho}=\frac{V_\circ}{\rho}\left(\hat{i}'\sin\frac{V_\circ t}{b}+\hat{j}'\cos\frac{V_\circ t}{b}\right)$$

The total rotation of the coordinate axes is represented by:

$$\vec{\omega}=\vec{\omega}_1+\vec{\omega}_2=\frac{V_\circ}{\rho}\left(\hat{i}'\sin\frac{V_\circ t}{b}+\hat{j}'\cos\frac{V_\circ t}{b}\right)+\hat{k}'\frac{V_\circ}{b}$$

$$\dot{\vec{\omega}}=\frac{V_\circ^2}{\rho b}\left(\hat{i}'\cos\frac{V_\circ t}{b}-\hat{j}'\sin\frac{V_\circ t}{b}\right)$$

$$\dot{\vec{\omega}}\times\vec{r}'=\frac{V_\circ^2}{\rho}\left(\hat{k}'\cos^2\frac{V_\circ t}{b}+\hat{k}'\sin^2\frac{V_\circ t}{b}\right)=\frac{V_\circ^2}{\rho}\hat{k}'$$

$$\vec{\omega}\times\dot{\vec{r}}'=0$$
$$\vec{\omega}\times\vec{r}'=\frac{V_\circ b}{\rho}\left(\hat{k}'\sin\frac{V_\circ t}{b}\cos\frac{V_\circ t}{b}-\hat{k}'\sin\frac{V_\circ t}{b}\cos\frac{V_\circ t}{b}\right)+V_\circ\left(\hat{j}'\sin\frac{V_\circ t}{b}-\hat{i}'\cos\frac{V_\circ t}{b}\right)$$
$$\vec{\omega}\times\left(\vec{\omega}\times\vec{r}'\right)=\frac{V_\circ^2}{\rho}\left(\hat{k}'\sin^2\frac{V_\circ t}{b}+\hat{k}'\cos^2\frac{V_\circ t}{b}\right)+\frac{V_\circ^2}{b}\left(-\hat{i}'\sin\frac{V_\circ t}{b}-\hat{j}'\cos\frac{V_\circ t}{b}\right)$$
$$\vec{\omega}\times\left(\vec{\omega}\times\vec{r}'\right)=\hat{k}'\frac{V_\circ^2}{\rho}-\hat{n}\frac{V_\circ^2}{b}$$

Since the origin of the coordinate system is traveling in a circle of radius ρ:

$$\vec{A}_\circ=\hat{k}'\frac{V_\circ^2}{\rho}$$
$$\ddot{\vec{r}}=\ddot{\vec{r}}'+\dot{\vec{\omega}}\times\vec{r}'+2\vec{\omega}\times\dot{\vec{r}}'++\vec{\omega}\times\left(\vec{\omega}\times\vec{r}'\right)+\vec{A}_\circ$$
$$\ddot{\vec{r}}=\hat{k}'\frac{V_\circ^2}{\rho}+\hat{k}'\frac{V_\circ^2}{\rho}-\hat{n}\frac{V_\circ^2}{b}+\hat{k}'\frac{V_\circ^2}{\rho}$$
$$\ddot{\vec{r}}=3\frac{V_\circ^2}{\rho}\hat{k}'-\frac{V_\circ^2}{b}\hat{n}$$

With appropriate change in coordinate notation, this is the same result as obtained in Example 5.2.2.

CHAPTER 6
GRAVITATIONAL AND CENTRAL FORCES

6.1 $m = \rho V = \rho \frac{4}{3}\pi r_s^3$

$$r_s = \left(\frac{3m}{4\pi\rho}\right)^{\frac{1}{3}}$$

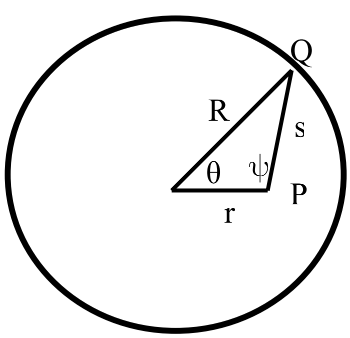
$$F = \frac{Gmm}{(2r_s)^2} = \frac{Gm^2}{4}\left(\frac{4\pi\rho}{3m}\right)^{\frac{2}{3}} = \frac{G}{4}\left(\frac{4\pi\rho}{3}\right)^{\frac{2}{3}} m^{\frac{4}{3}}$$

$$\frac{F}{W} = \frac{F}{mg} = \frac{Gm^2}{4g}\left(\frac{4\pi\rho}{3}\right)^{\frac{2}{3}} m^{\frac{1}{3}}$$

$$\frac{F}{W} = \frac{6.672\times10^{-11} N\cdot m^2\cdot kg^{-2}}{4\times9.8\, m\cdot s^{-2}}\left(\frac{4\pi\times11.35\, g\cdot cm^{-3}}{3}\times\frac{1kg}{10^3 g}\times\frac{10^6 cm^3}{1m^3}\right)^{\frac{2}{3}}\times(1kg)^{\frac{1}{3}}$$

$$\frac{F}{W} = 2.23\times10^{-9}$$

6.2 (a) The derivation of the force is identical to that in Section 6.2 except here r < R. This means that in the last integral equation, (6.2.7), the limits on u are R – r to R + r.



$$F = \frac{GmM}{4Rr^2}\int_{R-r}^{R+r}\left(1+\frac{r^2-R^2}{s^2}\right)ds$$

$$= \frac{GmM}{4Rr^2}\left[R+r-(R-r)+\frac{R^2-r^2}{R+r}-\frac{R^2-r^2}{R-r}\right]$$

$$F = \frac{GmM}{4Rr^2}\left[2r+R-r-(R+r)\right] = 0$$

(b) Again the derivation of the gravitational potential energy is identical to that in Example 6.7.1, except that the limits of integration on s are $(R-r)\rightarrow(R+r)$.

$$\phi = -G\frac{2\pi\rho R^2}{rR}\int_{R-r}^{R+r} ds$$

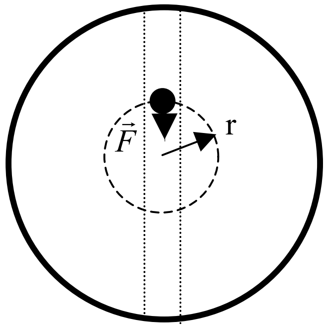
$$= -G\frac{2\pi\rho R^2}{rR}\left[R+r-(R-r)\right]$$

$$\phi = -G\frac{4\pi R^2\rho}{R} = -G\frac{M}{R}$$

For $r < R$, ϕ is independent of r. It is constant inside the spherical shell.

6.3 $\vec{F} = -\frac{GMm}{r^2}\hat{e}_r$

The gravitational force on the particle is due only to the mass of the earth that is inside the particle's instantaneous displacement from the center of the earth, r. The net effect of the mass of the earth outside r is zero (See Problem 6.2).



$$M = \frac{4}{3}\pi r^3 \rho$$
$$\vec{F} = -\frac{4}{3} G\pi\rho m r \hat{e}_r = -kr\hat{e}_r$$

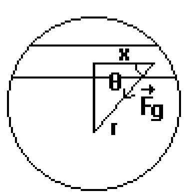
The force is a linear restoring force and induces simple harmonic motion.

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{3}{4G\pi\rho}}$$

The period depends on the earth's density but is independent of its size.
At the surface of the earth,

$$mg = \frac{GMm}{R_e^2} = \frac{Gm}{R_e^2}\cdot\frac{4}{3}\pi R_e^3 \rho$$
$$\frac{4G\pi\rho}{3} = \frac{g}{R_e}$$
$$T = 2\pi\sqrt{\frac{R_e}{g}} = 2\pi\sqrt{\frac{6.38\times10^6 m}{9.8 m\cdot s^{-2}}}\times\frac{1hr}{3600s} \approx 1.4hr$$

6.4



$$\vec{F}_g = -\frac{GMm}{r^2}\hat{e}_r \text{, where } M = \frac{4}{3}\pi r^3 \rho$$

The component of the gravitational force perpendicular to the tube is balanced by the normal force arising from the side of the tube. The component of force along the tube is

$$F_x = F_g \cos\theta$$

The net force on the particle is …

$$\vec{F} = -\hat{i}\frac{4}{3}G\pi\rho m r\cos\theta$$
$$r\cos\theta = x$$
$$\vec{F} = -\hat{i}\frac{4}{3}G\pi\rho m x = -\hat{i}kx$$

As in problem 6.3, the motion is simple harmonic with a period of 1.4 hours.

6.5 $$\frac{GMm}{r^2}=\frac{mv^2}{r} \quad \text{so} \quad v^2=\frac{GM}{r}$$

for a circular orbit r, v is constant.

$$T=\frac{2\pi r}{v}$$

$$T^2=\frac{4\pi^2 r^2}{v^2}=\frac{4\pi^2}{GM}r^3 \propto r^3$$

6.6 (a) $T=\frac{2\pi r}{v}$

From Example 6.5.3, the speed of a satellite in circular orbit is …

$$v=\left(\frac{gR_e^2}{r}\right)^{\frac{1}{2}}$$

$$T=\frac{2\pi r^{\frac{3}{2}}}{g^{\frac{1}{2}}R_e}$$

$$r=\left(\frac{T^2 gR_e^2}{4\pi^2}\right)^{\frac{1}{3}}$$

$$\frac{r}{R_e}=\left(\frac{T^2 g}{4\pi^2 R_e}\right)^{\frac{1}{3}}=\left(\frac{24^2\,hr^2\times 3600^2\,s^2\cdot hr^{-2}\times 9.8\,m\cdot s^{-2}}{4\pi^2 6.38\times 10^6\,m}\right)^{\frac{1}{3}}$$

$$\frac{r}{R_e}=6.62\approx 7$$

(b) $$T=\frac{2\pi r^{\frac{3}{2}}}{g^{\frac{1}{2}}R_e}=\frac{2\pi\left(60R_e\right)^{\frac{3}{2}}}{g^{\frac{1}{2}}R_e}=2\pi\sqrt{\frac{60^3 R_e}{g}}$$

$$=2\pi\left(\frac{60^3\times 6.38\times 10^6\,m}{9.8\,m\cdot s^{-2}\times 3600^2\,s^2\cdot hr^{-2}\times 24^2\,hr^2\cdot day^{-2}}\right)^{\frac{1}{2}}$$

$$T=27.27\,day\approx 27\,day$$

6.7 From Example 6.5.3, the speed of a satellite in a circular orbit just above the earth's surface is …

$$v=\sqrt{gR_e}$$

$$T=\frac{2\pi R_e}{v}=2\pi\sqrt{\frac{R_e}{g}}$$

This is the same expression as derived in Problem 6.3 for a particle dropped into a hole drilled through the earth. $T \approx 1.4$ hours.

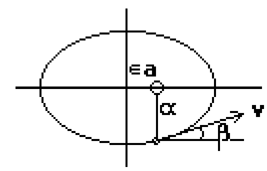
6.8 The Earth's orbit about the Sun is counter-clockwise as seen from, say, the north star. It's coordinates on approach at the latus rectum are $(x, y) = (\varepsilon a, -\alpha)$.

The easiest way to solve this problem is to note that $\varepsilon = \frac{1}{60}$ is small. The orbit is almost circular!

$$\therefore \frac{GM_S m}{r^2} = \frac{mv^2}{r} \text{ and } v^2 = \frac{GM_S}{r}$$

with $r = \alpha \approx a \approx b$ when $\varepsilon \approx 0$

$$v \approx \left(\frac{GM_S}{\alpha}\right) = 3 \cdot 10^4 \frac{m}{s}$$



More exactly

$$|\vec{r} \times \vec{v}| = \alpha v \cos\beta = l \text{, but } \alpha = \frac{ml^2}{k} \qquad \text{(equation 6.5.19)}$$

Since $k = GM_S m$ $\qquad \alpha = \frac{l^2}{GM_S}$, $\qquad$ hence $l = \alpha v \cos\beta = (\alpha GM_S)^{\frac{1}{2}}$

Or $\quad v = \left(\frac{GM_S}{\alpha}\right)^{\frac{1}{2}} \frac{1}{\cos\beta}$

The angle β can be calculated as follows:

$$\frac{y^2}{b^2} + \frac{x^2}{a^2} = 1 \qquad \text{(see appendix C)}$$

$$\therefore \frac{dy}{dx} = -\frac{b^2}{a^2}\frac{x}{y} \qquad \text{and at } (x, y) = (\varepsilon a, -\alpha)$$

so $\quad \frac{dy}{dx} = \frac{b^2}{a^2}\frac{\varepsilon a}{\alpha} = \frac{b^2}{a^2}\frac{\varepsilon}{(1-\varepsilon^2)} = \varepsilon$ since $1 - \varepsilon^2 = \frac{b^2}{a^2}$

here $\quad \frac{dy}{dx} = \tan\beta = \varepsilon \qquad$ or $\beta \approx \varepsilon$ (small ε)

and $\quad v = \left(\frac{GM_S}{\alpha}\right)^{\frac{1}{2}} \frac{1}{\cos\varepsilon} \approx \left(\frac{GM_S}{\alpha}\right)^{\frac{1}{2}}$ as before.

6.9 $\quad F(r) = F_s + F_d$

$$F_s = -\frac{GMm}{r^2}$$

$$F_d = -\frac{GM_d m}{r^2}$$

The net effect of the dust outside the planet's radius is zero (Problem 6.2). The mass of the dust inside the planet's radius is:

$$M_d = \frac{4}{3}\pi r^3 \rho$$
$$F(r) = -\frac{GMm}{r^2} - \frac{4}{3}\pi\rho m G r$$

6.10 $u = \frac{1}{r} = \frac{1}{r_\circ} e^{-k\theta}$

$$\frac{du}{d\theta} = -\frac{k}{r_\circ} e^{-k\theta}$$
$$\frac{d^2u}{d\theta^2} = \frac{k^2}{r_\circ} e^{-k\theta} = k^2 u$$

From equation 6.5.10 …

$$\frac{d^2u}{du^2} + u = k^2 u + u = -\frac{1}{ml^2u^2} f\left(u^{-1}\right)$$
$$f\left(u^{-1}\right) = -ml^2\left(k^2+1\right)u^3$$
$$f(r) = -\frac{ml^2\left(k^2+1\right)}{r^3}$$

The force varies as the inverse cube of r.

From equation 6.5.4, $r^2\dot{\theta} = l$

$$\frac{d\theta}{dt} = \frac{l}{r_\circ^2} e^{-2k\theta}$$
$$e^{2k\theta} d\theta = \frac{l}{r_\circ^2} dt$$
$$\frac{1}{2k} e^{2k\theta} = \frac{lt}{r_\circ^2} + C$$
$$\theta = \frac{1}{2k} \ln\left(\frac{2klt}{r_\circ^2} + C'\right)$$

θ varies logarithmically with t.

6.11 $f(r) = \frac{k}{r^3} = ku^3$

From equation 6.5.10 …

$$\frac{d^2u}{d\theta^2} + u = -\frac{1}{ml^2u^2} \cdot ku^3 = -\frac{ku}{ml^2}$$
$$\frac{d^2u}{d\theta^2} + \left(1 + \frac{k}{ml^2}\right)u = 0$$

If $\left(1+\dfrac{k}{ml^2}\right)<0$, $\quad \dfrac{d^2u}{d\theta^2}-cu=0$, $\quad c>0$, for which $u=ae^{b\theta}$ is a solution.

If $\left(1+\dfrac{k}{ml^2}\right)=0$, $\quad \dfrac{d^2u}{d\theta^2}=0$

$$\frac{du}{d\theta}=C_1$$

$$u=c_1\,\theta+c_2$$

$$r=\frac{1}{c_1\,\theta+c_2}$$

If $\left(1+\dfrac{k}{ml^2}\right)>0$, $\quad \dfrac{d^2u}{d\theta^2}+cu=0$, $\quad c>0$

$$u=A\cos\left(\sqrt{c}\theta+\delta\right)$$

$$r=\left[A\cos\left(\sqrt{1+\frac{k}{ml^2}}\theta+\delta\right)\right]^{-1}$$

6.12 $\quad u=\dfrac{1}{r}=\dfrac{1}{r_\circ\cos\theta}$

$$\frac{du}{d\theta}=\frac{\sin\theta}{r_\circ\cos^2\theta}$$

$$\frac{d^2u}{d\theta^2}=\frac{1}{r_\circ}\left(\frac{1}{\cos\theta}+\frac{2\sin^2\theta}{\cos^3\theta}\right)=\frac{1}{r_\circ\cos\theta}\left(1+\frac{2-2\cos^2\theta}{\cos^2\theta}\right)=\frac{1}{r_\circ\cos\theta}\left(\frac{2}{\cos^2\theta}-1\right)$$

$$\frac{d^2u}{d\theta^2}=u\left(2r_\circ^2u^2-1\right)=2r_\circ^2u^3-u$$

Substituting into equation 6.5.10 …

$$2r_\circ^2u^3-u+u=-\frac{1}{ml^2u^2}f\left(u^{-1}\right)$$

$$f\left(u^{-1}\right)=-2r_\circ^2ml^2u^5$$

$$f\left(r\right)=-\frac{2r_\circ^2ml^2}{r^5}$$

6.13 From Chapter 1, the transverse component of the acceleration is … $a_\theta=r\ddot{\theta}+2\dot{r}\dot{\theta}$
If this term is nonzero, then there must be a transverse force given by …

$$f(\theta)=m(r\ddot{\theta}+2\dot{r}\dot{\theta})$$

For $\quad r=a\theta$, and $\quad \theta=bt$

$$f\left(\theta\right)=2mab^2\neq 0$$

Since $\quad f\left(\theta\right)\neq 0$, the force is not a central field.

For $r = a\theta$, and the force to be central, try $\theta = bt^n$

$$f(\theta) = m\left[2ab^2n^2t^{2n-2} + ab^2n(n-1)t^{2n-2}\right]$$

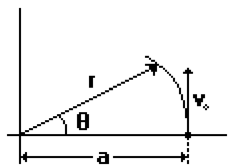
For a central field … $f(\theta) = 0$

$$2n + (n-1) = 0$$

$$n = \frac{1}{3}$$

$$\theta = bt^{\frac{1}{3}}$$

6.14



(a)

Calculating the potential energy

$$-\frac{dv}{dr} = f(r) = -k\left(\frac{4}{r^3} + \frac{a^2}{r^5}\right)$$

Thus, $V = -k\left(\frac{2}{r^2} + \frac{a^2}{4r^4}\right)$

The total energy is …

$$E = T_\circ + V_\circ = \frac{1}{2}v_\circ^2 - k\left(\frac{2}{a^2} + \frac{1}{4a^2}\right) = \frac{1}{2}\left(\frac{9k}{2a^2}\right) - \frac{9k}{4a^2} = 0$$

Its angular momentum is …

$$l^2 = a^2v_\circ^2 = \frac{9k}{2} = constant = r^4\dot{\theta}^2$$

Its KE is …

$$T = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\dot{\theta}^2 = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{l^2}{r^4}$$

The energy equation of the orbit is …

$$T + V = 0 = \frac{1}{2}\left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{l^2}{r^4} - k\left(\frac{2}{r^2} + \frac{a^2}{4r^4}\right)$$

$$= \left[\left(\frac{dr}{d\theta}\right)^2 + r^2\right]\frac{9k}{4r^4} - k\left(\frac{2}{r^2} + \frac{a^2}{4r^4}\right)$$

or $\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{9}\left(a^2 - r^2\right)$

Letting $r = a\cos\phi$ then $\frac{dr}{d\theta} = -a\sin\phi\frac{d\phi}{d\theta}$

So $\left(\frac{d\phi}{d\theta}\right)^2 = \frac{1}{9}$ $\quad\therefore \phi = \frac{1}{3}\theta$

Thus $r = a\cos\frac{1}{3}\theta$ $\quad\left(r = a @ \theta = 0^\circ\right)$

(b) at $\theta = \frac{3\pi}{2}$ $\quad r \to 0$ the origin of the force. To find how long it takes …

$$\dot{\theta} = \frac{l}{r^2} = \frac{av_\circ}{a^2\cos^2\frac{1}{3}\theta} = \frac{v_\circ}{a\cos^2\frac{1}{3}\theta}$$

$$dt = \frac{a}{v_\circ}\cos^2\frac{1}{3}\theta\, d\theta$$

$$T = \int_0^{\frac{3\pi}{2}} \frac{a}{v_\circ}\cos^2\frac{1}{3}\theta\, d\theta = \frac{3a}{v_\circ}\int_0^{\frac{\pi}{2}}\cos^2\phi\, d\phi = \frac{3\pi a}{4v_\circ}$$

Since $v_\circ = \left(\frac{9k}{2a^2}\right)^{\frac{1}{2}}$

$$T = \frac{3}{4}\pi a^2\left(\frac{2}{9k}\right)^{\frac{1}{2}} = \frac{\pi a^2}{4}\left(\frac{2}{k}\right)^{\frac{1}{2}}$$

(c) Since the particle falls into the center of the force

$v \to \infty$ $\quad$ (since $l = vr_\perp = const$)

6.15 From Example 6.5.4 … $\quad \frac{v_\circ}{v_c} = \left(\frac{2r_1}{r_1 + r_\circ}\right)^{\frac{1}{2}}$

Letting $V = \frac{v_\circ}{v_c}$ we have $V = \left(\frac{2}{1+\frac{r_\circ}{r_1}}\right)^{\frac{1}{2}}$

So: $\quad \frac{dV}{dr_1} = \frac{1}{2V}\left[-2\left(1+\frac{r_\circ}{r_1}\right)^{-2}\left(-\frac{r_\circ}{r_1^2}\right)\right]$

Thus $\quad \frac{\frac{dV}{V}}{\left(\frac{dr_1}{r_1}\right)} = \frac{1}{\left(\frac{2}{1+\frac{r_\circ}{r_1}}\right)}\frac{1}{\left(1+\frac{r_\circ}{r_1}\right)^2}\frac{r_\circ}{r_1} = \frac{1}{2}\frac{1}{\left(1+\frac{r_\circ}{r_1}\right)}\frac{r_\circ}{r_1}$

(a) $$\frac{\left(\frac{dV}{V}\right)}{\left(\frac{dr_1}{r_1}\right)} \approx \frac{1}{2}\frac{r_\circ}{r_1} \quad \text{(b)} \quad \left(\frac{dr_1}{r_1}\right) = \frac{2r_1}{r_\circ}\left(\frac{dV}{V}\right) = 2(60)1\% = 120\%!$$

The approximation of a differential has broken down – a correct result can be obtained by calculating finite differences, but the implication is clear – a 1% error in boost causes rocket to miss the moon by a huge factor --- ~ 2!

6.16 From section 6.5, $\varepsilon = 0.967$ and $r_\circ = 55\times10^6\,\text{mi}$.

From equations 6.5.21a&b, $r_1 = r_\circ \dfrac{1+\varepsilon}{1-\varepsilon}$

$$a = \frac{1}{2}(r_\circ + r_1) = \frac{r_\circ}{1-\varepsilon} = \frac{55\times10^6\,mi}{1-0.967}\times\frac{1\,AU}{93\times10^6\,mi}$$
$$a = 17.92\,AU$$

From equation 6.6.5, $\tau = ca^{\frac{3}{2}}$

$$\tau = 1yr\cdot AU^{-\frac{3}{2}}\times 17.92^{\frac{3}{2}}\,AU^{\frac{3}{2}}$$
$$\tau = 75.9\,yr$$

From equation 6.5.21a and 6.5.19 …

$$\varepsilon = \frac{\alpha}{r_0} - 1 = \frac{ml^2}{kr_0} - 1$$

$$\varepsilon = \frac{mr_\circ v_\circ^2}{k} - 1 \quad \text{and} \quad k = GMm$$

$$v_\circ = \left[\frac{GM}{r_\circ}(\varepsilon+1)\right]^{\frac{1}{2}}$$

From Example 6.5.3 we can translate the factor GM into the more convenient

$GM = a_e v_e^2$ … with a_e the radius of a circular orbit and v_e the orbital speed …

$$v_\circ = \left[\frac{a_e v_e^2}{r_\circ}(\varepsilon+1)\right]^{\frac{1}{2}} = \left[\frac{93\times10^6\,mi}{55\times10^6\,mi}(1.967)\right]^{\frac{1}{2}} v_e$$

$$v_0 = 1.824\,v_e$$

Since l is constant … $r_1 v_1 = r_\circ v_\circ$

$$v_1 = \frac{r_\circ}{r_1}v_\circ = \frac{1-\varepsilon}{1+\varepsilon}v_\circ = \frac{1-.967}{1.967}\times 1.824\,v_e = 0.0306\,v_e$$

$$v_e \approx \frac{2\pi a_e}{\tau} = \frac{2\pi\times 93\times10^6\,mi}{1\,yr\times 365\,day\cdot yr^{-1}\times 24\,hr\cdot day^{-1}} = 66,705\ mph$$

$$v_\circ = 1.22\times10^5\,mph \quad \text{and} \quad v_1 = 2.04\times10^3\,mph$$

6.17 From Example 6.10.1 …

$$\varepsilon = \left[1 + \left(q^2 - \frac{2}{d}\right)(qd\sin\phi)^2\right]^{\frac{1}{2}} \qquad \text{where } q = \frac{v}{v_e} \text{ and } d = \frac{r}{a_e}$$

are dimensionless ratios of the comet's speed and distance from the Sun in terms of the Earth's orbital speed and radius, respectively (q and d are the same as the factors V and R in Example 6.10.1). ϕ is the angle between the comet's orbital velocity and direction vector towards the Sun (see Figure 6.10.1).

The orbit is hyperbolic, parabolic, or elliptic as ε is > , = , or < 1 …

i.e., as $\left(q^2 - \frac{2}{R}\right)$ is > , = , or < 0.

$\left(q^2 - \frac{2}{R}\right)$ is > , = , or < 0 as $q^2 d$ is > , = , or < 2.

6.18 Since l is constant, v_{max} occurs at $r_\circ$ and v_{min} occurs at r_1, i.e. $v_{max} = v_\circ$ and $v_{min} = v_1$ and form the constancy of l … $v_1 r_1 = v_\circ r_\circ$

$$v_{min} v_{max} = v_1 v_\circ = \frac{r_\circ}{r_1} v_\circ^2$$

$$r_\circ v_\circ^2 = \frac{k}{m}(\varepsilon + 1) \qquad \text{(See Example 6.5.4)}$$

From equation 6.6.5 … $\frac{k}{m} = GM_\odot = \left(\frac{2\pi a}{\tau}\right)^2 a$

$$v_{min} v_{max} = \left(\frac{2\pi a}{\tau}\right)^2 \cdot \frac{a(\varepsilon+1)}{r_1}$$

From equation 6.5.21a&b … $r_1 = r_\circ \frac{1+\varepsilon}{1-\varepsilon}$. With $2a = r_\circ + r_1$:

$$\frac{a(\varepsilon+1)}{r_1} = \frac{1}{2}\frac{(r_\circ + r_1)(\varepsilon+1)}{r_1} = \frac{1}{2}\left(\frac{r_\circ}{r_1} + 1\right)(\varepsilon+1) = \frac{1}{2}\left[\left(\frac{1-\varepsilon}{1+\varepsilon}\right) + 1\right](\varepsilon+1) = 1$$

$$v_{min} v_{max} = \left(\frac{2\pi a}{\tau}\right)^2$$

6.19 As a result of the impulse, the speed of the planet <u>instantaneously</u> changes; its orbital radius does not, so there is no change in its potential energy V. The instantaneous change in its total orbital energy E is due to the change in its kinetic energy, T, only, so

$$\delta E = \delta T = \delta\left(\frac{1}{2}mv^2\right) = mv\delta v = mv^2\frac{\delta v}{v} = 2T\frac{\delta v}{v}$$

$$\frac{\delta E}{T} = 2\frac{\delta v}{v}$$

But the total orbital energy is

$$E=-\frac{k}{2a} \qquad \text{So} \qquad \delta E=\frac{k}{2a^2}\delta a$$

Since planetary orbits are nearly circular

$$V\sim-\frac{k}{a} \qquad \text{and} \qquad T\sim\frac{k}{2a}$$

Thus, $\delta E\cong T\frac{\delta a}{a}$ and $\frac{\delta E}{T}=\frac{\delta a}{a}$

We obtain $\frac{\delta a}{a}=2\frac{\delta v}{v}$

6.20 (a) $\bar{V}=\frac{1}{\tau}\int_0^\tau Vdt$

$$V(r)=-\frac{k}{r}$$

From equation 6.5.4, $l=r^2\dot{\theta}$

$$\frac{d\theta}{dt}=\frac{l}{r^2} \text{ or } dt=\frac{r^2d\theta}{l}$$

$$\int_0^\tau Vdt=-\int_0^{2\pi}\frac{kr}{l}d\theta$$

From equation 6.5.18a …

$$r=\frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$$

$$\int_0^\tau Vdt=-\frac{ka\left(1-\varepsilon^2\right)}{l}\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}$$

From equation 6.6.4 … $\tau=\frac{2\pi a^2}{l}\sqrt{1-\varepsilon^2}$

$$\bar{V}=-\frac{k\sqrt{1-\varepsilon^2}}{2\pi a}\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}$$

$$\int_0^{2\pi}\frac{d\theta}{1+\varepsilon\cos\theta}=\frac{2\pi}{\sqrt{1-\varepsilon^2}}, \ \varepsilon^2<1 \ \therefore \bar{V}=-\frac{k}{a}$$

(b) This problem is an example of the *virial theorem* which, for a bounded, periodic system, relates the time average of the quantity $\int_0^\tau\sum_i\bar{p}\cdot\bar{r}_i$ to its kinetic energy T. We will derive it for planetary motion as follows:

$$\frac{1}{\tau}\int_0^\tau\dot{\bar{p}}\cdot\bar{r}dt=\frac{1}{\tau}\int_0^\tau m\ddot{\bar{r}}\cdot\bar{r}dt=\frac{1}{\tau}\int_0^\tau\bar{F}\cdot\bar{r}dt$$

Integrate LHS by parts

$$\frac{1}{\tau}\left[m\dot{\bar{r}}\cdot\bar{r}\right]\Big|_0^\tau - \frac{1}{\tau}\int_0^\tau m\dot{\bar{r}}^2 dt = \frac{1}{\tau}\int_0^\tau \bar{F}\cdot\bar{r}dt$$

The first term is zero – since the quantity has the same value at 0 and τ.

Thus $2\langle T\rangle = -\langle \bar{F}\cdot\bar{r}\rangle$ where $\langle\ \rangle$ denote time average of the quantity within brackets.

but $-\langle \bar{r}\cdot\bar{F}\rangle = \langle r\cdot\bar{\nabla}V\rangle = \left\langle r\frac{dV}{dr}\right\rangle = \left\langle \frac{k}{r}\right\rangle = -\langle V\rangle$

hence $2\langle T\rangle = -\langle V\rangle$

but $\langle E\rangle = \langle T\rangle + \langle V\rangle = -\frac{\langle V\rangle}{2} + \langle V\rangle = \frac{\langle V\rangle}{2}$

hence $\langle V\rangle = 2\langle E\rangle$ but $E = -\frac{k}{2a} = constant$

and $\langle E\rangle = \frac{1}{\tau}\int_0^\tau E dt = E = -\frac{k}{2a}$ so $2E = -\frac{k}{a}$

Thus: $\langle V\rangle = -\frac{k}{a}$ as before and therefore $\langle T\rangle = -\frac{1}{2}\langle V\rangle = \frac{k}{2a}$

6.21 The energy of the initial orbit is

$$\frac{1}{2}mv^2 - \frac{k}{r} = E = -\frac{k}{2a}$$

(1) $$v^2 = \frac{k}{m}\left(\frac{2}{r} - \frac{1}{a}\right)$$

Since $r_a = a(1+\varepsilon)$ at apogee, the speed v_1, at apogee is

$$v_1^2 = \frac{k}{m}\left(\frac{2}{a(1+\varepsilon)} - \frac{1}{a}\right) = \frac{k}{ma}\frac{(1-\varepsilon)}{(1+\varepsilon)}$$

To place satellite in circular orbit, we need to boost its speed to v_c such that

$$\frac{1}{2}mv_c^2 - \frac{k}{r_a} = -\frac{k}{2r_a}$$ since the radius of the orbit is r_a

$$v_c^2 = \frac{k}{mr_a} = \frac{k}{ma(1+\varepsilon)}$$

Thus, the boost in speed $\Delta v_1 = v_c - v_1$

(2) $$\Delta v_1 = \left[\frac{k}{ma(1+\varepsilon)}\right]^{\frac{1}{2}}\left[1-(1-\varepsilon)^{\frac{1}{2}}\right]$$

Now we solve for the semi-major axis *a* and the eccentricity ε of the first orbit. From (1) above, at launch $v = v_\circ$ at $r = R_E$, so

$$v_\circ^2 = \frac{k}{m}\left(\frac{2}{R_E} - \frac{1}{a}\right)$$

and solving for a

$$a = \frac{R_E}{2 - mv_\circ^2 \frac{R_E}{k}} \quad \text{noting that } \ldots$$

(3) $$\frac{k}{mR_E} = \frac{GM_E}{R_E} = gR_E$$

$$a = \frac{R_E}{\left(2 - \frac{v_\circ^2}{gR_E}\right)} = \frac{R_E}{1.426} = \underline{4.49 \cdot 10^3}\ km$$

The eccentricity ε can be found from the angular momentum per unit mass, l, equation 6.5.19, and the data on ellipses defined in figure 6.5.1 …

$$l = r^2\dot{\theta} = v_\circ\left(R_E \sin\theta_\circ\right) = \left[\frac{k\alpha}{m}\right]^{\frac{1}{2}} = \left[\frac{ka\left(1-\varepsilon^2\right)}{m}\right]^{\frac{1}{2}}$$

where $v_\circ$, $\theta_\circ$ are the launch velocity, angle

Solving for ε (using (3) above)

$$\varepsilon^2 = 1 - \frac{v_\circ^2}{gR_E}\left(2 - \frac{v_\circ^2}{gR_E}\right)\sin^2\theta_\circ = 0.795$$

$$\therefore \underline{\varepsilon = 0.892}$$

Inserting these values for a, ε into (2) and using (3) gives

(a) $$\Delta v_1 = \left[gR_E\left(\frac{R_E/a}{1+\varepsilon}\right)\right]^{\frac{1}{2}}\left[1 - \left(1-\varepsilon\right)^{\frac{1}{2}}\right] = \underline{4.61 \cdot 10^3}\ km \cdot s^{-1}$$

(b) $$h = a\left(1+\varepsilon\right) - R_E = \underline{2.09 \cdot 10^3}\,km$$ {altitude above the Earth … at perigee}

6.22 $$f'(r) = -k\left(\frac{-be^{-br}}{r^2} - 2\frac{e^{-br}}{r^3}\right) = k\frac{e^{-br}}{r^2}\left(b + \frac{2}{r}\right)$$

$$\frac{f'(a)}{f(a)} = -\left(b + \frac{2}{a}\right)$$

From equation 6.14.3, $\psi = \pi\left[3 + a\frac{f'(a)}{f(a)}\right]^{-\frac{1}{2}} = \pi\left[3 - \left(ab+2\right)\right]^{-\frac{1}{2}}$

$$\psi = \frac{\pi}{\sqrt{1-ab}}$$

6.23 From Problem 6.9, $f(r) = -\dfrac{GMm}{r^2} - \dfrac{4}{3}\pi\rho m G r$

$$f'(r) = \frac{2GMm}{r^3} - \frac{4}{3}\pi\rho m G$$

$$\frac{f'(a)}{f(a)} = \frac{2GMma^{-3} - \frac{4}{3}\pi\rho m G}{-GMma^{-2} - \frac{4}{3}\pi\rho m G a} = \frac{-2 + \frac{4\pi\rho a^3}{3M}}{a\left(1 + \frac{4\pi\rho a^3}{3M}\right)}$$

From equation 6.14.3, $\psi = \pi\left[3 + a\dfrac{f'(a)}{f(a)}\right]^{-\frac{1}{2}}$

$$\psi = \pi\left[3 + \frac{-2 + \frac{4\pi\rho a^3}{3M}}{1 + \frac{4\pi\rho a^3}{3M}}\right]^{-\frac{1}{2}} = \pi\left[\frac{1 + 4\left(\frac{4\pi\rho a^3}{3M}\right)}{1 + \frac{4\pi\rho a^3}{3M}}\right]^{-\frac{1}{2}}$$

$$\psi = \pi\left(\frac{1+c}{1+4c}\right)^{\frac{1}{2}}, \qquad c = \frac{4\pi\rho a^3}{3M}$$

6.24 We differentiate equation 6.11.1b to obtain $m\ddot{r} = -\dfrac{dU(r)}{dr}$

For a circular orbit at $r = a$, $\ddot{r} = 0$ so

$$\frac{dU}{dr}\Big|_{r=a} = 0$$

For small displacements x from $r = a$,

$r = x + a$ and $\ddot{r} = \ddot{x}$

From Appendix D …

$$f(x+a) = f(a) + xf'(a) + \frac{x^2}{2}f''(a) + \ldots$$

Taking $f(r)$ to be $\dfrac{dU}{dr}$, $f'(r) = \dfrac{d^2U}{dr^2}$

Near $r = a$ …

$$\frac{dU}{dr} = \frac{dU}{dr}\Big|_{r=a} + x\frac{d^2U}{dr^2}\Big|_{r=a} + \ldots$$

$$m\ddot{x} = -x\frac{d^2U}{dr^2}\Big|_{r=a}$$

This represents a "restoring force," i.e., stable motion, so long as $\dfrac{d^2U}{dr^2} > 0$ at $r = a$.

6.25 $f'(r) = \frac{2k}{r^3} + \frac{4\varepsilon}{r^5}$

From equation 6.13.7, the condition for stability is $f(a) + \frac{a}{3} f'(a) < 0$

$$-\frac{k}{a^2} - \frac{\varepsilon}{a^4} + \frac{a}{3}\left(\frac{2k}{a^3} + \frac{4\varepsilon}{a^5}\right) < 0$$

$$-\frac{k}{3a^2} + \frac{\varepsilon}{3a^4} < 0$$

$$\frac{\varepsilon}{a^2} < k$$

$$a > \left(\frac{\varepsilon}{k}\right)^{\frac{1}{2}}$$

6.26 (a) $f(r) = -k\frac{e^{-br}}{r^2}$

$$f'(r) = -ke^{-br}\left(-\frac{b}{r^2} - \frac{2}{r^3}\right) = k\frac{e^{-br}}{r^2}\left(b + \frac{2}{r}\right)$$

From equation 6.13.7, the condition for stability is $f(a) + \frac{a}{3} f'(a) < 0$

$$-k\frac{e^{-ba}}{a^2} + \frac{a}{3}k\frac{e^{-ba}}{a^2}\left(b + \frac{2}{a}\right) < 0$$

$$-k\frac{e^{-ba}}{3a^2} + k\frac{be^{-ba}}{3a} < 0$$

$$b < \frac{1}{a} \qquad a < \frac{1}{b}$$

(b) $f(r) = -\frac{k}{r^3}$

$$f'(r) = \frac{3k}{r^4}$$

$$f(a) + \frac{a}{3} f'(a) = -\frac{k}{a^3} + \frac{a}{3}\left(\frac{3k}{a^4}\right) = 0$$

Since $f(a) + \frac{a}{3} f'(a)$ is not less than zero, the orbit is not stable.

6.27 (See Figure 6.10.1) From equation 6.5.18a $r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$

and the data on ellipses in Figure 6.5.1 $p = a(1-\varepsilon)$ so

$$r = p\frac{(1+\varepsilon)}{1+\varepsilon\cos\theta}$$

For a parabolic orbit, $\varepsilon = 1$

The comet intersects earth's orbit at $r = a$.

$$a = \frac{2p}{1+\cos\theta}$$

$$\cos\theta = -1 + \frac{2p}{a}$$

6.28 (See Figure 6.10.1) $T = \int dt$ along the comet's trajectory inside earth's orbit

From equation 6.5.4, $r^2\dot{\theta} = r^2\frac{d\theta}{dt} = l$ so $dt = \frac{r^2 d\theta}{l}$

$$T = \int \frac{r^2 d\theta}{l}$$

From equation 6.5.18a $r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$

and the data on ellipses in Figure 6.5.1 $p = a(1-\varepsilon)$ so

$$r = p\frac{(1+\varepsilon)}{1+\varepsilon\cos\theta}$$

From equation 6.5.18b, with $\varepsilon = 1$ for a parabolic orbit:

$$r = \frac{2p}{1+\cos\theta}$$

At $\theta = \frac{\pi}{2}$ the distance to the comet is $r = \alpha = \frac{2p}{1+\cos\frac{\pi}{2}} = 2p$

From equation 6.5.19, $\alpha = \frac{ml^2}{k}$, where $k = GMm$, so $p = \frac{l^2}{2GM}$

As shown in Example 6.5.3, $GM = av_e^2$

For a circular orbit, $v_e = \frac{2\pi a}{1\ yr}$

$$l = (2GMp)^{\frac{1}{2}} = (2ap)^{\frac{1}{2}} v_e = (2a)^{\frac{3}{2}} p^{\frac{1}{2}} \pi\ yr^{-1}$$

$$T = \int_{-\theta_\circ}^{+\theta_\circ} \frac{r^2 d\theta}{l} = \int_{-\theta_\circ}^{+\theta_\circ} \frac{4p^2}{(1+\cos\theta)^2} \cdot (2a)^{-\frac{3}{2}} p^{-\frac{1}{2}} \pi^{-1} d\theta\ yr$$

where $\theta_\circ = \cos^{-1}\left(-1 + \frac{2p}{a}\right)$ from Problem 6.27

$$T=\frac{\sqrt{2}p^{\frac{3}{2}}}{\pi a^{\frac{3}{2}}}\int_{-\theta_\circ}^{+\theta_\circ}\frac{d\theta}{(1+\cos\theta)^2}\ yr$$

From a table of integrals, $\int\frac{dx}{(1+\cos x)^2}=\frac{1}{2}\tan\frac{x}{2}+\frac{1}{6}\tan^3\frac{x}{2}$

$$T=\frac{\sqrt{2}}{\pi}\left(\frac{p}{a}\right)^{\frac{3}{2}}\left[\tan\frac{\theta_\circ}{2}+\frac{1}{3}\tan^3\frac{\theta_\circ}{2}\right]yr$$

$$\tan\frac{x}{2}=\left(\frac{1-\cos x}{1+\cos x}\right)^{\frac{1}{2}}$$

$$\tan\frac{\theta_\circ}{2}=\left(\frac{2-\frac{2p}{a}}{\frac{2p}{a}}\right)^{\frac{1}{2}}=\left(\frac{a-p}{p}\right)^{\frac{1}{2}}$$

$$T=\frac{\sqrt{2}}{\pi}\left(\frac{p}{a}\right)^{\frac{3}{2}}\left[\left(\frac{a-p}{p}\right)^{\frac{1}{2}}+\frac{1}{3}\left(\frac{a-p}{p}\right)^{\frac{3}{2}}\right]yr$$

$$=\frac{\sqrt{2}}{\pi}\left(\frac{p}{a}\right)^{\frac{3}{2}}\left(\frac{a-p}{p}\right)^{\frac{1}{2}}\left(1+\frac{a-p}{3p}\right)yr$$

$$T=\frac{\sqrt{2}}{3\pi}\left(\frac{2p}{a}+1\right)\left(1-\frac{p}{a}\right)^{\frac{1}{2}}\ yr$$

T is a maximum when $(2p+a)(a-p)^{\frac{1}{2}}$ is a maximum.

$$\frac{d}{dp}\left[(2p+a)^2(a-p)\right]=2(2p+a)(2)(a-p)+(2p+a)^2(-1)$$

$$=(2p+a)(3a-6p)$$

T is a maximum when $p=\frac{a}{2}$.

$$T=\frac{\sqrt{2}}{3\pi}(2)\left(\frac{1}{2}\right)^{\frac{1}{2}}=\frac{2}{3\pi}\ yr=77.5\ day$$

When $p=0.6a$

$$T=\frac{\sqrt{2}}{3\pi}(2.2)\sqrt{.04}=0.2088\,yr=76.2\,day$$

6.29 $V(r)=-\frac{k}{r}-\frac{k\varepsilon}{r^3}$

$$f(r) = -\frac{dV}{dr} = \frac{k}{r^2} + \frac{3k\varepsilon}{r^4} = \frac{k}{r^4}\left(r^2 + 3\varepsilon\right)$$

$$f'(r) = -\frac{2k}{r^3} - \frac{12k\varepsilon}{r^5} = -\frac{2k}{r^5}\left(r^2 + 6\varepsilon\right)$$

$$\frac{f'(a)}{f(a)} = -\frac{2}{a}\left(\frac{r^2 + 6\varepsilon}{r^2 + 3\varepsilon}\right)$$

From equation 6.14.3, $\psi = \pi\left[3 + a\frac{f'(a)}{f(a)}\right]^{-\frac{1}{2}}$

$$\psi = \pi\left[3 - 2\left(\frac{r^2 + 6\varepsilon}{r^2 + 3\varepsilon}\right)\right]^{-\frac{1}{2}} = \pi\sqrt{\frac{r^2 + 3\varepsilon}{r^2 - 3\varepsilon}}$$

For $\quad \varepsilon = \frac{2}{5}R\Delta R$, $R = 4000\,mi$, $\Delta R = 13\,mi$

$$\varepsilon = \frac{2}{5}(4000)(13) = 2.08\times 10^4\,mi^2$$

For $\quad r \approx R$, $r^2 = 1.6\times 10^7\,mi^2$

$$\psi = 1.0039\pi = 180.7^\circ$$

6.30 $\quad V_{rel} = -\frac{k}{r} - \frac{1}{2m_\circ c^2}\left(E + \frac{k}{r}\right)^2$

$$f(r) = -\frac{dV}{dr} = -\frac{k}{r^2} + \frac{2}{2m_\circ c^2}\left(E + \frac{k}{r}\right)\left(-\frac{k}{r^2}\right)$$

$$f(r) = -\frac{k}{r^2}\left[1 + \frac{1}{m_\circ c^2}\left(E + \frac{k}{r}\right)\right]$$

$$f'(r) = \frac{2k}{r^3}\left[1 + \frac{1}{m_\circ c^2}\left(E + \frac{k}{r}\right)\right] - \frac{k}{r^2}\left(\frac{1}{m_\circ c^2}\right)\left(-\frac{k}{r^2}\right)$$

$$f'(r) = \frac{k}{r^3}\left[2 + \frac{1}{m_\circ c^2}\left(2E + \frac{3k}{r}\right)\right]$$

$$\frac{f'(a)}{f(a)} = -\frac{1}{a}\left[\frac{2 + \frac{1}{m_\circ c^2}\left(2E + \frac{3k}{a}\right)}{1 + \frac{1}{m_\circ c^2}\left(E + \frac{k}{a}\right)}\right]$$

$$3 + a\frac{f'(a)}{f(a)} = \frac{\left[3 + \frac{3}{m_\circ c^2}\left(E + \frac{k}{a}\right) - 2 - \frac{1}{m_\circ c^2}\left(2E + \frac{3k}{a}\right)\right]}{\left[1 + \frac{1}{m_\circ c^2}\left(E + \frac{k}{a}\right)\right]}$$

$$= \frac{\left[1+\frac{E}{m_\circ c^2}\right]}{\left[1+\frac{1}{m_\circ c^2}\left(E+\frac{k}{a}\right)\right]}$$

$$3+a\frac{f'(a)}{f(a)} = \left[\frac{m_\circ c^2+E}{m_\circ c^2+E+\frac{k}{a}}\right]$$

$$\psi = \pi\left[3+a\frac{f'(a)}{f(a)}\right]^{-\frac{1}{2}}$$

$$\psi = \pi\left[1+\frac{\frac{k}{a}}{m_\circ c^2+E}\right]^{\frac{1}{2}}$$

6.31 From equation 6.5.18a $r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$ … (Here θ is the polar angle of conic section trajectories as illustrated by the coordinates in Figure 6.5.1)
… and the data on ellipses in Figure 6.5.1 $r_0 = a(1-\varepsilon)$ so

$$r_{com} = r_\circ \frac{1+\varepsilon}{1+\varepsilon\cos\theta}$$

From equation 6.5.18b $r = \frac{\alpha}{1+\varepsilon\cos\theta}$ and at $\theta = 0^\circ$ $r_0 = \frac{\alpha}{1+\varepsilon}$

And from equation 6.5.19 $\alpha = \frac{ml^2}{k}$ so $r_0 = \frac{ml^2}{k(1+\varepsilon)}$

$$\frac{m}{k} = \frac{m}{GMm} = \frac{1}{GM}$$

From Example 6.5.3, $GM = a_e v_e^2$ and $l^2 = r_{com}^2 v_{com}^2 \sin^2\phi$

$$r_{com} = \frac{r_{com}^2 v_{com}^2 \sin^2\phi(1+\varepsilon)}{a_e v_e^2(1+\varepsilon)(1+\varepsilon\cos\theta)}$$

$$1 = RV^2\sin^2\phi\frac{1}{1+\varepsilon\cos\theta}$$

$$\cos\theta = \frac{1}{\varepsilon}\left(RV^2\sin^2\phi - 1\right)$$

$$\sin\theta = \left(1-\cos^2\theta\right)^{\frac{1}{2}} = \left[1-\frac{1}{\varepsilon^2}\left(RV^2\sin^2\phi-1\right)^2\right]^{\frac{1}{2}}$$

$$\sin\theta = \frac{1}{\varepsilon}\left[\varepsilon^2 - \left(RV^2\sin^2\phi\right)^2 + 2\left(RV^2\sin^2\phi\right) - 1\right]^{\frac{1}{2}}$$

Again from Example 6.5.3 …

$$\varepsilon = \left[1 + \left(V^2 - \frac{2}{R}\right)\left(RV\sin\phi\right)^2\right]^{\frac{1}{2}}$$

$$\varepsilon^2 = 1 + \left(RV^2\sin\phi\right)^2 - 2RV^2\sin^2\phi$$

$$\sin\theta = \frac{1}{\varepsilon}\left[\left(RV^2\sin\phi\right)^2 - \left(RV^2\sin^2\phi\right)^2\right]^{\frac{1}{2}}$$

$$\sin\theta = \frac{1}{\varepsilon}RV^2\sin\phi\,\cos\phi$$

$$\frac{\cos\theta}{\sin\theta} = \frac{RV^2\sin^2\phi - 1}{RV^2\sin\phi\cos\phi}$$

$$\cot\theta = \tan\phi - \frac{2}{RV^2\sin 2\phi}$$

$$\theta = \cot^{-1}\left(\tan\phi - \frac{2}{RV^2}\csc 2\phi\right)$$

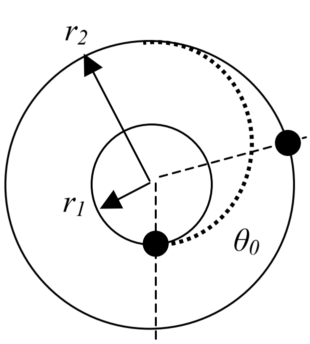
For $V = 0.5,\ R = 4,\ \phi = 30^\circ$:

$$\theta = \cot^{-1}\left(\tan 30^\circ - \frac{2}{4(.5)^2}\csc 60^\circ\right)$$

$$= \cot^{-1}\left(\frac{1}{\sqrt{3}} - \frac{2}{\frac{\sqrt{3}}{2}}\right) = \cot^{-1}\left(-\sqrt{3}\right)$$

$$\theta = -30^\circ$$

6.32



The picture at left shows the orbital transfer and the position of the two satellites at the moment the transfer is initiated. Satellite B is "ahead" of satellite A by the angle θ_0

$a = \frac{r_1 + r_2}{2}$ is the semi-major axis of the elliptical transfer orbit.

From Kepler's 3rd law (Equation 6.6.5) applied to objects in orbit about Earth …

$$\tau^2 = \frac{4\pi^2}{GM_E{}^2}a^3$$

The time to intercept is …

$$T_t = \frac{\tau}{2} = \pi\frac{1}{\sqrt{GM_E}}a^{\frac{3}{2}} = \frac{\pi}{R_E\sqrt{g}}a^{\frac{3}{2}} \qquad \text{since } \frac{GM_E}{R_E{}^2} = g$$

Letting $r_1 = R_E + h_1$ and $r_2 = R_E + h_2$ where h_1 and h_2 are the heights of the 2 satellites above the ground. Inserting these into the above gives …

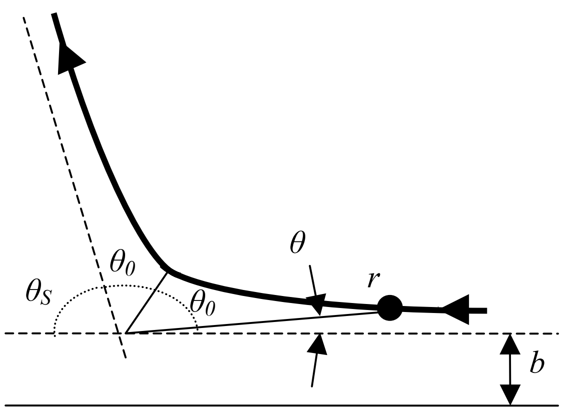
$$T_t = \frac{\pi}{R_E\sqrt{g}} R_E^{\frac{3}{2}} \left(2 + \frac{h_1 + h_2}{R_E}\right)^{\frac{3}{2}} = \frac{\pi}{\sqrt{R_E g}} \left(2 + \frac{h_1 + h_2}{R_E}\right)^{\frac{3}{2}}$$

From Example 6.6.2, R_E = 6371 km, h_1 = 200 mi = 324 km and $h_2 = r_2 - R_E$ = 42,400 km - 36,029 km. Putting in the numbers …

T_t = 4.79 hr

(b) Thus, $\theta_0 = 180^0 \left(1 - \frac{T_t}{12}\right) = 108^0$

6.33



The potential for the inverse-cube force law is … $V(r) = \frac{k}{2r^2}$

Letting $u = r^{-1}$, we have (Equation 6.9.3)

$$\frac{1}{2} m l^2 \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] + V\left(u^{-1}\right) = E$$

$$\frac{du}{d\theta} = \sqrt{\frac{2(E-V)}{ml^2} - u^2}$$

$$d\theta = \frac{du}{\sqrt{\frac{2(E-V)}{ml^2} - u^2}} = \frac{ml}{\sqrt{2m(E-V) - m^2 l^2 u^2}} du$$

Now, integrating from $r = \infty\ (u = 0)$ up to $r = r_{\min}\ (u = u_{\max})$ …

$$\theta_0 = \int_0^{u_{\max}} \frac{ml}{\sqrt{2m(E-V) - m^2 l^2 u^2}} du$$

But $E = \frac{1}{2} m v_0^2$, $l = b v_0$, so …

$$\theta_0 = \int_0^{u_{\max}} \frac{m b v_0}{\sqrt{2m\left(\frac{1}{2} m v_0^2 - \frac{1}{2} k u^2\right) - m^2 b^2 v_0^2 u^2}} du$$

$$\theta_0 = \int_0^{u_{\max}} \frac{m b v_0}{\sqrt{2m\left(\frac{1}{2} m v_0^2 - \frac{1}{2} k u^2\right) - m^2 b^2 v_0^2 u^2}} du$$

Before evaluating this integral, we need to find $u_{\max}\left(=r_{\min}{}^{-1}\right)$, in other words, the distance of closest approach to the scattering center.

$$E=T\left(r_{\min}\right)+V\left(r_{\min}\right)=\frac{1}{2}mv^2+\frac{1}{2}\frac{k}{r_{\min}{}^2}=\frac{1}{2}mv_0{}^2$$

But, the angular momentum per unit mass *l* is …

$l=bv_0=r_{\min}v$ and substituting for *v* into the above gives …

$$\frac{ml^2}{r_{\min}{}^2}+\frac{k}{r_{\min}{}^2}=mv_0{}^2 \text{ so … } \frac{1}{r_{\min}{}^2}=\frac{mv_0{}^2}{ml^2+k}=u_{\max}{}^2$$

Solving for $u_{\max}$ …

$$u_{\max}=\frac{1}{\sqrt{b^2+\dfrac{k}{mv_0{}^2}}}$$

Now we evaluate the integral for θ_0 …

$$\theta_0=\int_0^{u_{\max}}\frac{b}{\sqrt{1-\left(b^2+\dfrac{k}{mv_0{}^2}\right)u^2}}du=\frac{b}{\left(b^2+\dfrac{k}{mv_0{}^2}\right)}\sin^{-1}\frac{u}{u_{\max}}\Bigg|_0^{u_{\max}}=\frac{b}{\left(b^2+\dfrac{k}{mv_0{}^2}\right)}\frac{\pi}{2}$$

Solving for *b* …

$$b\left(\theta_0\right)=\frac{k}{mv_0{}^2}\frac{2\theta_0}{\sqrt{\pi^2-4\theta_0{}^2}}$$

But $\theta_0=\frac{1}{2}\left(\pi-\theta_S\right)$. Thus, we have …

$$b\left(\theta_S\right)=\frac{k}{mv_0{}^2}\frac{\pi-\theta_S}{\sqrt{\theta_S\left(2\pi-\theta_S\right)}}$$

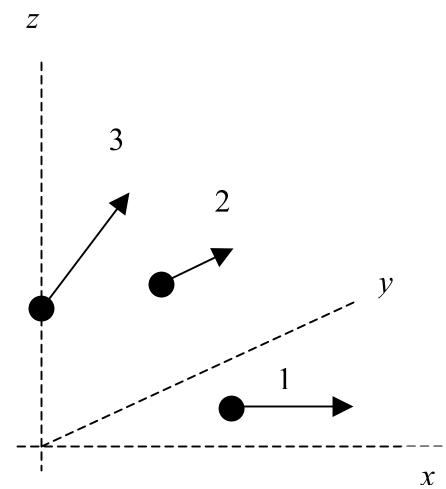
We can now compute the differential cross section …

$$\sigma\left(\theta_S\right)=\frac{b}{\sin\theta_S}\left|\frac{db}{d\theta_S}\right|=\frac{k\pi^2\left(\pi-\theta_S\right)}{mv_0{}^2\theta_S{}^2\left(2\pi-\theta_S\right)^2\sin\theta_S}$$

Since $d\Omega=2\pi\sin\theta_s\,d\theta_s$ we get …

$$\sigma\left(\theta_S\right)d\Omega=2\pi\left|bdb\right|=\frac{k\pi^3}{E}\left[\frac{\left(\pi-\theta_S\right)}{\left(2\pi-\theta_S\right)^2\theta_S{}^2}\right]d\theta_S$$

Chapter 7
Dynamics of Systems of Particles



7.1 From eqn. 7.1.1, $\vec{r}_{cm} = \frac{1}{m}\sum_i m_i \vec{r}_i$

$$\vec{r}_{cm} = \frac{1}{3}\left(\vec{r}_1 + \vec{r}_2 + \vec{r}_3\right) = \frac{1}{3}\left(\hat{i} + \hat{j} + \hat{j} + \hat{k} + \hat{k}\right)$$

$$\vec{r}_{cm} = \frac{1}{3}\left(\hat{i} + 2\hat{j} + 2\hat{k}\right)$$

$$\vec{v}_{cm} = \frac{d}{dt}\vec{r}_{cm} = \frac{1}{3}\left(\vec{v}_1 + \vec{v}_2 + \vec{v}_3\right) = \frac{1}{3}\left(2\hat{i} + \hat{j} + \hat{i} + \hat{j} + \hat{k}\right)$$

$$\vec{v}_{cm} = \frac{1}{3}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$$

From eqn 7.1.3, $\vec{p} = \sum_i m_i \vec{v}_i = \vec{v}_1 + \vec{v}_2 + \vec{v}_3$

$$\vec{p} = 3\hat{i} + 2\hat{j} + \hat{k}$$

7.2 (a) From eqn. 7.2.15, $T = \sum_i \frac{1}{2} m_i v_i^2$

$$T = \frac{1}{2}\left[2^2 + 1^2 + \left(1^2 + 1^2 + 1^2\right)\right] = 4$$

(b) From Prob. 7.1, $\vec{v}_{cm} = \frac{1}{3}\left(3\hat{i} + 2\hat{j} + \hat{k}\right)$

$$\frac{1}{2} m v_{cm}^2 = \frac{1}{2} \times 3 \times \frac{1}{9}\left(3^2 + 2^2 + 1^2\right) = 2\frac{1}{3}$$

(c) From eqn.7.2.8, $\vec{L} = \sum_i \vec{r}_i \times m\vec{v}_i$

$$\vec{L} = \left[\left(\hat{i} + \hat{j}\right) \times 2\hat{i}\right] + \left[\left(\hat{j} + \hat{k}\right) \times \hat{j}\right] + \left[\hat{k} \times \left(\hat{i} + \hat{j} + \hat{k}\right)\right]$$

$$\vec{L} = \left(-2\hat{k}\right) + \left(-\hat{i}\right) + \left(\hat{j} - \hat{i}\right) = -2\hat{i} + \hat{j} - 2\hat{k}$$

7.3 $\vec{v}_\circ = \vec{v}_b - \vec{v}_g$

Since momentum is conserved and the bullet and gun were initially at rest:

$$0 = m\vec{v}_b + M\vec{v}_g$$

$$\vec{v}_g = -\gamma \vec{v}_b, \qquad \gamma = \frac{m}{M}$$

$$\vec{v}_\circ = \left(1 + \gamma\right)\vec{v}_b$$

$$\vec{v}_b = \frac{\vec{v}_\circ}{1 + \gamma}$$

$$\vec{v}_g = -\frac{\gamma \vec{v}_\circ}{1+\gamma}$$

7.4 Momentum is conserved: $mv_\circ = m\left(\frac{v_\circ}{2}\right) + Mv_{blk}$

$$v_{blk} = \frac{1}{2}\gamma v_\circ \qquad \gamma = \frac{m}{M}$$

$$T_i - T_f = \frac{1}{2}mv_\circ^2 - \left[\frac{1}{2}m\left(\frac{v_\circ}{2}\right)^2 + \frac{1}{2}M\left(\frac{\gamma v_\circ}{2}\right)^2\right]$$

$$= \frac{1}{2}mv_\circ^2\left(1 - \frac{1}{4} - \frac{1}{m}M\frac{\gamma^2}{4}\right)$$

$$\frac{T_i - T_f}{T_i} = \frac{3}{4} - \frac{\gamma}{4}$$

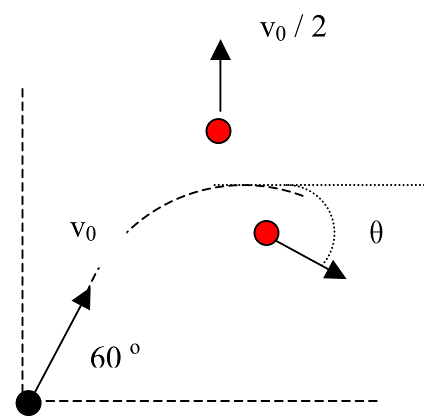
7.5 At the top of the trajectory:

$$\vec{v} = \hat{i}v_\circ \cos 60^\circ = \hat{i}\frac{v_\circ}{2}$$

Momentum is conserved:

$$\hat{i}m\frac{v_\circ}{2} = \hat{j}\left(\frac{m}{2}\right)\left(\frac{v_\circ}{2}\right) + \frac{m}{2}\vec{v}_2$$

$$\vec{v}_2 = \hat{i}v_\circ - \hat{j}\frac{v_\circ}{2}$$



Direction: $\theta = \tan^{-1}\left(\frac{-\frac{v_\circ}{2}}{v_\circ}\right) = 26.6^\circ$ below the horizontal.

Speed: $v_2 = \left[v_\circ^2 + \left(\frac{v_\circ}{2}\right)^2\right]^{\frac{1}{2}} = 1.118v_\circ$

7.6 When a ball reaches the floor, $\frac{1}{2}mv^2 = mgh$.

As a result of the bounce, $\frac{v'}{v} = \varepsilon$.

The height of the first bounce: $mgh' = \frac{1}{2}mv'^2$

$$h' = \frac{v'^2}{2g} = \frac{\varepsilon^2 v^2}{2g} = \varepsilon^2 h$$

Similarly, the height of the second bounce, $h'' = \varepsilon^2 h' = \varepsilon^4 h$

Total distance $= h + 2\varepsilon^2 h + 2\varepsilon^4 h + \ldots = h\left(-1 + \sum_{n=0}^{\infty} 2\varepsilon^{2n}\right)$

Now $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$, $|r| < 1$.

$$\left(-1 + \sum_{n=0}^{\infty} 2\varepsilon^{2n}\right) = -1 + \frac{2}{1-\varepsilon^2} = \frac{1+\varepsilon^2}{1-\varepsilon^2}$$

total distance $= h\left(\frac{1+\varepsilon^2}{1-\varepsilon^2}\right)$

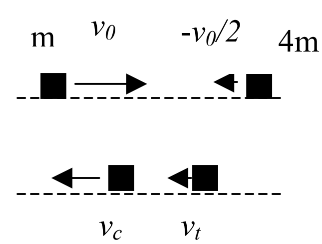
For the first fall, $\frac{1}{2} g t_\circ^2 = h$, so $t_\circ = \sqrt{\frac{2h}{g}}$

For the fall from height h': $t_1 = \sqrt{\frac{2h'}{g}} = \varepsilon\sqrt{\frac{2h}{g}}$

Accounting for equal rise and fall times:

Total time $= \sqrt{\frac{2h}{g}}\left(1 + 2\varepsilon + 2\varepsilon^2 + \ldots\right) = \sqrt{\frac{2h}{g}}\left(-1 + \sum_{n=0}^{\infty} 2\varepsilon^n\right)$

Total time $= \sqrt{\frac{2h}{g}}\left(\frac{1+\varepsilon}{1-\varepsilon}\right)$



7.7 From eqn. 7.5.5:

$$\dot{x}_1' = \frac{(m_1 - \varepsilon m_2)\dot{x}_1 + (m_2 + \varepsilon m_2)\dot{x}_2}{m_1 + m_2}$$

$$\dot{x}_2' = \frac{(m_1 + \varepsilon m_1)\dot{x}_1 + (m_2 - \varepsilon m_1)\dot{x}_2}{m_1 + m_2}$$

$$v_c = \frac{\left(m - \frac{1}{4}4m\right)v_\circ + \left(4m + \frac{1}{4}4m\right)\left(-\frac{v_\circ}{2}\right)}{m + 4m} = \frac{0 + 5m\left(-\frac{v_\circ}{2}\right)}{5m} = -\frac{v_\circ}{2}$$

$$v_t = \frac{\left(m + \frac{1}{4}m\right)v_\circ + \left(4m - \frac{1}{4}m\right)\left(-\frac{v_\circ}{2}\right)}{m + 4m}$$

$$= \frac{\frac{5}{4}mv_\circ + \frac{15}{4}m\left(-\frac{v_\circ}{2}\right)}{5m} = -\frac{v_\circ}{8}$$

Both car and truck are traveling in the initial direction of the truck with speeds $\frac{v_\circ}{2}$ and $\frac{v_\circ}{8}$, respectively.

7.8 From eqn. 7.2.15, $T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$

Meanwhile:

$$\frac{1}{2} m v_{cm}^2 + \frac{1}{2} \mu v^2 = \frac{1}{2} m \left(\frac{m_1 \vec{v}_1 + m_2 \vec{v}_2}{m} \right)^2 + \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} \left(\vec{v}_1 - \vec{v}_2 \right)^2$$

$$= \frac{1}{2m} \left[m_1^2 v_1^2 + m_2^2 v_2^2 + 2 m_1 m_2 \vec{v}_1 \cdot \vec{v}_2 + m_1 m_2 \left(v_1^2 + v_2^2 - 2 \vec{v}_1 \cdot \vec{v}_2 \right) \right]$$

$$= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

Therefore, $T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} \mu v^2$

7.9 From Prob. 7.8, $T = \frac{1}{2} m v_{cm}^2 + \frac{1}{2} \mu v^2$

$Q = T - T'$ and since $v_{cm} = v'_{cm}$:

$$Q = \frac{1}{2} \mu v^2 - \frac{1}{2} \mu v'^2$$

From eqn. 7.5.4, $\varepsilon = \frac{v'}{v}$

$$Q = \frac{1}{2} \mu v^2 \left(1 - \varepsilon^2 \right)$$

7.10 Conservation of momentum:

$$m_1 v_1 = m_1 v'_1 + m_2 v'_2$$

$$v'_1 = v_1 - \frac{m_2}{m_1} v'_2$$

$$v_1'^2 = v_1^2 - \frac{2 m_2}{m_1} v_1 v'_2 + \frac{m_2^2}{m_1^2} v_2'^2$$

Conservation of energy:

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2$$

$$\frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 v_1^2 - m_2 v_1 v'_2 + \frac{m_2^2}{2 m_1} v_2'^2 + \frac{1}{2} m_2 v_2'^2$$

$$\frac{m_2}{2} \left(\frac{m_2}{m_1} + 1 \right) v_2'^2 - m_2 v_1 v'_2 = 0$$

$$v'_2 = \frac{2 m_1 v_1}{m_1 + m_2}$$

$$T_1 - T'_1 = \frac{1}{2} m_1 v_1^2 - \frac{1}{2} m_1 v_1'^2 = \frac{1}{2} m_2 v_2'^2 = \frac{2 m_1^2 m_2}{\left(m_1 + m_2 \right)^2} v_1^2$$

$$\frac{T_1 - T_1'}{T_1} = \frac{\frac{2m_1\mu}{m}v_1^2}{\frac{1}{2}m_1 v_1^2} = \frac{4\mu}{m}$$

7.11 From eqn. 7.2.14, $\vec{L} = \vec{r}_{cm} \times m\vec{v}_{cm} + \sum_i \vec{r}'_i \times m_i \vec{v}'_i$

$$\sum_i \vec{r}'_i \times m_i \vec{v}'_i = \vec{r}'_1 \times m_1 \vec{v}'_1 + \vec{r}'_2 \times m_2 \vec{v}'_2$$

From eqn. 7.3.2, $\vec{R} = \vec{r}'_1\left(1 + \frac{m_1}{m_2}\right) = \vec{r}'_1\left(\frac{m_1 + m_2}{m_2}\right) = \frac{m_1}{\mu}\vec{r}'_1$

Since from eqn. 7.3.1, $\vec{r}'_1 = -\frac{m_2}{m_1}\vec{r}'_2$

$$\vec{R} = -\frac{m_2}{\mu}\vec{r}'_2$$

$$\sum_i \vec{r}'_i \times m_i \vec{v}'_i = \frac{\mu}{m_1}\vec{R} \times m_1 \vec{v}'_1 + \left(-\frac{\mu}{m_2}\right)\vec{R} \times m_2 \vec{v}'_2$$

$$= \mu\vec{R} \times \left(\vec{v}'_1 - \vec{v}'_2\right) = \vec{R} \times \mu\vec{v}$$

$$\vec{L} = \vec{r}_{cm} \times m\vec{v}_{cm} + \vec{R} \times \mu\vec{v}$$

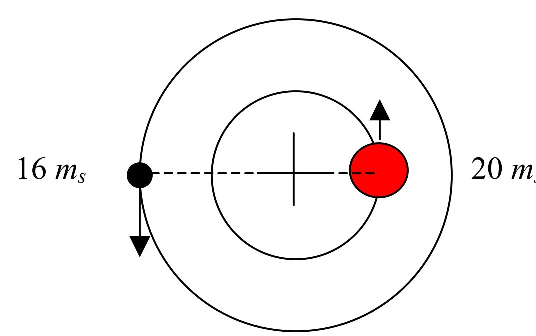
7.12 Let m_s = mass of Sun and a_e = semi-major axis of Earth's orbit then from eqn. 7.3.9c,

$$\tau = \left(\frac{m_1}{m_s} + \frac{m_2}{m_s}\right)^{-\frac{1}{2}}\left(\frac{a}{a_e}\right)^{\frac{3}{2}} yr$$

$$\frac{a}{a_e} = \tau^{\frac{2}{3}}\left(\frac{m_1}{m_s} + \frac{m_2}{m_s}\right)^{\frac{1}{3}}$$

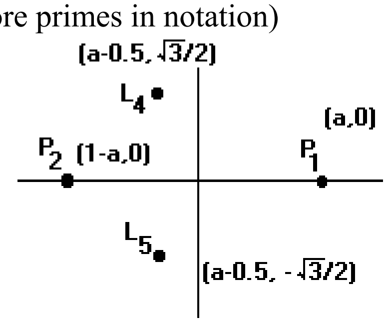
$$= \left(5.6\,da \times \frac{1\,yr}{365\,da}\right)^{\frac{2}{3}}(20+16)^{\frac{1}{3}}\,yr^{-\frac{2}{3}}$$

$$a = 0.20\,a_e = \frac{1}{5}a_e$$



7.13 (Ignore primes in notation)

a)



The coordinates of the two primaries, P1 and P2, are shown at left – along with the coordinates of L_4 and L_5.

b) $V(x,y) = -\dfrac{(1-\alpha)}{\left[(x-\alpha)^2+y^2\right]^{\frac{1}{2}}} - \dfrac{\alpha}{\left[(x+1-\alpha)^2+y^2\right]^{\frac{1}{2}}} - \dfrac{x^2+y^2}{2}$ (7.4.13)

$$\frac{\partial V}{\partial x} = \frac{(1-\alpha)(x-\alpha)}{[\ ]^{\frac{3}{2}}} + \frac{\alpha(x+1-\alpha)}{[\ ]^{\frac{3}{2}}} - x$$

Now $x = \alpha - 0.5$ at L_4 and L_5

also, each bracket term in the denominator equals 1 at L_4, L_5

$$\frac{\partial V}{\partial x} = (1-\alpha)(\alpha - 0.5 - \alpha) + \alpha(\alpha - 0.5 + 1 - \alpha) - (\alpha - 0.5)$$

$$= -0.5 + 0.5\alpha + 0.5\alpha - \alpha + 0.5 \equiv 0$$

$$\frac{\partial V}{\partial y} = \frac{(1-\alpha)y}{[\ ]^{\frac{3}{2}}} + \frac{\alpha y}{[\ ]^{\frac{3}{2}}} - y$$

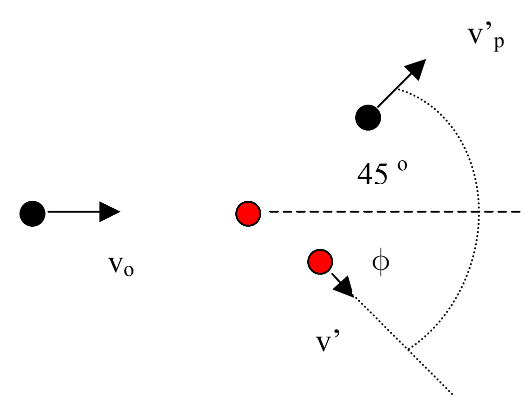
Again, the denominator in brackets equals 1 @ L_4, L_5

So, $$\frac{\partial V}{\partial y} = (1-\alpha)\left(\pm\frac{\sqrt{3}}{2}\right) + \alpha\left(\pm\frac{\sqrt{3}}{2}\right) - \left(\pm\frac{\sqrt{3}}{2}\right)$$

$$= \pm\frac{\sqrt{3}}{2} \mp \alpha\frac{\sqrt{3}}{2} \pm \alpha\frac{\sqrt{3}}{2} \mp \frac{\sqrt{3}}{2} \equiv 0$$

Thus $\bar{\nabla} V(x,y) = \hat{i}\dfrac{\partial V}{\partial x} + \hat{j}\dfrac{\partial V}{\partial y} \equiv 0$ at L_4, L_5.

7.14 Conservation of momentum:



$$m_p \vec{v}_\circ = m_p \vec{v}'_p + 4m_p \vec{v}'_\alpha$$

$$v_\circ = v'_p \cos 45^\circ + 4v'_\alpha \cos\phi$$

$$4v'_\alpha \cos\phi = v_\circ - \frac{v'_p}{\sqrt{2}}$$

$$0 = v'_p \sin 45^\circ - 4v'_\alpha \sin\phi$$

$$4v'_\alpha \sin\phi = \frac{v'_p}{\sqrt{2}}$$

$$16v'^2_\alpha = v_\circ^2 - \sqrt{2}\, v_\circ v'_p + v'^2_p$$

Conservation of energy:

$$\frac{1}{2}m_p v_\circ^2 = \frac{1}{2}m_p v'^2_p + \frac{1}{2}4m_p v'^2_\alpha$$

$$16v'^2_\alpha = 4v_\circ^2 - 4v'_p$$

Subtracting: $0 = -3v_\circ^2 - \sqrt{2}\, v_\circ v'_p + 5v'^2_p$

$$v'_p = \frac{\sqrt{2}v_\circ \pm \sqrt{2v_\circ^2 + 60v_\circ^2}}{10} = \frac{v_\circ}{10}\left(\sqrt{2} \pm \sqrt{62}\right)$$

$v'_p > 0$, so the positive square root is used.

$$v'_p = 0.9288\, v_\circ$$

$$v'_{px} = v'_{py} = \frac{v'_p}{\sqrt{2}} = 0.657\, v_\circ$$

$$v'_\alpha = \frac{1}{2}\left(v_\circ^2 - v_p'^2\right)^{\frac{1}{2}} = \frac{v_\circ}{2}\left(1 - .9288^2\right)^{\frac{1}{2}}$$

$$v'_\alpha = 0.1853 v_\circ$$

$$\tan\phi = \frac{\frac{v'_p}{\sqrt{2}}}{v_\circ - \frac{v'_p}{\sqrt{2}}} = \frac{v'_p}{\sqrt{2}v_\circ - v'_p} = \frac{.9288}{\sqrt{2} - .9288}$$

$$\phi = \tan^{-1} 1.9134 = 62.41^\circ$$

$$v'_{\alpha x} = v'_\alpha \cos\phi = 0.086\, v_\circ$$

$$v'_{\alpha y} = -v'_\alpha \sin\phi = -0.164\, v_\circ$$

7.15 Conservation of energy:

$$\frac{1}{2} m_p v_\circ^2 = \frac{1}{2} m_p v_p'^2 + \frac{1}{2} 4 m_p v_\alpha'^2 + \frac{1}{4}\left(\frac{1}{2} m_p v_\circ^2\right)$$

$$16 v_\alpha'^2 = 3 v_\circ^2 - 4 v_p'^2$$

From the conservation of momentum eqn of Prob. 7.14:

$$16 v_\alpha'^2 = v_\circ^2 - \sqrt{2} v_\circ v'_p + v_p'^2$$

Subtracting: $0 = -2v_\circ^2 - \sqrt{2} v_\circ v'_p + 5 v_p'^2$

$$v'_p = \frac{\sqrt{2}v_\circ \pm \sqrt{2v_\circ^2 + 40v_\circ^2}}{10} = \frac{v_\circ}{10}\left(\sqrt{2} \pm \sqrt{42}\right)$$

Using the positive square root, since $v'_p > 0$:

$$v'_p = 0.7895\, v_\circ$$

$$v'_{px} = v'_{py} = \frac{v'_p}{\sqrt{2}} = 0.558\, v_\circ$$

$$v'_\alpha = \left(\frac{3}{16} v_\circ^2 - \frac{1}{4} v_p'^2\right)^{\frac{1}{2}} = \frac{v_\circ}{2}\left(.75 - .7895^2\right)^{\frac{1}{2}}$$

$$v'_\alpha = 0.1780 v_\circ$$

From the conservation of momentum eqns of Prob. 7.14:

$$\tan\phi = \frac{v'_p}{\sqrt{2}v_\circ - v'_p} = \frac{.7895}{\sqrt{2} - .7895}$$

$$\phi = \tan^{-1} 1.2638 = 51.65^\circ$$
$$v'_{\alpha x} = v'_\alpha \cos\phi = 0.110\, v_\circ$$
$$v'_{\alpha y} = -v'_\alpha \sin\phi = -0.140\, v_\circ$$

7.16 From eqn. 7.6.14, $\tan\phi_1 = \dfrac{\sin\theta}{\gamma + \cos\theta}$ ϕ_1 and θ are the scattering angles in the Lab and C.M. frames respectively.

From eqn. 7.6.16, for Q = 0, $\gamma = \dfrac{m_1}{m_2}$

$$\tan 45^\circ = 1 = \frac{\sin\theta}{\frac{1}{4} + \cos\theta}$$

$$\frac{1}{4} + \cos\theta = \sin\theta \qquad \text{and squaring } \ldots$$

$$\frac{1}{16} + \frac{1}{2}\cos\theta + \cos^2\theta = 1 - \cos^2\theta$$

$$2\cos^2\theta + \frac{1}{2}\cos\theta - \frac{15}{16} = 0$$

$$\cos\theta = \frac{-\frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{15}{2}}}{4} = -.125 \pm .696$$

Since $0 < \theta < \dfrac{\pi}{2}$, $\quad \theta = \cos^{-1} .571 \approx 55.2^\circ$

7.17 From eqn. 7.6.14, $\tan\phi_1 = \dfrac{\sin\theta}{\gamma + \cos\theta}$

From eqn. 7.6.18, $\gamma = \dfrac{m_1}{m_2}\left[1 - \dfrac{Q}{T}\left(1 + \dfrac{m_1}{m_2}\right)\right]^{-\frac{1}{2}}$

$$\gamma = \frac{1}{4}\left[1 - \frac{1}{4}\left(1 + \frac{1}{4}\right)\right]^{-\frac{1}{2}} = 0.3015$$

$$\tan 45^\circ = \frac{\sin\theta}{.3015 + \cos\theta}$$

$.3015 + \cos\theta = \sin\theta$ (since $\sin\theta > \cos\theta$, $\theta > 45^\circ$)

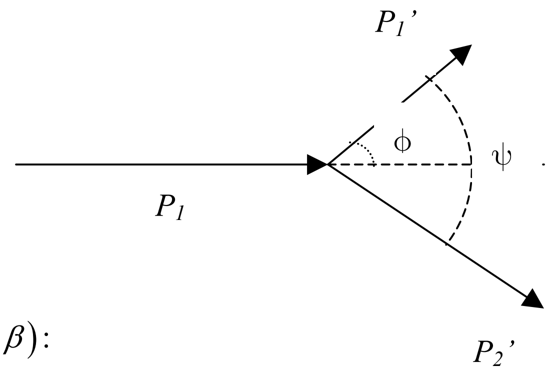
$$.3015^2 = \sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta$$

Using the identity $2\sin\theta\cos\theta = \sin 2\theta$

$$\sin 2\theta = 1 - .3015^2 = 0.9091$$

Since $\theta > 45^\circ$, $2\theta > 90^\circ$: $\quad 2\theta = \sin^{-1} .9091 = 114.62^\circ$

$$\theta = 57.3^\circ$$



7.18 Conservation of momentum:

$$P_1 = P_1' \cos\phi + P_2' \cos(\psi - \phi)$$

$$0 = P_1' \sin\phi - P_2' \sin(\psi - \phi)$$

From Appendix B for $\sin(\alpha+\beta)$ and $\cos(\alpha+\beta)$:

$$P_1 = P_1' \cos\phi + P_2' (\cos\psi\cos\phi + \sin\psi\sin\phi)$$

$$0 = P_1' \sin\phi - P_2' (\sin\psi\cos\phi - \cos\psi\sin\phi)$$

$$P_1^2 = P_1'^2 \cos^2\phi + P_2'^2 \left(\cos^2\psi\cos^2\phi + 2\cos\psi\cos\phi\sin\phi\sin\psi + \sin^2\psi\sin^2\phi\right)$$
$$+2P_1'P_2'\left(\cos^2\phi\cos\psi + \cos\phi\sin\psi\sin\phi\right)$$

$$0 = P_1'^2 \sin^2\phi + P_2'^2 \left(\sin^2\psi\cos^2\phi - 2\sin\psi\cos\phi\cos\psi\sin\phi + \cos^2\psi\sin^2\phi\right)$$
$$-2P_1'P_2'\left(\sin\phi\sin\psi\cos\phi - \cos\psi\sin^2\phi\right)$$

Adding: $P_1^2 = P_1'^2 + P_2'^2 + 2P_1'P_2'\cos\psi$

Conservation of energy:

$$\frac{P_1^2}{2m} = \frac{P_1'^2}{2m} + \frac{P_2'^2}{2m} + Q$$

$$Q = \frac{1}{2m}\left(P_1^2 - P_1'^2 - P_2'^2\right) = \frac{1}{2m}\left(2P_1'P_2'\cos\psi\right)$$

$$Q = \frac{P_1'P_2'\cos\psi}{m}$$

7.19 $T_1 = \frac{1}{2}m_1v_1^2$ $\qquad T_1' = \frac{1}{2}m_1v_1'^2$

let $r = \frac{T_1'}{T_1} = \frac{v_1'^2}{v_1^2}$ … ratio of scattered particle to incident particle energy

Looking at Figure 7.6.2 …

$$\bar{v}_1' \cdot \bar{v}_1' = (v_1' - v_{cm}) \cdot (v_1' - v_{cm})$$

$$\bar{v}_1'^2 = v_1'^2 + v_{cm}^2 - 2v_1'v_{cm}\cos\phi_1$$

hence $v_1'^2 = \bar{v}_1'^2 - v_{cm}^2 + 2v_1'v_{cm}\gamma$ $\qquad$ where $\gamma = \cos\phi_1$

$$\therefore r = \frac{\bar{v}_1'^2}{v_1^2} - \frac{v_{cm}^2}{v_1^2} + \frac{2v_1'v_{cm}\gamma}{v_1^2}$$

but $\bar{v}_1' = \bar{v}_1$ …the center of mass speeds of the incident and scattered particle are the same.

$$\frac{\bar{v}_1'}{v_1} = \frac{m_2}{m_2 + m_1} = \frac{\alpha}{1+\alpha}$$ …from equation 7.6.12 where $\alpha = \frac{m_2}{m_1}$

$$\frac{v_{cm}}{v_1} = \frac{m_1}{m_2 + m_1} = \frac{1}{1+\alpha} \qquad \text{Equation 7.6.11}$$

Thus

$$r = \frac{\alpha^2}{(1+\alpha)^2} - \frac{1}{(1+\alpha)^2} + \frac{2\gamma}{(1+\alpha)} \qquad \frac{v_1'}{v_1} = \frac{\alpha^2}{(1+\alpha)^2} - \frac{1}{(1+\alpha)^2} + \frac{2\gamma}{(1+\alpha)} r^{\frac{1}{2}}$$

Simplifying

$$r - \frac{2\gamma}{1+\alpha} r^{\frac{1}{2}} + \left(\frac{1-\alpha}{1+\alpha}\right) = 0$$

Let $x^2 = r$ and solving the resulting quadratic for x

$$x = \frac{\gamma}{1+\alpha} + \frac{1}{1+\alpha}\left[\gamma^2 - \left(1-\alpha^2\right)\right]^{\frac{1}{2}}$$

Squaring

$$r = x^2 = \frac{1}{(1+\alpha)^2}\left[2\gamma^2 + \alpha^2 - 1 + 2\gamma\left(\gamma^2 + \alpha^2 - 1\right)^{\frac{1}{2}}\right]$$

Now $\frac{\Delta T_1}{T_1} = 1 - r = 1 - \frac{1}{(1+\alpha)^2}\left[2\gamma^2 + \alpha^2 - 1 + 2\gamma\left(\gamma^2 + \alpha^2 - 1\right)^{\frac{1}{2}}\right]$

And, after a little algebra, we get the desired solution

$$\frac{\Delta T_1}{T_1} = \frac{2}{1+\alpha} - \frac{2\gamma}{(1+\alpha)^2}\left[\gamma + \sqrt{\gamma^2 + \alpha^2 - 1}\right]$$

7.20 From Equation 7.6.15 … $\gamma = \frac{m_1 v_1}{\bar{v}_1'(m_1 + m_2)} = \frac{m_1}{m_2(1 + m_1/m_2)}\frac{v_1}{\bar{v}_1'}$

Now we solve for $\frac{v_1}{\bar{v}_1'}$ …

$v_1 = \left(\frac{2T}{m_1}\right)^{\frac{1}{2}}$ and now solving for $\bar{v}_1'$ starting with Equation 7.6.9 …

$\frac{1}{2}\mu\bar{v}_1'^2 = \frac{1}{2}\mu\bar{v}_1^{\,2} - Q$ and using $\bar{v}_1' = \frac{m_2}{m_1 + m_2} v_1$ we get …

$$= \frac{1}{2} m_1 v_1^{\,2} \frac{1}{(1 + m_1/m_2)} - Q = \frac{T}{(1 + m_1/m_2)} - Q$$

$$\bar{v}_1'^2 = \frac{2}{m_1(1 + m_1/m_2)}\left[\frac{T}{(1 + m_1/m_2)} - Q\right].$$

Thus, solving for γ …

$$\gamma = \frac{m_1}{m_2} \frac{\left(2T/m_1\right)^{\frac{1}{2}}}{\left(2/m_1\right)^{\frac{1}{2}} \left(1 + m_1/m_2\right)^{\frac{1}{2}} \left[\frac{T}{\left(1 + m_1/m_2\right)} - Q \right]^{\frac{1}{2}}}$$

Finally…

$$\gamma = \frac{m_1}{m_2} \frac{1}{\left[1 - \frac{Q\left(1 + m_1/m_2\right)}{T} \right]^{\frac{1}{2}}}$$

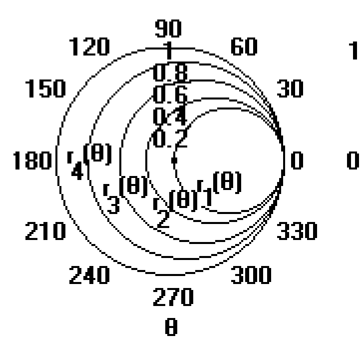
7.21 The time of flight, $\tau = \text{constant}$—so $\tau = \frac{r}{v_1'}$ but from problem 7.19 above

$$r = v_1' \tau = \frac{v_1 \tau}{1+\alpha} \left[\gamma + \sqrt{\gamma^2 + \alpha^2 - 1} \right]$$

As an example, let $v_1 \tau = 1$ and we have

$r_1 = \gamma$	$\alpha = 1$	pp scattering
$r_2 = \frac{1}{3}\left[\gamma + \sqrt{\gamma^2 + 3} \right]$	$\alpha = 2$	p – D
$r_3 = \frac{1}{5}\left[\gamma + \sqrt{\gamma^2 + 15} \right]$	$\alpha = 4$	p – He
$r_4 = \frac{1}{13}\left[\gamma + \sqrt{\gamma^2 + 143} \right]$	$\alpha = 12$	p – C

Below is a polar plot of these four curves.



7.22 From eqn. 7.7.6, $F_u - F_g = m\dot{v} + v\dot{m}$

since $v = \text{constant}$, $\dot{v} = 0$

$\dot{m} = \lambda \dot{z} = \lambda v$, $\qquad \lambda = \text{mass per unit length}$

$F_g = (\lambda z) g$

$$F_u = \lambda z g + (\lambda v) v = g\lambda \left(z + \frac{v^2}{g} \right)$$

F_u is equal to the weight of a length $z + \frac{v^2}{g}$ of chain.

7.23 $m = \frac{4}{3}\pi r^3 \rho$

$\dot{m} = 4\pi r^2 \rho \dot{r} \quad \propto \quad \pi r^2 \dot{z}$ where $v = \dot{z}$

$\dot{r} = k\dot{z}$ k a constant of proportionality

$r = r_\circ + kz$

From eqn. 7.7.6, $mg = m\dot{v} + v\dot{m}$

$$\frac{4}{3}\pi r^3 \rho g = \frac{4}{3}\pi r^3 \rho \ddot{z} + 4\pi r^2 \rho \left(k\dot{z}\right)\dot{z}$$

$$g = \ddot{z} + \frac{3k\dot{z}^2}{r}$$

$$\ddot{z} = g - \frac{3\dot{z}^2}{z + \frac{r_\circ}{k}}$$

For $r_\circ = 0$, $\ddot{z} = g - \frac{3\dot{z}^2}{z}$

A series solution is used for this differential equation:

$$\dot{z}^2 = \sum_{n=0}^{\infty} a_n z^n$$

$$\ddot{z} = \frac{d\dot{z}}{dt} = \frac{d\dot{z}}{dz}\cdot\frac{dz}{dt} = \dot{z}\frac{d\dot{z}}{dz} = \frac{1}{2}\frac{d(\dot{z}^2)}{dz}$$

$$\frac{d(\dot{z}^2)}{dz} = \sum_n a_n n z^{n-1}$$

$$\frac{\dot{z}^2}{z} = \sum_n a_n z^{n-1}$$

$$\therefore \quad \ddot{z} = \frac{1}{2}\sum_n a_n n z^{n-1} = g - 3\sum_n a_n z^{n-1}$$

For $n = 1$: $\frac{1}{2}a_1 = g - 3a_1$

$$a_1 = \frac{2}{7}g$$

For $n \neq 1$: $\frac{1}{2}na_n = -3a_n$

Since n is an integer, $a_n = 0$ for $n \neq 1$

$$\dot{z}^2 = \frac{2}{7}g\,z$$

$$\ddot{z} = g - \frac{3}{z}\left(\frac{2}{7}g\,z\right) = \frac{g}{7}$$

7.24 From eqn. 7.7.6, $mg = m\dot{v} + v\dot{m}$, where m and v refer to the portion of the chain hanging over the edge of the table.

$m = \lambda z$ and $v = \dot{z}$ where λ is the mass per unit length of chain

$\dot{m} = \lambda \dot{z}$ and $\dot{v} = \ddot{z}$

$$\ddot{z} = \frac{d\dot{z}}{dt} = \frac{d\dot{z}}{dz}\cdot\frac{dz}{dt} = \dot{z}\frac{d\dot{z}}{dz} = \frac{1}{2}\frac{d(\dot{z}^2)}{dz}$$

$$\lambda z g = \lambda z\left(\frac{1}{2}\frac{d(\dot{z}^2)}{dz}\right) + \dot{z}(\lambda\dot{z})$$

$$\ddot{z} = \frac{1}{2}\frac{d(\dot{z}^2)}{dz} = g - \frac{\dot{z}^2}{z}$$

Because of the initial condition $z_\circ = b \neq 0$, a normal power series solution to this differential equation (…as in Prob. 7.22) does not work. Instead, we use the Method of Frobenius …

$$\dot{z}^2 = \sum_{n=0}^{\infty} a_n z^{n+s}$$

$$\frac{d(\dot{z}^2)}{dz} = \sum_n a_n (n+s) z^{n+s-1}$$

$$\frac{\dot{z}^2}{z} = \sum_n a_n z^{n+s-1}$$

$$\ddot{z} = \frac{1}{2}\sum_n a_n (n+s) z^{n+s-1} = g - \sum_n a_n z^{n+s-1}$$

Equality can be attained for $a_n \neq 0$ at $n = 0$ and $n = 3$

…otherwise $a_n = 0$ $\quad n \neq 0,3$

For $n = 0$, $\frac{1}{2}a_\circ s = -a_\circ$

$$s = -2$$

$$\ddot{z} = \frac{1}{2}\sum_n a_n (n-2) z^{n-3} = g - \sum_n a_n z^{n-3}$$

For $n = 3$, $\frac{1}{2}a_3 = g - a_3$

$$a_3 = \frac{2}{3}g$$

For all $n \neq 0$, 3: $\frac{1}{2}a_n(n-2) = -a_n$

$$a_n = 0, \qquad n \neq 0,3.$$

$$\dot{z}^2 = a_\circ z^{-2} + \frac{2}{3}gz$$

At $t=0$, $\dot{z}=0$, and $z=b$

$$0=\frac{a_{\circ}}{b^2}+\frac{2gb}{3}$$

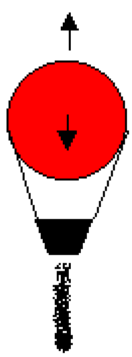
$$a_{\circ}=-\frac{2}{3}gb^3$$

$$\dot{z}^2=-\frac{2}{3}g\frac{b^3}{z^2}+\frac{2}{3}gz$$

At $z=a$, $\dot{z}^2=\frac{2}{3}g\left(a-\frac{b^3}{a^2}\right)=\frac{2g}{3a^2}\left(a^3-b^3\right)$

$$\dot{z}=\left[\frac{2g}{3a^2}\left(a^3-b^3\right)\right]^{\frac{1}{2}}$$

7.25 Initially, the upward buoyancy force balances the weight of the balloon and sand.



$$F_B-\left(M+m_{\circ}\right)g=0 \qquad (1)$$

Let $m=m(t)$ – the mass of sand at time t where $0\le t\le t_{\circ}$

$$m=m_{\circ}\left(1-\frac{t}{t_{\circ}}\right) \qquad (2)$$

The velocity of sand relative to the balloon is zero upon release so $\bar{V}=0$ in equation 7.7.5 … there is no upward "rocket-thrust."

As sand is released, the net upward force is the difference between the initial buoyancy force, F_B, and the weight of the balloon and remaining sand. Let y be the subsequent displacement of the balloon, so equation 7.7.5 reduces to $F = ma$

$$F_B-\left(M+m\right)g=\left(M+m\right)\frac{dv}{dt}$$

and using (1) and (2) above we get

$$\frac{dv}{dt}=\frac{m_{\circ}gt}{\left(M+m_{\circ}\right)t_{\circ}-m_{\circ}t}=-g+\frac{\left(M+m_{\circ}\right)gt_{\circ}}{\left(M+m_{\circ}\right)t_{\circ}-m_{\circ}t}$$

whose solution is:

$$v=\frac{dy}{dt}=-gt-\frac{\left(M+m_{\circ}\right)gt_{\circ}}{m_{\circ}}\ln\left(1-\frac{m_{\circ}t}{\left(M+m_{\circ}\right)t_{\circ}}\right)$$

$$y=C-\int\left(gt+\frac{g}{k}\ln\left(1-kt\right)\right)dt\,, \qquad k=\frac{m_{\circ}}{t_{\circ}\left(M+m_{\circ}\right)}$$

$$=C-\frac{1}{2}gt^2-\frac{gt}{k}\ln\left(1-kt\right)-g\int\frac{tdt}{1-kt}$$

Integrating by parts

$$= C - \frac{1}{2}gt^2 - \frac{gt}{k}\ln(1-kt) - \frac{g}{k}\int\left(-1 + \frac{1}{1-kt}\right)dt$$

$$= C - \frac{1}{2}gt^2 - \frac{gt}{k}\ln(1-kt) + \frac{gt}{k} + \frac{g}{k^2}\ln(1-kt)$$

$$= C - \frac{1}{2}gt^2 + \frac{g}{k^2}(1-kt)\ln(1-kt) + \frac{gt}{k}$$

but $y = 0$ at $t = 0$ so $C = 0$

$$y = \frac{gt}{k} - \frac{1}{2}gt^2 + \frac{g}{k^2}(1-kt)\ln(1-kt)$$

and at $t = t_\circ$

(a) $$H = \frac{gt_\circ^2}{2m_\circ^2}\left[(2M + m_\circ)m_\circ + 2M(M + m_\circ)\ln\left(\frac{M}{M + m_\circ}\right)\right]$$

(b) $$v = \frac{gt_\circ}{m_\circ}\left[(M + m_\circ)\ln\frac{(M + m_\circ)}{M} - m_\circ\right]$$

(c) letting $\varepsilon = \frac{m_\circ}{M} << 1$ we have

$$H = \frac{gt_\circ^2}{2\varepsilon^2}\left[(2+\varepsilon)\varepsilon - 2(1+\varepsilon)\ln(1+\varepsilon)\right]$$

$$= \frac{gt_\circ^2}{2\varepsilon^2}\left[2\varepsilon + \varepsilon^2 - 2(1+\varepsilon)\left(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \ldots\right)\right]$$

$$H \cong \frac{gt_\circ^2}{6}\varepsilon$$

Similarly:

$$v = \frac{gt_\circ}{\varepsilon}\left[(1+\varepsilon)\ln(1+\varepsilon) - \varepsilon\right]$$

$$= \frac{gt_\circ}{\varepsilon}\left[(1+\varepsilon)\left(\varepsilon - \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} - \ldots\right) - \varepsilon\right]$$

$$\approx \frac{1}{2}gt_\circ\varepsilon$$

(d) $H = 327m$; $v = 9.8ms^{-1}$

7.26 $\dot{m} = -k$ or $m = m_\circ - kt$

Burn-out occurs at time $T = \frac{\varepsilon m_\circ}{k}$

So – the rocket equation (7.7.7) becomes

$$m\frac{dv}{dt} = -V\dot{m}$$ (-) since V is oppositely directed to $\dot{v}$

$$\frac{dv}{dt} = +\frac{Vk}{m_\circ - kt}$$

Thus

$$v = Vk\int \frac{dt}{m_\circ - kt} = -V\ln\left(m_\circ - kt\right) + C_1 \qquad \text{where } C_1 \text{ is a constant.}$$

Now, $v = 0 @ t = 0$ so $C_1 = V\ln m_\circ$

Hence $v = -V\ln\left[\frac{(m_\circ - kt)}{m_\circ}\right], \qquad 0 \le t \le \frac{\varepsilon m_\circ}{k}$

Let y be the displacement at the time t so

$$y = -V\int \ln\left[\frac{(m_\circ - kt)}{m_\circ}\right]dt + C_2$$

Integrating the above expression by parts

$$y = -Vt\ln\left[\frac{(m_\circ - kt)}{m_\circ}\right] - Vk\int \frac{t\,dt}{m_\circ - kt} + C_2$$

$$= -Vt\ln\left[\frac{(m_\circ - kt)}{m_\circ}\right] + V\int\left(1 - \frac{m_\circ}{m_\circ - kt}\right)dt + C_2$$

$$= -Vt\ln\left[\frac{(m_\circ - kt)}{m_\circ}\right] + Vt + \frac{Vm_\circ}{k}\ln\left(m_\circ - kt\right) + C_2$$

since $y = 0$ at $t = 0$, $C_2 = \frac{-Vm_\circ \ln m_\circ}{k}$ and we have

$$y = Vt + \frac{V}{k}\left(m_\circ - kt\right)\ln\left[\frac{(m_\circ - kt)}{m_\circ}\right]$$

At burn-out $t = T = \frac{\varepsilon m_\circ}{k}$ so

(a) $\qquad y(\varepsilon) = D = \frac{m_\circ V}{k}\left[\varepsilon + (1-\varepsilon)\ln(1-\varepsilon)\right]$

(b) $\qquad \varepsilon$ cannot exceed 1.0 although it can approach 1.0 for small payloads

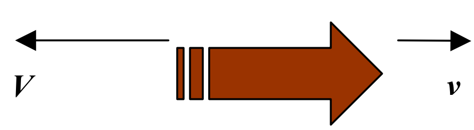
Thus $y_{\max} = \lim_{\varepsilon \to 1} y(\varepsilon) = \frac{m_\circ V}{k}$

7.27 From eqn. 7.7.5, $-k\vec{v} = m\dot{\vec{v}} - \vec{V}\dot{m}$

Since $\vec{V}$ is opposite in direction from $\vec{v}$

$$-kv = m\dot{v} + V\dot{m}$$

$$v = -\frac{m}{k}\dot{v} - \frac{\dot{m}}{k}V$$



$\alpha = \left|\frac{\dot{m}}{k}\right|$ and since $\dot{m} < 0$, $\alpha = -\frac{\dot{m}}{k}$

$$v-\alpha V=\alpha\frac{m}{\dot m}\dot v=\alpha\frac{m}{\dot m}\frac{dv}{dt}=\alpha\frac{m}{\dot m}\frac{dv}{dm}\frac{dm}{dt}=\alpha m\frac{dv}{dm}$$

$$\frac{dm}{\alpha m}=\frac{dv}{v-V\alpha}$$

$$\frac{1}{\alpha}\int_{m_\circ}^{m}\frac{dm}{m}=\int_0^v\frac{dv}{v-V\alpha}$$

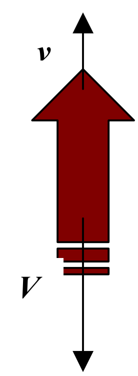
$$\frac{1}{\alpha}\ln\left(\frac{m}{m_\circ}\right)=\ln\left(\frac{v-V\alpha}{-V\alpha}\right)$$

$$\left(\frac{m}{m_\circ}\right)^{\frac{1}{\alpha}}=-\frac{v}{V\alpha}+1$$

$$v=V\alpha\left[1-\left(\frac{m}{m_\circ}\right)^{\frac{1}{\alpha}}\right]$$

7.28 From eqn. 7.7.5, $-m\vec{g}=m\dot{\vec{v}}-\vec{V}\dot m$

since $\vec{V}$ is opposite in direction from $\vec{v}$,



$$-mg=m\dot v+V\dot m$$

$$-mgdt=mdv+Vdm$$

$$\dot m=\frac{dm}{dt}\qquad \text{so } dt=\frac{dm}{\dot m}$$

$$-mg\frac{dm}{\dot m}=mdv+Vdm$$

$$dv=-dm\left(\frac{g}{\dot m}+\frac{V}{m}\right)$$

$$\int_0^{v_e}dv=-\int_{m_\circ}^{m_p}dm\left(\frac{g}{\dot m}+\frac{V}{m}\right)\qquad m_p=\text{payload mass}$$

$$v_e=\frac{g}{\dot m}\left(m_\circ-m_p\right)+V\ln\frac{m_\circ}{m_p}$$

$$m_f=m_\circ-m_p\approx m_\circ\qquad m_f=\text{fuel mass}$$

$$v_e=\frac{g}{\dot m}m_f+V\ln\left(1+\frac{m_f}{m_p}\right)$$

$$\ln\left(1+\frac{m_f}{m_p}\right)=\frac{v_e}{V}-\frac{g}{V}\frac{m_\circ}{\dot m}$$

$$\frac{m_f}{m_p}=\exp\left(\frac{v_e}{V}-\frac{g}{V}\frac{m_\circ}{\dot m}\right)-1$$

For $V = kv_e$, $\quad \dfrac{m_f}{m_p} = \exp\left(\dfrac{1}{k} - \dfrac{g}{kv_e}\dfrac{m_\circ}{\dot{m}}\right) - 1$

From chap. 2, Section 2.3 … $\quad v_e \approx 11\dfrac{km}{s}$

For $|\dot{m}| = 0.01 m_\circ s^{-1}$ and $k = \dfrac{1}{4}$:

$$\frac{m_f}{m_p} = \exp\left[4 - \frac{9.8}{\frac{1}{4}(11{,}000)(-.01)}\right] - 1$$

$$\frac{m_f}{m_p} = 77$$

7.29 We can use Equation 7.7.9 to calculate the final velocity attained by the ion rocket during the 100 hour burn. Assuming the rocket starts from rest (even if the ion rocket is turned on while in Earth orbit, the initial rocket speed $v_0 \approx 10^{-4} c \approx 0$. Thus …

$v \approx V \ln \dfrac{m_0}{m_p}$ and $m_0 = m_F + m_p = 2m_p + m_p = 3m_p$

$v \approx V \ln 3 = 0.1099 c$. The final rocket velocity is a little more than 10% c.

$$T = \frac{L}{0.1099c} = \frac{4\,LY}{0.1099\,LY/yr} = 36.4\,yr$$

7.30 We again use Equation 7.7.9 …

$v \approx V_{ion} \ln \dfrac{m_0}{m_p} = V_{ion} \ln \dfrac{m_F + m_p}{m_p} = V_{ion} \ln 2$ for the ion rocket. For the chemical rocket …

$v \approx V_{chem} \ln \dfrac{m_F + m_p}{m_p}$. Setting these two equations equal …

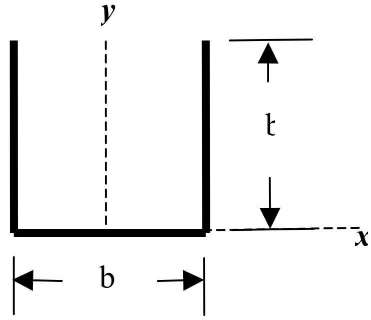
$V_{chem} \ln \dfrac{m_F + m_p}{m_p} = V_{ion} \ln 2$. Solving for m_F …

$\dfrac{m_F + m_p}{m_p} = 2^{(V_{ion}/V_{chem})} = 2^{\left(0.1c/10^{-5}c\right)} = 2^{10^4} \approx 10^{500}$ which demonstrates the virtue of ejecting mass at high velocity!

CHAPTER 8
MECHANICS OF RIGID BODIES: PLANAR MOTION

8.1 (a) For each portion of the wire having a mass $\frac{m}{3}$ and centered at

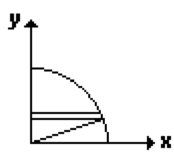
$\left(-\frac{b}{2},\frac{b}{2}\right)$, $(0,0)$, and $\left(\frac{b}{2},\frac{b}{2}\right)$ …



$$x_{cm}=\frac{1}{m}\left[-\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)+0+\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)\right]=0$$

$$y_{cm}=\frac{1}{m}\left[\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)+0+\left(\frac{b}{2}\right)\left(\frac{m}{3}\right)\right]=\frac{b}{3}$$

(b)



$$ds=xdy=\left(b^2-y^2\right)^{\frac{1}{2}}dy$$

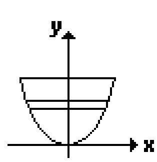
$$y_{cm}=\frac{1}{m}\int_0^b \rho y\left(b^2-y^2\right)^{\frac{1}{2}}dy$$

$$y_{cm}=\frac{-\frac{\rho}{2}\int_{y=0}^{y=b}\left(b^2-y^2\right)^{\frac{1}{2}}d\left(b^2-y^2\right)}{\frac{1}{4}\pi b^2\rho}$$

$$y_{cm}=\frac{4b}{3\pi}$$

From symmetry, $x_{cm}=\frac{4b}{3\pi}$

(c)



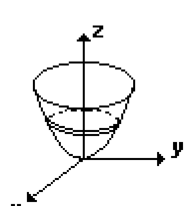
The center of mass is on the y-axis.

$$ds=2xdy=2\left(by\right)^{\frac{1}{2}}dy$$

$$y_{cm}=\frac{\int_0^b 2\rho y\left(by\right)^{\frac{1}{2}}dy}{\int_0^b 2\rho\left(by\right)^{\frac{1}{2}}dy}=\frac{\int_0^b y^{\frac{3}{2}}dy}{\int_0^b y^{\frac{1}{2}}dy}$$

$$y_{cm}=\frac{3b}{5}$$

(d)



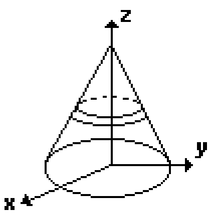
The center of mass is on the z-axis.

$$dv=\pi r^2dz=\pi\left(x^2+y^2\right)dz=\pi bzdz$$

$$z_{cm} = \frac{\int_0^b \rho z \pi b z dz}{\int_0^b \rho \pi b z dz} = \frac{\int_0^b z^2 dz}{\int_0^b z dz}$$

$$z_{cm} = \frac{2}{3} b$$

(e)



The center of mass is on the z-axis.
α is the half-angle of the apex of the cone. $r_\circ$ is the radius of the base at $z = 0$ and r is radius of a circle at some arbitrary z in a plane parallel to the base.

$$\tan \alpha = \frac{r_\circ}{b} = \frac{r}{b - z} \text{, a constant}$$

$$dv = \pi r^2 dz = \pi \left(b - z\right)^2 \tan^2 \alpha \, dz$$

$$m = \rho \frac{1}{3} \pi r_\circ^2 b = \frac{1}{3} \pi \rho b^3 \tan^2 \alpha$$

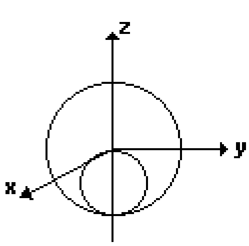
$$z_{cm} = \frac{\int_0^b \rho z \pi \left(b - z\right)^2 \tan^2 \alpha \, dz}{\frac{1}{3} \pi \rho b^3 \tan^2 \alpha} = \frac{3}{b^3} \int_0^b \left(b^2 z - 2bz^2 + z^3\right) dz$$

$$z_{cm} = \frac{b}{4}$$

8.2 $$x_{cm} = \frac{\int \rho x dx}{\int \rho dx} = \frac{\int_0^b c x^2 dx}{\int_0^b c x dx}$$

$$x_{cm} = \frac{2b}{3}$$

8.3 The center of mass is on the z-axis. Consider the sphere with the cavity to be made



of a (i) solid sphere of radius a and mass M_s, with its center of mass at $z = 0$, and (ii) a solid sphere the size of the cavity, with mass $-M_c$ and center of mass at $z = -\frac{a}{2}$. The actual sphere with the cavity has a mass $m = M_s - M_c$ and center of mass z_{cm}.

$$0 = \frac{1}{M_s} \left[M_c \left(-\frac{a}{2} \right) + m \, z_{cm} \right]$$

$$M_s = \frac{4}{3} \pi a^3 \rho \text{ , } M_c = \frac{4}{3} \pi \left(\frac{a}{2} \right)^3 \rho$$

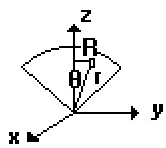
$$0 = \frac{1}{a^3} \left\{ \left(\frac{a}{2} \right)^3 \left(-\frac{a}{2} \right) + \left[a^3 - \left(\frac{a}{2} \right)^3 \right] z_{cm} \right\}$$

$$z_{cm} = \frac{a}{14}$$

8.4 (a) $I_z = \sum_i m_i R_i^2 = \frac{m}{3}\left[\left(\frac{b}{2}\right)^2 + 0 + \left(\frac{b}{2}\right)^2\right]$

$$I_z = \frac{mb^2}{6}$$

(b)



$$ds = rd\theta dr\ ,\ R = r\sin\theta$$

$$I_z = \int R^2 \rho ds$$

$$I_z = \rho \int_{r=0}^{r=b} r^2\, rdr \int_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} \sin^2\theta\, d\theta$$

$$I_z = \frac{\rho b^4}{4} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\theta d\theta$$

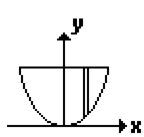
$$\left[\int \sin^2\theta d\theta = \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]$$

$$I_z = \frac{\rho b^4}{4}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$m = \frac{1}{4}\rho\pi b^2$$

$$I_z = \frac{mb^2}{4\pi}(\pi - 2)$$

(c)



$$ds = hdx = \left(b - \frac{x^2}{b}\right)dx$$

Where the parabola intersects the line $y = b$,

$$x = (by)^{\frac{1}{2}} = \pm b$$

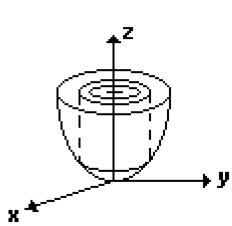
$$I_y = \int_{-b}^{b} x^2 \rho\left(b - \frac{x^2}{b}\right)dx = \rho\int_{-b}^{b}\left(bx^2 - \frac{x^4}{b}\right)dx$$

$$I_y = \frac{4}{15}\rho b^4$$

$$m = \int_{-b}^{b} \rho\left(b - \frac{x^2}{b}\right)dx = \frac{4}{3}\rho b^2$$

$$I_y = \frac{1}{5}mb^2$$

(d)



$$dv = 2\pi RhdR$$

$$h = b - z$$

$$R = \left(x^2 + y^2\right)^{\frac{1}{2}} = (bz)^{\frac{1}{2}}$$

$$dR = \frac{1}{2}\left(\frac{b}{z}\right)^{\frac{1}{2}} dz$$

$$I_z = \int R^2 \rho dv = \int_0^b bz\rho 2\pi (bz)^{\frac{1}{2}} (b-z) \frac{1}{2}\left(\frac{b}{z}\right)^{\frac{1}{2}} dz$$

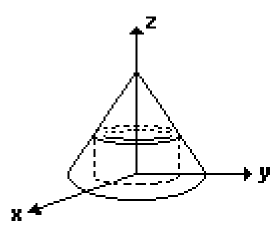
$$I_z = \pi\rho b^2 \int_0^b \left(bz - z^2\right) dz = \frac{1}{6}\pi\rho b^5$$

$$m = \int \rho dv = \int_0^b \rho 2\pi (bz)^{\frac{1}{2}} (b-z) \frac{1}{2}\left(\frac{b}{z}\right)^{\frac{1}{2}} dz$$

$$m = \pi\rho b \int_0^b (b-z) dz = \frac{1}{2}\pi\rho b^3$$

$$I_z = \frac{1}{3} m b^2$$

(e)



α is the half-angle of the apex of the cone. $r_\circ$ is the radius of the base at $z = 0$ and r is radius of a circle at some arbitrary z in a plane parallel to the base.

$$\tan\alpha = \frac{R_\circ}{b} = \frac{R}{b-z}\text{, a constant}$$

$$dv = 2\pi R h dR = 2\pi \frac{(b-z)R_\circ}{b} z dR$$

Since $R = \frac{(\mathrm{b-z})R_\circ}{b}$, $dR = -\frac{R_\circ}{b}dz$, and the limits of integration for $R = 0 \to R_\circ$ correspond to $z = b \to 0$

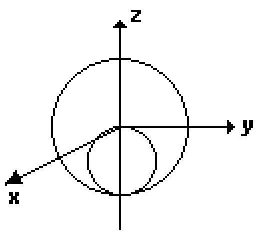
$$I_z = \int R^2 \rho dv = \int_b^0 \frac{(b-z)^2 R_\circ^2}{b^2} \rho 2\pi \frac{(b-z)R_\circ}{b} z\left(-\frac{R_\circ}{b}\right) dz$$

$$I_z = +2\pi\rho \frac{R_\circ^4}{b^4} \int_0^b \left(b^3 z - 3b^2 z^2 + 3bz^3 - z^4\right) dz = \frac{1}{10}\pi\rho R_\circ^4 b$$

$$m = \rho \frac{1}{3}\pi R_\circ^2 b$$

$$I_z = \frac{3}{10} m R_\circ^2$$

8.5 Consider the sphere with the cavity to be made of a (i) solid sphere of radius a and and mass M_s, with its center of mass at $z = 0$, and (ii) a solid sphere the size of the cavity, with mass $-M_c$ and center of mass at $z = -\frac{a}{2}$. The actual sphere with the cavity has a mass $m = M_s - M_c$ and center of mass z_{cm}.



$$m = M_s - M_c = \frac{4}{3}\pi a^3 \rho - \frac{4}{3}\pi\left(\frac{a}{2}\right)^3 \rho = \frac{7}{8}\frac{4}{3}\pi a^3 \rho$$

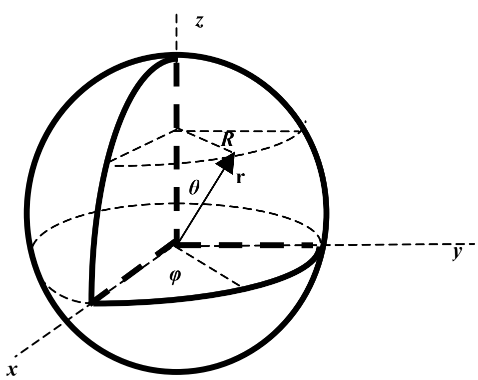
$$M_s = \frac{8}{7}m \text{ and } M_c = \frac{1}{7}m$$

From eqn. 8.3.2, $\quad I_s = I_c + I$

$$\frac{2}{5}M_s a^2 = \frac{2}{5}M_c\left(\frac{a}{2}\right)^2 + I$$

$$I = \frac{2}{5}\left(\frac{8}{7}m\right)a^2 - \frac{2}{5}\left(\frac{1}{7}m\right)\frac{a^2}{4} = \frac{31}{70}ma^2$$

8.6 The moment of inertia about one of the straight edges is $I_z = \int R^2 \rho dv$ where $R^2 = x^2 + y^2$. From Appendix F …



$$dv = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$R^2 = x^2 + y^2 = r^2 \sin^2\theta$$

Let a = radius of sphere

$$I_z = \int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\frac{\pi}{2}}\int_{\phi=0}^{\phi=\frac{\pi}{2}} r^2 \sin^2\theta \rho r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$I_z = \rho\frac{\pi}{2}\int_{r=0}^{r=a}\int_{\theta=0}^{\theta=\frac{\pi}{2}} r^4 \sin^3\theta dr d\theta$$

$$I_z = \frac{1}{10}\rho\pi a^5 \int_0^{\frac{\pi}{2}} \sin^3\theta d\theta$$

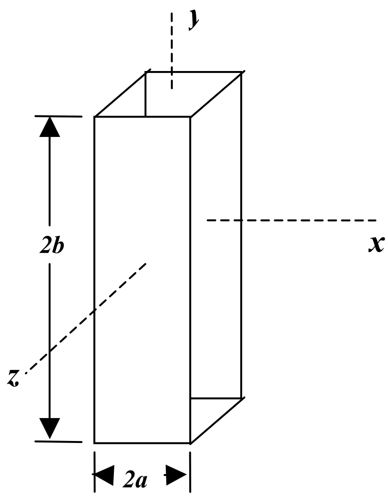
$$\left[\int \sin^3\theta d\theta = \frac{\cos^3\theta}{3} - \cos\theta\right]$$

$$I_z = \frac{2}{30}\rho\pi a^5$$

$$m = \frac{1}{8}\frac{4}{3}\pi a^3 \rho = \frac{1}{6}\pi a^3 \rho$$

$$I_z = \frac{2}{5}ma^2$$

8.7 For a rectangular parallelepiped:



$dv = hdxdy$ h is the length of the box in the z-direction

$$R = \left(x^2 + y^2\right)^{\frac{1}{2}}$$

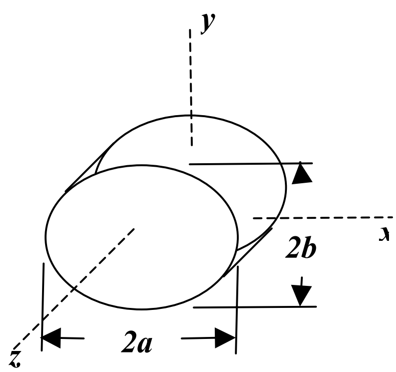
$$I_z = \int R^2 \rho dv = \int_{x=-a}^{x=a} \int_{y=-b}^{y=b} \left(x^2 + y^2\right) \rho h dxdy$$

$$I_z = \rho h \int_{-a}^{a} \left(2bx^2 + \frac{2b^3}{3}\right) dx = \frac{4}{3} \rho hab\left(a^2 + b^2\right)$$

$$m = \rho(2a)(2b)h = 4\rho abh$$

$$I_z = \frac{m}{3}\left(a^2 + b^2\right)$$

For an elliptic cylinder:



Again $dv = hdxdy$, and $R = \left(x^2 + y^2\right)^{\frac{1}{2}}$

On the surface, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$x = \pm a\left(1 - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$$

$$I_z = \int R^2 dv = \int_{y=-b}^{y=b} \int_{x=-a\left(1-\frac{y^2}{b^2}\right)^{\frac{1}{2}}}^{x=a\left(1-\frac{y^2}{b^2}\right)^{\frac{1}{2}}} \left(x^2 + y^2\right) \rho h dxdy$$

$$I_z = \rho h \int_{-b}^{b} \left[\frac{2}{3} a^3 \left(1 - \frac{y^2}{b^2}\right)^{\frac{3}{2}} + 2a\left(1 - \frac{y^2}{b^2}\right)^{\frac{1}{2}} y^2\right] dy$$

$$= 2\rho h \frac{a}{b}\left[\frac{a^2}{3b^2} \int_{-b}^{b} \left(b^2 - y^2\right)^{\frac{3}{2}} dy + \int_{-b}^{b} \left(b^2 - y^2\right)^{\frac{1}{2}} y^2 dy\right]$$

From a table of integrals:

$$\int \left(b^2 - y^2\right)^{\frac{3}{2}} dy = \frac{y}{4}\left(b^2 - y^2\right)^{\frac{3}{2}} + \frac{3}{8} b^2 y \left(b^2 - y^2\right)^{\frac{1}{2}} + \frac{3}{8} b^4 \sin^{-1} \frac{y}{b}$$

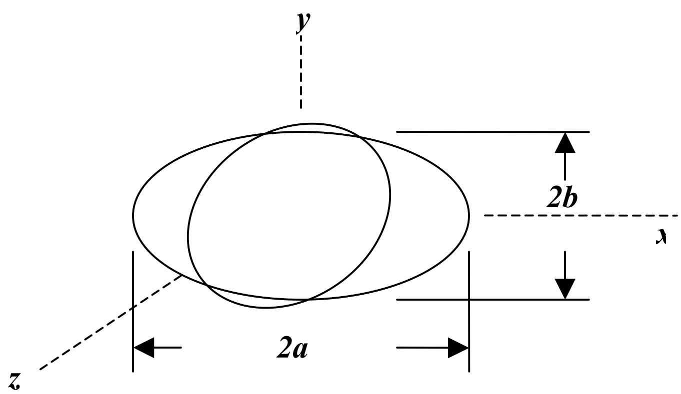
$$\int \left(b^2 - y^2\right)^{\frac{1}{2}} y^2 dy = -\frac{y}{4}\left(b^2 - y^2\right)^{\frac{3}{2}} + \frac{b^2 y}{8}\left(b^2 - y^2\right)^{\frac{1}{2}} + \frac{b^4}{8} \sin^{-1} \frac{y}{b}$$

$$I_z = 2\rho h \frac{a}{b}\left[\frac{a^2}{3b^2}\left(\frac{3}{8} b^4 \pi\right) + \frac{b^4}{8}\pi\right] = \frac{1}{4} \rho h \pi ab\left(a^2 + b^2\right)$$

$$m = \rho h(\pi ab)$$

$$I_z = \frac{m}{4}(a^2 + b^2)$$

For an ellipsoid: $dv = hdxdy$, $R^2 = x^2 + y^2$ and on the surface, $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



$$z = \pm c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$$

$$h = 2|z| = 2c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$$

In the xy plane $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$x = \pm a\left(1 - \frac{y^2}{b^2}\right)^{\frac{1}{2}}$$

$$I_z = R^2 \rho dv = \int_{y=-b}^{y=b} \int_{x=-a\left(1-\frac{y^2}{b^2}\right)^{\frac{1}{2}}}^{x=a\left(1-\frac{y^2}{b^2}\right)^{\frac{1}{2}}} (x^2 + y^2)\rho 2c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{\frac{1}{2}} dxdy$$

$$I_z = \frac{2\rho c}{a}\int_{y=-b}^{y=b}\left\{\int\left[\left(a^2 - \frac{a^2y^2}{b^2}\right) - x^2\right]^{\frac{1}{2}} x^2 dx + y^2 \int\left[\left(a^2 - \frac{a^2y^2}{b^2}\right) - x^2\right]^{\frac{1}{2}} dx\right\}$$

From a table of integrals:

$$\int (k^2 - x^2)^{\frac{1}{2}} x^2 dx = -\frac{x}{4}(k^2 - x^2)^{\frac{3}{2}} + \frac{k^2 x}{8}(k^2 - x^2)^{\frac{1}{2}} + \frac{k^4}{8}\sin^{-1}\frac{x}{k}$$

$$\int (k^2 - x^2)^{\frac{1}{2}} dx = \frac{x}{2}(k^2 - x^2)^{\frac{1}{2}} + \frac{k^2}{2}\sin^{-1}\frac{x}{k}$$

$$I_z = \frac{2\rho c}{a}\int_{-b}^{b}\left[\frac{a^4}{8}\left(1 - \frac{y^2}{b^2}\right)^2 \pi + y^2\frac{a^2}{2}\left(1 - \frac{y^2}{b^2}\right)\pi\right]dy$$

$$I_z = \rho\pi ac\int_{-b}^{b}\left[\frac{a^2}{4}\left(1 - \frac{2y^2}{b^2} + \frac{y^4}{b^4}\right) + y^2 - \frac{y^4}{b^2}\right]dy$$

$$I_z = \frac{4}{15}\rho\pi abc(a^2 + b^2)$$

For an ellipsoid, $m = \rho\frac{4}{3}\pi abc$, so $I_z = \frac{m}{5}(a^2 + b^2)$

8.8 (See Figure 8.4.1) Note that $l'+l$ is the distance from $0'$ to 0, defined as d

From eqn. 8.4.13, $k_{cm}^2 = ll'$

$$k_{cm}^2 + l^2 = ll' + l^2 = l\left(l'+l\right)$$

$$k_{cm}^2 + l^2 = ld$$

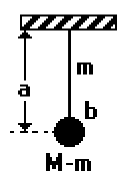
From eqn. 8.4.9b, $k^2 = k_{cm}^2 + l^2$

$$k^2 = ld$$

From eqn. 8.4.6, $T_\circ = 2\pi\sqrt{\frac{k^2}{gl}}$

$$T_\circ = 2\pi\sqrt{\frac{d}{g}}$$

8.9



Period of a simple pendulum: $T = 2\pi\sqrt{\frac{a}{M}}$

Period of real pendulum: $T_\circ = 2\pi\sqrt{\frac{I}{Mgl}}$ (eqn. 8.4.5)

Where $I =$ moment of inertia
$l =$ distance to CM of physical pendulum
$a =$ distance to CM of *bob*
$b =$ radius of *bob*

location of CM of physical pendulum:

$$l = \frac{m\frac{(a-b)}{2} + (M-m)a}{m+(M-m)} = a - \frac{m}{M}\left(\frac{a}{2}+\frac{b}{2}\right) = a - \frac{m}{2M}(a+b)$$

$$l = a\left[1 - \frac{m}{2M} - \frac{m}{2M}\frac{b}{a}\right]$$

Moment of inertia:

$$I_{bob} = (M-m)a^2 + \frac{2}{5}(M-m)b^2 = (M-m)\left(a^2 + \frac{2}{5}b^2\right)$$

$$= Ma^2\left(1-\frac{m}{M}\right)\left(1+\frac{2}{5}\frac{b^2}{a^2}\right)$$

$$I_{rod} = \frac{1}{3}m(a-b)^2 = \frac{1}{3}Ma^2\left[\frac{m}{M}\left(1-\frac{b}{a}\right)^2\right]$$

$$\therefore I = I_{bob} + I_{rod} = Ma^2\left[\left(1-\frac{m}{M}\right)\left(1+\frac{2}{5}\frac{b^2}{a^2}\right) + \frac{1}{3}\frac{m}{M}\left(1-\frac{b}{a}\right)^2\right]$$

letting $\alpha = \frac{m}{M}$ and $\beta = \frac{b}{a}$

$$T_\circ = 2\pi \left\{ \frac{Ma^2\left[(1-\alpha)\left(1+\frac{2}{5}\beta^2\right)+\frac{1}{3}\alpha(1-\beta)^2\right]}{Mga\left[1-\frac{\alpha}{2}-\frac{\alpha\beta}{2}\right]} \right\}^{\frac{1}{2}}$$

(a) $$\frac{T_\circ}{T} = \sqrt{\frac{(1-\alpha)\left(1+\frac{2}{5}\beta^2\right)+\frac{1}{3}\alpha(1-\beta)^2}{\left(1-\frac{\alpha}{2}-\frac{\alpha\beta}{2}\right)}} \approx 1-\frac{1}{12}\alpha \text{ to } 1^{st} \text{ \underline{order} in } \alpha$$

(b) $m = 10g$ $M = 1kg$ $a = 1.27m$ $b = 5cm$

$\alpha = 0.01$ $\beta = 0.0394$

$$\frac{T_\circ}{T} \approx 1-\frac{1}{12}\alpha = 0.9992$$ (actually 0.9994 using complete expression)

8.10 The period of the "seconds" pendulum is

$$T_2 = 2\pi\sqrt{\frac{I}{Mgl}} = 2\,s$$

The period of the modified pendulum is

$$T' = 2\pi\sqrt{\frac{I'}{M'gl'}} = 2\left(\frac{n}{n-20}\right)$$

where I', M', l' refer to parameters of pendulum *with m attached* and $n\ (= 24\times 60\times 60)$ is the number of seconds in a day.

$$I' = I + ml_m^2 \qquad l' = \frac{Ml + ml_m}{M}$$

where l_m is the distance of the attached mass m from the pivot point.

So $$\left(1-\frac{20}{n}\right)^{-2} = \frac{\pi^2 I'}{M'gl'} = \frac{\pi^2 I + \pi^2 ml_m^2}{(Ml + ml_m)g}$$

$$= \frac{Mgl + \pi^2 ml_m^2}{(Ml + ml_m)g}$$

Thus $$\left(1-\frac{20}{n}\right)^{2} = \frac{(Ml + ml_m)g}{(Mgl + \pi^2 ml_m^2)}$$

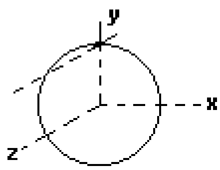
Or $$1-\frac{40}{n} \approx \frac{\left(1+\frac{\alpha l_m}{l}\right)}{\left(1+\frac{\pi^2\alpha l_m^2}{gl}\right)} \qquad \alpha = \frac{m}{M}$$

Solving for α gives the approximate result

$$\alpha = \frac{m}{M} \cong \frac{\frac{40}{n}\frac{l}{l_m}}{\left(\frac{\pi^2 l_m}{g} - 1\right)}$$

Letting $l_m = 1.3m$; $l = 1.0m$; we obtain $\alpha \cong 1.15 \cdot 10^{-3}$

8.11 (a) $I_{\perp cm} = ma^2$ (all mass in rim)



$$I_{\perp rim} = ma^2 + ma^2 = 2ma^2$$

$$\therefore T = 2\pi\sqrt{\frac{I_{\perp rim}}{mga}} = 2\pi\sqrt{\frac{2a}{g}}$$

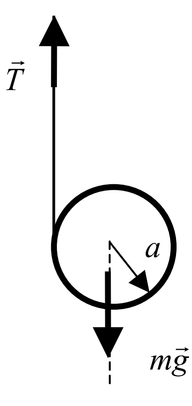
(b) $I_z = I_x + I_y = 2I_{\| cm} = ma^2 \qquad (= I_{\perp cm})$

$$\therefore I_{\| cm} = \frac{ma^2}{2}$$

hence $I_{\| rim} = \frac{ma^2}{2} + ma^2 = \frac{3}{2}ma^2$

$$\therefore T = 2\pi\sqrt{\frac{I_{\| rim}}{mga}} = 2\pi\sqrt{\frac{3a}{2g}}$$

8.12



$$m\ddot{x}_{cm} = mg - T$$

$$I_{cm}\dot{\omega} = aT$$

$$\ddot{x}_{cm} = a\dot{\omega}$$

$$I_{cm} = \frac{2}{5}ma^2$$

$$m\ddot{x}_{cm} = mg - \frac{I_{cm}\dot{\omega}}{a} = mg - \frac{1}{a}\left(\frac{2}{5}ma^2\frac{\ddot{x}_{cm}}{a}\right) = mg - \frac{2}{5}m\ddot{x}_{cm}$$

$$\ddot{x}_{cm} = \frac{5}{7}g$$

8.13 When two men hold the plank, each supports $\frac{mg}{2}$.

When one man lets go: $mg - R = m\ddot{x}_{cm}$ and $R\frac{l}{2} = I_{cm}\dot{\omega}$

From table 8.3.1, $I_{cm} = \frac{ml^2}{12}$

$$\dot{\omega} = \frac{Rl}{2} \cdot \frac{12}{ml^2} = \frac{6R}{ml}$$

$$\ddot{x}_{cm} = \frac{l}{2}\dot{\omega} = \frac{3R}{m}$$

$$mg - R = m\left(\frac{3R}{m}\right) = 3R$$

$$R = \frac{mg}{4}$$

$$\ddot{x}_{end} = l\dot{\omega} = l\frac{6R}{ml} = \frac{6}{m}\left(\frac{mg}{4}\right)$$

$$\ddot{x}_{end} = \frac{3}{2}g$$

8.14 For a solid sphere:

$$M_s = \frac{4}{3}\pi a^3 \rho \text{ and } I_s = \frac{2}{5}M_s a^2$$

$$\left(k_{cm}^2\right)_s = \frac{2}{5}a^2$$

For subscript c representing a solid sphere the size of the cavity, from eqn. 8.3.2:

$$I_s = I + I_c$$

$$I = \frac{2}{5}\left(\frac{4}{3}\pi a^3 \rho\right)a^2 - \frac{2}{5}\left[\frac{4}{3}\pi\left(\frac{a}{2}\right)^3 \rho\right]\left(\frac{a}{2}\right)^2 = \frac{2}{5}\cdot\frac{4}{3}\cdot\frac{31}{32}\pi a^5 \rho$$

$$m = \frac{4}{3}\pi a^3 \rho - \frac{4}{3}\pi\left(\frac{a}{2}\right)^3 \rho = \frac{4}{3}\cdot\frac{7}{8}\pi a^3 \rho$$

$$k_{cm}^2 = \frac{I}{m} = \frac{2}{5}\cdot\frac{31}{32}\cdot\frac{8}{7}a^2 = \frac{31}{70}a^2$$

From eqn. 8.6.11, for a sphere rolling down a rough inclined plane:

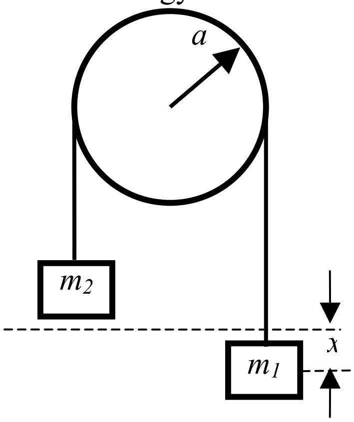
$$\ddot{x}_{cm} = \frac{g\sin\theta}{1+\left(\frac{k_{cm}^2}{a^2}\right)}$$

$$\frac{\ddot{x}}{\ddot{x}_s} = \frac{1+\frac{\left(k_{cm}^2\right)_s}{a^2}}{1+\frac{k_{cm}^2}{a^2}}$$

$$= \frac{1+\frac{2}{5}}{1+\frac{31}{70}} = \frac{\frac{7}{5}}{\frac{101}{70}}$$

$$\frac{\ddot{x}}{\ddot{x}_s} = \frac{98}{101}$$

8.15 Energy is conserved:

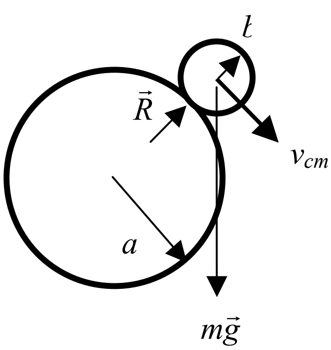


$$\frac{1}{2}m_1\dot{x}^2+\frac{1}{2}m_2\dot{x}^2+\frac{1}{2}I\left(\frac{\dot{x}}{2}\right)^2+m_2gx-m_1gx=E$$

$$m_1\dot{x}\ddot{x}+m_2\dot{x}\ddot{x}+\frac{I}{a^2}\dot{x}\ddot{x}+m_2g\dot{x}-m_1g\dot{x}=0$$

$$\ddot{x}=\frac{(m_1-m_2)g}{m_1+m_2+\dfrac{I}{a^2}}$$

8.16 While the cylinders are in contact:



$$f_r=\frac{mv_{cm}^2}{r}=mg\cos\theta-R$$

$r=a+b$, so $\dfrac{mv_{cm}^2}{a+b}=mg\cos\theta-R$

From conservation of energy:

$$mg(a+b)=\frac{1}{2}mv_{cm}^2+\frac{1}{2}I\omega^2+mg(a+b)\cos\theta$$

From table 8.3.1, $I=\dfrac{1}{2}ma^2$

$$\omega=\frac{v_{cm}}{a}$$

$$\frac{1}{2}mv_{cm}^2+\frac{1}{2}\left(\frac{1}{2}ma^2\right)\left(\frac{v_{cm}^2}{a^2}\right)=mg(a+b)(1-\cos\theta)$$

$$\frac{mv_{cm}^2}{a+b}=\frac{4}{3}mg(1-\cos\theta)$$

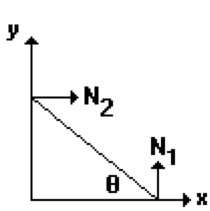
When the rolling cylinder leaves, $R=0$:

$$mg\cos\theta=\frac{4}{3}mg(1-\cos\theta)$$

$$\frac{7}{3}\cos\theta=\frac{4}{3}$$

$$\theta=\cos^{-1}\frac{4}{7}$$

8.17



$$m\ddot{x}=N_2$$

$$m\ddot{y}=N_1-mg$$

$$\frac{1}{2}mv_{cm}^2+\frac{1}{2}I\omega^2+mgy=mgy_\circ$$

$$x=\frac{l}{2}\cos\theta,\qquad \dot{x}=-\frac{l}{2}\dot{\theta}\sin\theta,\qquad \ddot{x}=\frac{l}{2}\left(\dot{\theta}^2\cos\theta+\ddot{\theta}\sin\theta\right)$$

$$y=\frac{l}{2}\sin\theta,\qquad \dot{y}=\frac{l}{2}\dot{\theta}\cos\theta,\qquad \ddot{y}=\frac{l}{2}\left(-\dot{\theta}^2\sin\theta+\ddot{\theta}\cos\theta\right)$$

$$v_{cm}^2=\dot{x}^2+\dot{y}^2=\left(-\frac{l}{2}\dot{\theta}\sin\theta\right)^2+\left(\frac{l}{2}\dot{\theta}\cos\theta\right)^2=\frac{l^2\dot{\theta}^2}{4}$$

$I=\dfrac{ml^2}{12}$, and $\omega=\dot{\theta}$

$$\frac{1}{2}m\frac{l^2\dot{\theta}^2}{4}+\frac{1}{2}\frac{ml^2}{12}\dot{\theta}^2+mg\frac{l}{2}\sin\theta=mg\frac{l}{2}\sin\theta_\circ$$

$$\frac{l}{3}\dot{\theta}^2=g\left(\sin\theta_\circ-\sin\theta\right)$$

$$\dot{\theta}=\left[\frac{3g}{l}\left(\sin\theta_\circ-\sin\theta\right)\right]^{\frac{1}{2}}$$

$$\ddot{\theta}=\frac{1}{2}\left[\frac{3g}{l}\left(\sin\theta_\circ-\sin\theta\right)\right]^{-\frac{1}{2}}\left(\frac{3g}{l}\right)\left(-\cos\theta\right)\dot{\theta}=-\frac{3g}{2l}\cos\theta$$

$$N_2=m\ddot{x}=-\frac{ml}{2}\left[\cos\theta\left(\frac{3g}{l}\right)\left(\sin\theta_\circ-\sin\theta\right)+\sin\theta\left(-\frac{3g}{2l}\right)\cos\theta\right]$$

Separation occurs when $N_2=0$:

$$\sin\theta_\circ-\sin\theta-\frac{1}{2}\sin\theta=0,\qquad \theta=\sin^{-1}\left(\frac{2}{3}\sin\theta_\circ\right)$$

8.18 $R_x=m\ddot{x}$

$$R_y-mg=m\ddot{y}$$

$$x=\frac{l}{2}\sin\theta,\qquad \dot{x}=\frac{l}{2}\dot{\theta}\cos\theta,\quad \ddot{x}=\frac{l}{2}\left(-\dot{\theta}^2\sin\theta+\ddot{\theta}\cos\theta\right)$$

$$y=\frac{l}{2}\cos\theta,\qquad \dot{y}=-\frac{l}{2}\dot{\theta}\sin\theta,\qquad \ddot{y}=-\frac{l}{2}\left(\dot{\theta}^2\cos\theta+\ddot{\theta}\sin\theta\right)$$

$$\frac{1}{2}mv_{cm}^2+\frac{1}{2}I\dot{\theta}^2+mg\frac{l}{2}\cos\theta=mg\frac{l}{2}$$

$$v_{cm}=\frac{l}{2}\dot{\theta},\qquad I=\frac{ml^2}{12}$$

$$\frac{m}{2}\frac{l^2\dot{\theta}^2}{4}+\frac{1}{2}\frac{ml^2}{12}\dot{\theta}^2 = mg\frac{l}{2}(1-\cos\theta)$$

$$\frac{l\dot{\theta}^2}{3} = g(1-\cos\theta)$$

$$\dot{\theta} = \left[\frac{3g}{l}(1-\cos\theta)\right]^{\frac{1}{2}}$$

$$\ddot{\theta} = \frac{1}{2}\left[\frac{3g}{l}(1-\cos\theta)\right]^{-\frac{1}{2}}\left(\frac{3g}{l}\right)\sin\theta\dot{\theta} = \frac{3g}{2l}\sin\theta$$

$$R_x = \frac{ml}{2}\left[(-\sin\theta)\left(\frac{3g}{l}\right)(1-\cos\theta)+\cos\theta\left(\frac{3g}{2l}\sin\theta\right)\right]$$

$$R_x = \frac{3mg}{4}\sin\theta(3\cos\theta-2)$$

$$R_y = mg-\frac{ml}{2}\left[\cos\theta\left(\frac{3g}{l}\right)(1-\cos\theta)+\sin\theta\left(\frac{3g}{2l}\sin\theta\right)\right]$$

$$R_y = mg-\frac{3mg}{2}\left(\cos\theta-\cos^2\theta+\frac{\sin^2\theta}{2}\right)$$

$$R_y = \frac{mg}{4}(3\cos\theta-1)^2$$

The reaction force constrains the tail of the rocket from sliding backward for $R_x > 0$:

$$3\cos\theta-2>0$$

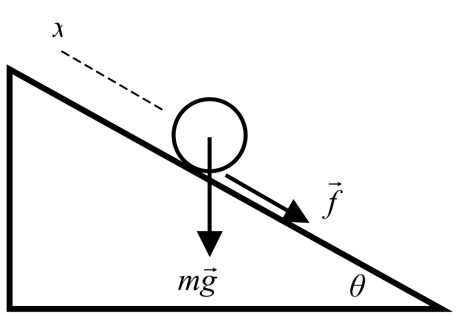
$$\theta < \cos^{-1}\frac{2}{3}$$

The rocket is constrained from sliding forward for $R_x < 0$:

$$\theta > \cos^{-1}\frac{2}{3}$$

8.19 $m\ddot{x} = -mg\sin\theta-\mu mg\cos\theta$

$$\ddot{x} = -g(\sin\theta+\mu\cos\theta)$$



Since acceleration is constant, $x = \dot{x}_\circ t+\frac{1}{2}\ddot{x}t^2$:

$$x = v_\circ t-\frac{gt^2}{2}(\sin\theta+\mu\cos\theta)$$

Meanwhile $(\mu mg\cos\theta)a = I\dot{\omega} = \frac{2}{5}ma^2\dot{\omega}$

$$\dot{\omega} = \frac{5}{2}\frac{\mu g\cos\theta}{a}$$

$$\omega = \frac{5}{2}\frac{\mu g\cos\theta}{a}t$$

The ball begins pure rolling when $v = a\omega$ …

$$v = v_\circ + \ddot{x}t = v_\circ - g\left(\sin\theta + \mu\cos\theta\right)t = a\frac{5}{2}\frac{\mu g\cos\theta}{a}t$$

$$t = \frac{2v_\circ}{g\left(2\sin\theta + 7\mu\cos\theta\right)}$$

At that time:

$$x = \frac{2v_\circ^2}{g\left(2\sin\theta + 7\mu\cos\theta\right)} - \frac{g}{2}\frac{4v_\circ^2\left(\sin\theta + \mu\cos\theta\right)}{g^2\left(2\sin\theta + 7\mu\cos\theta\right)^2}$$

$$x = \frac{2v_\circ^2}{g}\frac{\left(\sin\theta + 6\mu\cos\theta\right)}{\left(2\sin\theta + 7\mu\cos\theta\right)^2}$$

8.20 $m\ddot{x} = \mu mg$

$$\ddot{x} = \mu g$$

$\dot{x} = \mu g t$, and $x = \frac{1}{2}\mu g t^2$

$$I\dot{\omega} = \frac{2}{5}ma^2\dot{\omega} = -\mu mga$$

$$\dot{\omega} = -\frac{5}{2}\frac{\mu g}{a}$$

$$\omega = \omega_\circ - \frac{5}{2}\frac{\mu g}{a}t$$

Slipping ceases to occur when $v = a\omega$ …

$$\mu g t = a\omega_\circ - \frac{5}{2}\mu g t$$

$$t = \frac{2}{7}\frac{a\omega_\circ}{g}$$

$$x = \frac{1}{2}\mu g\left(\frac{2}{7}\frac{a\omega_\circ}{\mu g}\right)^2$$

$$x = \frac{2}{49}\frac{a^2\omega_\circ^2}{\mu g}$$

8.21 Let the moments of inertia of A and B be $I_a\left(=\frac{1}{2}M_a a^2\right)$ and $I_b\left(=\frac{1}{2}M_b b^2\right)$.

The angular velocity of A is $\dot{\alpha}$ while that of B is $\dot{\beta} - \dot{\alpha} + \dot{\phi}$ (remember that in two dimensions, angular velocity is the rate of change of an angle between an line or direction fixed to the body and one fixed in space). For rolling contact, lengths traveled along the perimeters of the disks A and B must be equal to the arc length traveled along the track C.

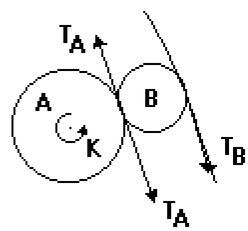
$$a\phi = b\beta = \left(a + 2b\right)\left(\alpha - \phi\right)$$

so that $\phi = \dfrac{(a+2b)\alpha}{2(a+b)}$ and $\beta = \dfrac{a(a+2b)\alpha}{2b(a+b)}$

After some algebra … the angular velocity of B is found to be …

$$\omega_B = \dot{\beta} - \dot{\alpha} + \dot{\phi} = \frac{a\dot{\alpha}}{2b}$$

For A, we take moments about O and for B we take moments about its center. Call T_A and T_B the components of the reaction forces tangent to A and B (the "upward-going" T_A acts on disk B. The "downward-going" T_A acts on disk A)



Thus $K - T_A a = I_A \ddot{\alpha}$ (Torque on Disk A)

$$-T_A b - T_B b = -I_B\left(\ddot{\beta} - \ddot{\alpha} + \ddot{\phi}\right) = -I_B a \frac{\ddot{\alpha}}{2b} \qquad \text{(Torque on Disk B)}$$

$$T_A - T_B = M_B (a+b)\left(\ddot{\alpha} - \ddot{\phi}\right) = \frac{1}{2} M_B a \ddot{\alpha} \qquad \text{(Force on Disk B)}$$

Eliminate T_A and T_B

$$K = \frac{\ddot{\alpha}}{4b^2}\left(4b^2 I_A + M_B a^2 b^2 + a^2 I_B\right)$$

Integrating this equation gives :

$$Kt = \frac{\dot{\alpha}}{4b^2}\left(4b^2 I_A + M_B a^2 b^2 + a^2 I_B\right)$$

Putting $\omega_A = \dot{\alpha}$ at $t = t_\circ$ gives

$$\omega_A = \frac{4b^2 K t_\circ}{\left(4b^2 I_A + M_B a^2 b^2 + a^2 I_B\right)}$$

Putting in values for I_A and I_B gives

$$\omega_A = \frac{4Kt_\circ}{a^2\left(2M_A + \frac{3}{2}M_B\right)}$$

Since the angular velocity of B is $\omega_B = \dot{\beta} - \dot{\alpha} + \dot{\phi} = \dfrac{a\dot{\alpha}}{2b}$, we have

$$\omega_B = \frac{a}{2b}\omega_A = \frac{2Kt_\circ}{ab\left(2M_A + \frac{3}{2}M_B\right)}$$

8.22 From section 8.7 (see Figure 8.7.1), the instantaneous center of rotation is the point O. If x is the distance from the center of mass to O and $\frac{l}{2}$ is the distance from the center of mass to the center of percussion O', then from eqn. 8.7.10 …

$$x\left(\frac{l}{2}\right)=\frac{I_{cm}}{M}=\frac{Ml^2}{12}\left(\frac{1}{M}\right)=\frac{l^2}{12}$$

$$x=\frac{l}{6}$$

8.23 In order that no reaction occurs between the table surface and the ball, the ball must approach and recede from its collision with the cushion by rolling without slipping. Using a prime to denote velocity and rotational velocity after the collision:

$$v'_{cm}=v_{cm}-\frac{\hat{P}}{m}$$

$$\omega'_{cm}=\omega_{cm}-\frac{\hat{P}}{I}(h-a)$$

The conditions for no reaction are $v_{cm}=a\omega_{cm}$ and $v'_{cm}=a\omega'_{cm}$.

$$a\omega_{cm}-\frac{\hat{P}}{m}=a\left[\omega_{cm}-\frac{\hat{P}}{I}(h-a)\right]$$

$$\frac{1}{m}=\frac{1}{I}a(h-a)$$

$$I=\frac{2}{5}ma^2$$

$$h=\frac{2ma^2}{5ma}+a=\frac{7}{5}a$$

$$a=\frac{d}{2}$$

$$h=\frac{7}{10}d$$

8.24 During the collision, angular momentum about the point 0 is conserved:

$$m'v_\circ l'=I\dot{\theta}$$

$$\dot{\theta}=\frac{m'v_\circ l'}{I}$$

After the collision, energy is conserved:

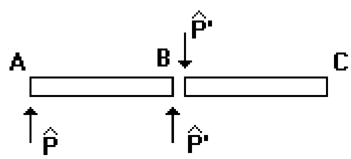
$$\frac{1}{2}I\dot{\theta}^2-mg\frac{l}{2}-m'gl'=-mg\frac{l}{2}\cos\theta_\circ-m'gl'\cos\theta_\circ$$

$$\frac{1}{2}\frac{m'^2v_\circ^2l'^2}{I}=g(1-\cos\theta_\circ)\left(\frac{ml}{2}+m'l'\right)$$

$$I = \frac{ml^2}{3} + m'l'^2$$

$$v_\circ = \frac{1}{m'l'}\left[2g\left(1-\cos\theta_\circ\right)\left(\frac{ml}{2}+m'l'\right)\left(\frac{ml^2}{3}+m'l'^2\right)\right]^{\frac{1}{2}}$$

8.25



The effect of rod BC acting on rod AB is impulse $+\hat{P}'$.
The effect of AB on BC is $-\hat{P}'$.

$$mv_1 = \hat{P} + \hat{P}' \qquad mv_2 = -\hat{P}'$$

$$I\omega_1 = \frac{l}{2}\left(\hat{P} - \hat{P}'\right) \qquad I\omega_2 = -\frac{l}{2}\hat{P}'$$

$$I = \frac{ml^2}{12}$$

$$\omega_1 = \frac{6}{ml}\left(\hat{P} - \hat{P}'\right) \qquad \omega_2 = -\frac{6}{ml}\hat{P}'$$

$$v_B = v_1 - \frac{l}{2}\omega_1 \qquad v_B = v_2 + \frac{l}{2}\omega_2$$

$$v_B = \frac{\hat{P} + \hat{P}'}{m} - \frac{l}{2}\left(\frac{6}{ml}\right)\left(\hat{P} - \hat{P}'\right)$$

$$v_B = -\frac{\hat{P}'}{m} + \frac{l}{2}\left(-\frac{6}{ml}\hat{P}'\right)$$

$$\hat{P} + \hat{P}' - 3\left(\hat{P} - \hat{P}'\right) = -\hat{P}' - 3\hat{P}'$$

$$8\hat{P}' = 2\hat{P}$$

$$\hat{P}' = \frac{\hat{P}}{4}$$

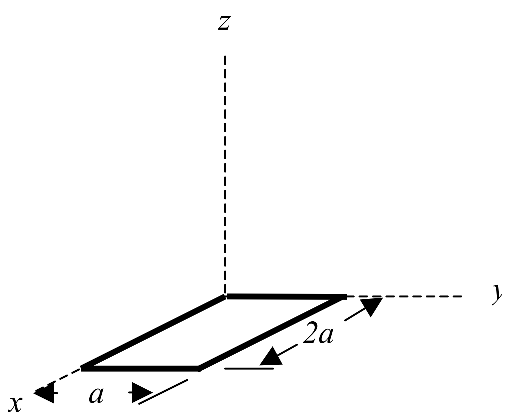
$$v_1 = \frac{5\hat{P}}{4m} \qquad v_2 = -\frac{\hat{P}}{4m}$$

$$\omega_1 = \frac{9\hat{P}}{2ml} \qquad \omega_2 = -\frac{3\hat{P}}{2ml}$$

$$v_B = -\frac{\hat{P}}{m}$$

CHAPTER 9
MOTION OF RIGID BODIES
IN THREE DIMENSIONS

9.1 (a) $I_{xx} = \int \left(y^2 + z^2\right) dm$



$dm = \rho dxdy$ and $m = 2a^2\rho$

$$I_{xx} = \int_{y=0}^{y=a} \int_{x=0}^{x=2a} \left(y^2 + 0\right) \rho dxdy$$

$$= \int_0^a 2a\rho y^2 dy$$

$$I_{xx} = \frac{2}{3}\rho a^4 = \frac{ma^2}{3}$$

$$I_{yy} = \int \left(x^2 + z^2\right) dm = \int_{y=0}^{y=a} \int_{x=0}^{x=2a} x^2 \rho dxdy$$

$$= \int_0^a \frac{8a^3\rho}{3} dy$$

$$I_{yy} = \frac{8a^4\rho}{3} = \frac{4ma^2}{3}$$

From the perpendicular axis theorem:

$$I_{zz} = I_{xx} + I_{yy} + \frac{5ma^2}{3}$$

$$I_{xy} = I_{yx} = -\int xydm = -\int_{y=0}^{y=a} \int_{x=0}^{x=2a} xy\rho dxdy$$

$$= -\int_0^a \frac{4a^2\rho}{2} ydy$$

$$I_{xy} = I_{yx} = -\rho a^4 = -\frac{ma^2}{2}$$

$$I_{xz} = I_{zx} = -\int xzdm = 0 = I_{yz} = I_{zy}$$

(b) $\cos\alpha = \frac{2}{\sqrt{5}}, \quad \cos\beta = \frac{1}{\sqrt{5}}, \quad \cos\gamma = 0$

From equation 9.1.10 …

$$I = \frac{ma^2}{3}\left(\frac{4}{5}\right) + \frac{4ma^2}{3}\left(\frac{1}{5}\right) + 2\left(-\frac{ma^2}{2}\right)\left(\frac{2}{\sqrt{5}}\right)\left(\frac{1}{\sqrt{5}}\right)$$

$$= \frac{2}{15} ma^2$$

(c) $\vec{\omega} = \omega\left(\hat{i}\cos\alpha + \hat{j}\cos\beta\right) = \frac{\omega}{\sqrt{5}}\left(2\hat{i} + \hat{j}\right)$

From equation 9.1.29 …

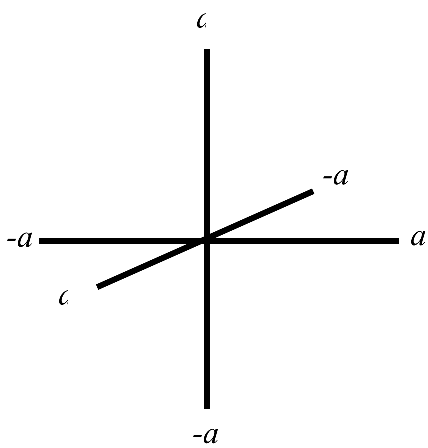
$$\vec{L} = \hat{i}\left[\frac{2\omega}{\sqrt{5}} \cdot \frac{ma^2}{3} + \frac{\omega}{\sqrt{5}}\left(-\frac{ma^2}{2}\right)\right] + \hat{j}\left[\frac{2\omega}{\sqrt{5}}\left(-\frac{ma^2}{2}\right) + \frac{\omega}{\sqrt{5}} \cdot \frac{4ma^2}{3}\right]$$

$$\vec{L} = \frac{\omega m a^2}{6\sqrt{5}}\left(\hat{i} + 2\hat{j}\right)$$

(d) From equation 9.1.32:

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{\omega}{2\sqrt{5}} \cdot \frac{\omega m a^2}{6\sqrt{5}}(2+2) = \frac{1}{15} m a^2 \omega^2$$

9.2



(a) $\vec{\omega} = \frac{\omega}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$

$$I_{xx} = 2I_{rod} = 2\frac{m(2a)^2}{12} = \frac{2}{3}ma^2 = I_{yy} = I_{zz}$$

$I_{xy} = -\int xydm = 0$ since, for each rod, either x or y or both are 0. The same is true for the other products of inertia.

From equation 9.1.29:

$$\vec{L} = \frac{2}{3}ma^2 \cdot \frac{\omega}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

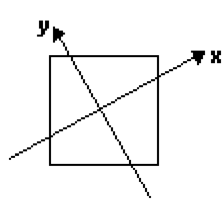
From equation 9.1.32,

$$T = \frac{1}{2}\frac{\omega}{\sqrt{3}}\frac{2ma^2\omega}{3\sqrt{3}}\left(1^2 + 1^2 + 1^2\right) = \frac{ma^2\omega^2}{3}$$

(b) From equation 9.1.10, with the moments of inertia equal to $\frac{2ma^2}{3}$ and the products of inertia equal to 0:

$$I = \frac{2ma^2}{3}\left(\cos^2\alpha + \cos^2\beta + \cos^2\gamma\right) = \frac{2}{3}ma^2$$

(c)



For the x-axis being any axis through the center of the lamina and in the plane of the lamina, and the y-axis also in the plane of the lamina …

$I_{xx} = I_{yy}$ due to the similar geometry of the mass distributions with respect to the x- and y-axes.

From the perpendicular axis theorem:

$$I_{zz} = I_{xx} + I_{yy}$$

$$I_{zz} = 2I_{xx}$$

From Table 8.3.1, $I_{zz} = \frac{ma^2}{6}$

$$I_{xx} = \frac{ma^2}{12}$$

9.3 (a) From equation 9.2.13, $\tan 2\theta = \dfrac{2I_{xy}}{I_{xx} - I_{yy}}$

From Prob. 9.1, $I_{xx} = \dfrac{ma^2}{3}$, $I_{yy} = \dfrac{4ma^2}{3}$, $I_{xy} = -\dfrac{ma^2}{2}$

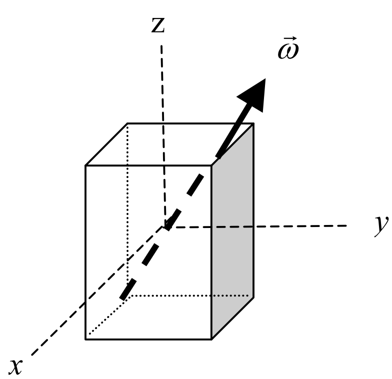
$$\tan 2\theta = 1$$

$$2\theta = 45^\circ \qquad \theta = 22.5^\circ$$

The 1-axis makes an angle of 22.5° with the x-axis.

(b) From symmetry, the principal axes are parallel to the sides of the lamina and perpendicular to the lamina, respectively.

9.4 (a) From symmetry, the coordinate axes are principal axes.



From Table 8.3.1:

$$I_1 = \frac{m}{12}\left[(2a)^2 + (3a)^2\right] = \frac{13}{12}ma^2$$

$$I_2 = \frac{m}{12}\left[a^2 + (3a)^2\right] = \frac{10}{12}ma^2$$

$$I_3 = \frac{m}{12}\left[a^2 + (2a)^2\right] = \frac{5}{12}ma^2$$

$$\vec{\omega} = \frac{\omega}{\sqrt{14}}\left(\hat{e}_1 + 2\hat{e}_2 + 3\hat{e}_3\right)$$

From equation 9.2.5,

$$T = \frac{1}{2}\left[\frac{13}{12}ma^2 \cdot \frac{\omega^2}{14} + \frac{10}{12}ma^2 \cdot \frac{4\omega^2}{14} + \frac{5}{12}ma^2 \cdot \frac{9\omega^2}{14}\right]$$

$$= \frac{7}{24}ma^2\omega^2$$

(b) From equation 9.2.4,

$$\vec{L} = \hat{e}_1\left(\frac{13}{12}ma^2 \cdot \frac{\omega}{\sqrt{14}}\right) + \hat{e}_2\left(\frac{10}{12}ma^2 \cdot \frac{2\omega}{\sqrt{14}}\right) + \hat{e}_3\left(\frac{5}{12}ma^2 \cdot \frac{3\omega}{\sqrt{14}}\right)$$

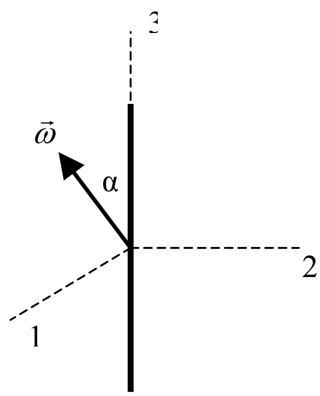
$$\vec{L} = \frac{ma^2\omega}{12\sqrt{14}}\left(13\hat{e}_1 + 20\hat{e}_2 + 15\hat{e}_3\right)$$

$$\cos\theta = \frac{\vec{\omega} \cdot \vec{L}}{|\omega||L|} = \frac{\left[(1)(13) + (2)(20) + (3)(15)\right]}{\left(1^2 + 2^2 + 3^2\right)^{\frac{1}{2}} \cdot \left(13^2 + 20^2 + 15^2\right)^{\frac{1}{2}}}$$

$$= \frac{98}{(11{,}116)^{\frac{1}{2}}} = 0.9295$$

$$\theta = 21.6^\circ$$

9.5 (a) Select coordinate axes such that the axis of the rod is the 3-axis, its center is at the



origin, and $\vec{\omega}$ lies in the 1, 3 plane.

From Table 8.3.1, $I_1 = I_2 = \frac{ml^2}{12}$, $I_3 = 0$

$\vec{\omega} = \omega\left(\hat{e}_1 \sin\alpha + \hat{e}_3 \cos\alpha\right)$

From equation 9.2.4,

$$\vec{L} = \frac{ml^2}{12}\omega\left(\hat{e}_1 \sin\alpha + 0 + 0\right) = \hat{e}_1 \frac{ml^2\omega}{12}\sin\alpha$$

$\vec{L}$ is perpendicular to the rod, and $\left|\vec{L}\right| = \frac{ml^2\omega}{12}\sin\alpha$

(b) Since $\vec{\omega}$ is constant, from equations. 9.3.5 …

$$N_1 = 0 + 0$$

$$N_2 = 0 + \left(\omega\cos\alpha\right)\left(\omega\sin\alpha\right)\left(\frac{ml^2}{12}\right)$$

$$N_3 = 0 + 0$$

$$\vec{N} = \hat{e}_2 \frac{ml^2\omega^2}{12}\sin\alpha\cos\alpha$$

$\vec{N}$ is perpendicular to the rod ($\hat{e}_3$ direction) and to $\vec{L}$ ($\hat{e}_1$ direction), and

$$\left|\vec{N}\right| = \frac{ml^2\omega^2}{12}\sin\alpha\cos\alpha$$

9.6 From Problem 9.4 … $\vec{\omega} = \frac{\omega}{\sqrt{14}}\left(\hat{e}_1 + 2\hat{e}_2 + 3\hat{e}_3\right)$

$I_1 = \frac{13}{12}ma^2$, $I_2 = \frac{10}{12}ma^2$, and $I_3 = \frac{5}{12}ma^2$

From eqns. 9.3.5:

$$N_1 = 0 + \frac{\omega^2}{14}(2)(3)\frac{ma^2}{12}(5-10) = -\frac{ma^2\omega^2}{28}(5)$$

$$N_2 = 0 + \frac{\omega^2}{14}(3)(1)\frac{ma^2}{12}(13-5) = \frac{ma^2\omega^2}{28}(4)$$

$$N_3 = 0 + \frac{\omega^2}{14}(1)(2)\frac{ma^2}{12}(10-13) = -\frac{ma^2\omega^2}{28}$$

$$\left|\vec{N}\right| = \frac{ma^2\omega^2}{28}\left(5^2 + 4^2 + 1^2\right)^{\frac{1}{2}} = \frac{ma^2\omega^2}{28}\sqrt{42}$$

9.7 Multiplying equations 9.3.5 by ω_1, ω_2, and ω_3, respectively …

$$0 = I_1\dot{\omega}_1\omega_1 + \omega_1\omega_2\omega_3\left(I_3 - I_2\right)$$

$$0 = I_2\dot{\omega}_2\omega_2 + \omega_1\omega_2\omega_3\left(I_1 - I_3\right)$$

$0 = I_3\dot{\omega}_3\omega_3 + \omega_1\omega_2\omega_3\left(I_2 - I_1\right)$

Adding, $0 = I_1\dot{\omega}_1\omega_1 + I_2\dot{\omega}_2\omega_2 + I_3\dot{\omega}_3\omega_3$

$$0 = \frac{d}{dt}\left[\frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)\right]$$

From equation 9.2.5, $0 = \frac{d}{dt}\left[T_{rot}\right]$

$T_{rot} = constant$

Multiplying equations 9.3.5 by $I_1\omega_1$, $I_2\omega_2$, and $I_3\omega_3$, respectively:

$0 = I_1^2\dot{\omega}_1\omega_1 + I_1\omega_1\omega_2\omega_3\left(I_3 - I_2\right)$

$0 = I_2^2\dot{\omega}_2\omega_2 + I_2\omega_1\omega_2\omega_3\left(I_1 - I_3\right)$

$0 = I_3^2\dot{\omega}_3\omega_3 + I_3\omega_1\omega_2\omega_3\left(I_2 - I_1\right)$

Adding, $0 = I_1^2\dot{\omega}_1\omega_1 + I_2^2\dot{\omega}_2\omega_2 + I_3^2\dot{\omega}_3\omega_3$

$$0 = \frac{1}{2}\frac{d}{dt}\left(I_1^2\omega_1^2 + I_2^2\omega_2^2 + I_3^2\omega_3^2\right)$$

From equation 9.2.4, $0 = \frac{d}{dt}\left(L^2\right)$

$L^2 = constant$

9.8 From equations 9.3.5 for zero torque …

$0 = I_1\dot{\omega}_1 + \omega_2\omega_3\left(I_3 - I_2\right)$

$0 = I_2\dot{\omega}_2 + \omega_1\omega_3\left(I_1 - I_3\right)$

From the perpendicular axis theorem, $I_3 = I_1 + I_2$

$0 = I_1\dot{\omega}_1 + \omega_2\omega_3 I_1$

$0 = I_2\dot{\omega}_2 - \omega_1\omega_3 I_2$

Multiplying by $\frac{\omega_1}{I_1}$ and $\frac{\omega_2}{I_2}$, respectively:

$0 = \omega_1\dot{\omega}_1 + \omega_1\omega_2\omega_3$

$0 = \omega_2\dot{\omega}_2 - \omega_1\omega_2\omega_3$

Adding, $0 = \omega_1\dot{\omega}_1 + \omega_2\dot{\omega}_2 = \frac{1}{2}\frac{d}{dt}\left(\omega_1^2 + \omega_2^2\right)$

$\omega_1^2 + \omega_2^2 = constant$

If $I_1 = I_2$, from the third of Euler's equations … $\left(I_3\dot{\omega}_3 = 0\right)$

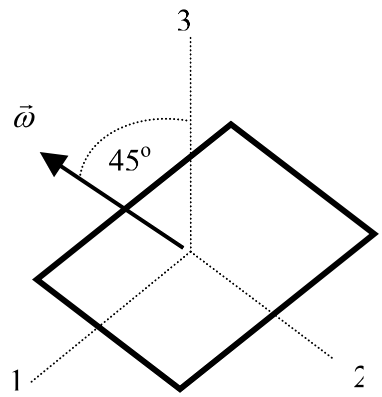
$\omega_3 = constant$

9.9 (a) From symmetry … $I_s = I_3$ and $I_1 = I_2 = I$

From the perpendicular axis theorem, $I_3 = I_1 + I_2$ or $I_s = 2I$

From eqn 9.5.8, $\Omega = \left(2 - 1\right)\omega\cos\alpha$

For $\alpha = 45^\circ$, $\Omega = \dfrac{\omega}{\sqrt{2}}$



For $\dfrac{2\pi}{\omega} = 1\,\text{s}$, $T_1 = \dfrac{2\pi}{\Omega} = \sqrt{2}\,s = 1.414\,\text{s}$

T_1 is the period of precession of $\vec{\omega}$ about $\hat{e}_3$.

From equation 9.6.12,

$$\dot{\phi} = \omega\left[1 + \left(\frac{I_s^2}{I^2} - 1\right)\cos^2\alpha\right]^{\frac{1}{2}} = \omega\left[1 + \left(\frac{2^2}{1^2} - 1\right)\left(\frac{1}{\sqrt{2}}\right)^2\right]^{\frac{1}{2}}$$

$$\dot{\phi} = \omega\sqrt{\frac{5}{2}}$$

$$T_2 = \frac{2\pi}{\dot{\phi}} = \sqrt{0.4}\,s = 0.632\,\text{s}$$

T_2 is the period of wobble of $\hat{e}_3$ about $\vec{L}$.

(b) $$I_1 = I_2 = I = \frac{m}{12}\left(a^2 + \frac{a^2}{16}\right) = \frac{17}{16}\cdot\frac{ma^2}{12}$$

$$I_3 = I_s = \frac{m}{12}\left(a^2 + a^2\right) = 2\cdot\frac{ma^2}{12}$$

From equation 9.5.8, $\Omega = \left(\dfrac{2}{\frac{17}{16}} - 1\right)\omega\dfrac{1}{\sqrt{2}} = \dfrac{15}{17}\dfrac{\omega}{\sqrt{2}}$

$$T_1 = \frac{2\pi}{\Omega} = \frac{17}{15}\sqrt{2}\,s = 1.603\,\text{s}$$

From equation 9.6.12, $\dot{\phi} = \omega\left[1 + \left(\dfrac{2^2\cdot 16^2}{17^2} - 1\right)\left(\dfrac{1}{\sqrt{2}}\right)^2\right]^{\frac{1}{2}}$

$$\dot{\phi} = 1.5072\,\omega$$

$$T_2 = \frac{2\pi}{\dot{\phi}} = \frac{1}{1.5072}\,s = 0.663\,\text{s}$$

9.10 From equation 9.6.10, $\tan\theta = \dfrac{I}{I_s}\tan\alpha$.

Since $I_s > I$, $\theta < \alpha$ and the angle between the axis of rotation $(\vec{\omega})$ and the invariable line $(\vec{L})$ is $\alpha - \theta$.

From your favorite table of trigonometric identities …

$$\tan(\alpha - \theta) = \frac{\tan\alpha - \tan\theta}{1 + \tan\alpha\tan\theta}$$

$$\tan(\alpha-\theta)=\frac{\left(1-\frac{I}{I_s}\right)\tan\alpha}{1+\frac{I}{I_s}\tan^2\alpha}$$

$$\alpha-\theta=\tan^{-1}\left[\frac{(I_s-I)\tan\alpha}{I_s+I\tan^2\alpha}\right]$$

9.11 $\alpha-\theta$ is a maximum for $\frac{I_s}{I}$ a maximum.

For $\frac{I_s}{I}=2$, $\tan(\alpha-\theta)=\frac{\tan\alpha}{2+\tan^2\alpha}$

$$\frac{d\tan(\alpha-\theta)}{d\tan\alpha}=\frac{1}{2+\tan^2\alpha}-\frac{2\tan^2\alpha}{\left(2+\tan^2\alpha\right)^2}=\frac{2-\tan^2\alpha}{\left(2+\tan^2\alpha\right)^2}$$

At the maximum, $\frac{d\tan(\alpha-\theta)}{d\tan\alpha}=0=2-\tan^2\alpha$

$$\alpha=\tan^{-1}\sqrt{2}=54.7^\circ$$

$$\tan(\alpha-\theta)\le\frac{\left(1-\frac{1}{2}\right)\sqrt{2}}{1+\frac{1}{2}\left(\sqrt{2}\right)^2}=\frac{\sqrt{2}}{4}$$

$$\alpha-\theta\le\tan^{-1}\left(\frac{\sqrt{2}}{4}\right)=19.5^\circ$$

9.12 (a) From Problem 9.10, for $I_s>I$, the angle between $\vec{\omega}$ and $\vec{L}$ is …

$$\alpha-\theta=\tan^{-1}\left[\frac{(I_s-I)\tan\alpha}{I_s+I\tan^2\alpha}\right]$$

From Problem 9.9(a), $I_s=2I$ and $\tan\alpha=\tan 45^\circ=1$

$$\alpha-\theta=\tan^{-1}\frac{1}{(2+1)}=\tan^{-1}\frac{1}{3}=18.4^\circ$$

(b) From Prob. 9.9(b), $I_s=2\cdot\frac{ma^2}{12}$ and $I=\frac{17}{16}\cdot\frac{ma^2}{12}$

$$\alpha-\theta=\tan^{-1}\left[\frac{\left(2-\frac{17}{16}\right)}{2+\frac{17}{16}}\right]=\tan^{-1}\frac{15}{49}=17.0^\circ$$

9.13 From Example 9.6.2, $\alpha = 0.2''$ and $\frac{I_s}{I} = 1.00327$

It follows from equation 9.6.10 that, for $I_s > I$, the angle between $\vec{\omega}$ and $\vec{L}$ is:

$$\alpha - \theta = \alpha - \tan^{-1}\left(\frac{I}{I_s}\tan\alpha\right)$$

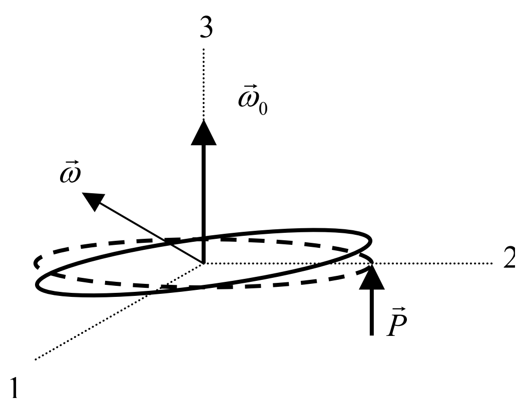
For α so small,

$$\tan\alpha \approx \alpha \text{, and } \tan^{-1}\left(\frac{I}{I_s}\tan\alpha\right) \approx \frac{I}{I_s}\tan\alpha \approx \frac{I\alpha}{I_s}$$

$$\alpha - \theta \approx \alpha - \frac{I\alpha}{I_s} = \frac{I_s - I}{I_s}\alpha$$

$$\alpha - \theta \approx \frac{.00327}{1.00327}0.2 = 0.00065 \text{ arcsec}$$

9.14 From Table 8.3.1, $I_s = \frac{ma^2}{2}$ and $I = \frac{ma^2}{4}$



$\vec{\omega}_\circ = \omega\hat{e}_3$ and $\vec{\hat{P}} = \frac{ma\omega}{4}\hat{e}_3$

Selecting the 2-axis through the point of impact, during the collision …

$$\int \vec{N}dt = \vec{r} \times \vec{p} = a\hat{e}_2 \times \frac{ma\omega}{4}\hat{e}_3 = \frac{ma^2\omega}{4}\hat{e}_1$$

The only non-zero component of $\vec{N}$ is N_1. During the collision $\omega_2 = 0$, so integrating the first of Euler's equations 9.3.5 …

$$\int N_1 dt = I_1\omega_1$$

Immediately after the collision,

$$\omega_1 = \frac{ma^2\omega}{4}\left(\frac{4}{ma^2}\right) = \omega$$

(ω_3 still is equal to ω, and $\omega_2 = 0$)

After the collision, $\vec{N} = 0$. From Prob. 9.8, with $I_1 = I_2$, for $\vec{N} = 0$,

$\omega_3 = constant$ and $\omega_1^2 + \omega_2^2 = constant$.

$$\tan\alpha = \frac{\left(\omega_1^2 + \omega_2^2\right)^{\frac{1}{2}}}{\omega_3} = constant$$

Using the values of ω_i *immediately after the collision* …

$$\tan\alpha = \frac{\left(\omega^2 + 0\right)^{\frac{1}{2}}}{\omega} = 1$$

$\alpha = 45^\circ$

From equation 9.5.8, with $\vec{\omega} = \vec{\omega}_\circ + \omega_1 \hat{e}_1 = \omega(\hat{e}_1 + \hat{e}_3)$:

$$\Omega = (2-1)(\sqrt{2}\omega)(\cos 45^\circ) = \omega$$

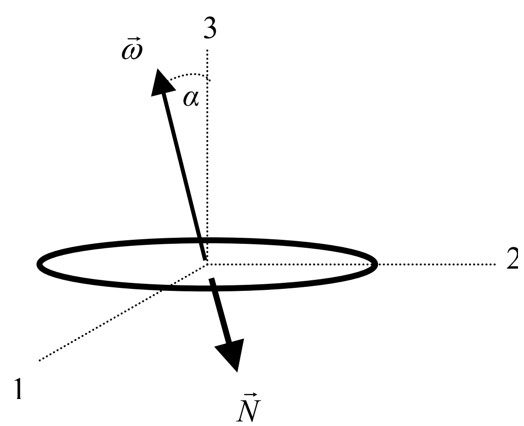
From equation 9.6.12 …

$$\dot{\phi} = \sqrt{2}\omega\left[1 + (2^2 - 1)(\cos^2 45^\circ)\right]^{\frac{1}{2}} = \omega\sqrt{5}$$

9.15 $\vec{N} = -c\vec{\omega} = -c(\omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3)$

Say the 3 axis is the symmetry axis, $I_3 = I_s$, $I_1 = I_2 = I$

For the third component of equations 9.3.5 …



$$-c\omega_3 = I_s \dot{\omega}_3 + 0$$

$$\frac{\dot{\omega}_3}{\omega_3} = -\frac{c}{I_s}$$

$$\ln \frac{\omega_3}{(\omega_3)_\circ} = -\frac{c}{I_s} t$$

$$\omega_3 = (\omega_3)_\circ e^{-\frac{ct}{I_s}}$$

For the first two components of equations 9.3.5 …

$-c\omega_1 = I\dot{\omega}_1 + \omega_2 \omega_3 (I_s - I)$, and

$-c\omega_2 = I\dot{\omega}_2 + \omega_1 \omega_3 (I - I_s)$

Rearranging terms and multiplying by ω_1 and ω_2 respectively …

$$\dot{\omega}_1 \omega_1 + \frac{c}{I}\omega_1^2 + \omega_1 \omega_2 \omega_3 (I_s - I) = 0$$

$$\dot{\omega}_2 \omega_2 + \frac{c}{I}\omega_2^2 - \omega_1 \omega_2 \omega_3 (I_s - I) = 0$$

Adding, $\dot{\omega}_1 \omega_1 + \dot{\omega}_2 \omega_2 + \frac{c}{I}(\omega_1^2 + \omega_2^2) = 0$

$$\frac{1}{(\omega_1^2 + \omega_2^2)} \frac{d}{dt}(\omega_1^2 + \omega_2^2) = -\frac{2c}{I}$$

$$\ln \frac{\omega_1^2 + \omega_2^2}{(\omega_1^2 + \omega_2^2)_\circ} = -\frac{2c}{I} t$$

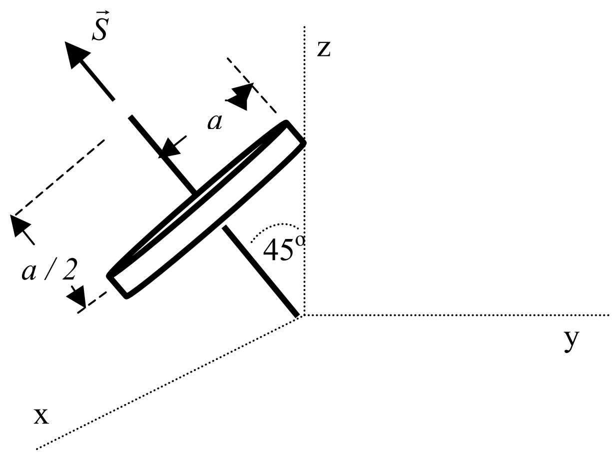
$$(\omega_1^2 + \omega_2^2) = (\omega_1^2 + \omega_2^2)_\circ e^{-\frac{2ct}{I}}$$

$$(\omega_1^2 + \omega_2^2)^{\frac{1}{2}} = (\omega_1^2 + \omega_2^2)_\circ^{\frac{1}{2}} e^{\frac{ct}{I}}$$

$$\tan\alpha = \frac{(\omega_1^2 + \omega_2^2)^{\frac{1}{2}}}{\omega_3} = (\tan\alpha_\circ) e^{-ct\left(\frac{1}{I} - \frac{1}{I_s}\right)}$$

For $I_s > I$, $\left(\frac{1}{I} - \frac{1}{I_s}\right) > 0$ and α decreases with time.

9.16 (a) From table 8.3.1… $I_s = \frac{ma^2}{2}$ about the symmetry (spin) axis.



$I_{cm\ disk} = m\frac{a^2}{4}$ about an axis through the center of mass of the disk and ⊥ to the spin axis. From the parallel axis theorem, equation 8.3.21, …

$I_{disk} = \frac{ma^2}{4} + m\left(\frac{a}{2}\right)^2$ about the pivot point.

$I_{rod} = \left(\frac{m}{2}\right)\frac{a^2}{3}$ moment of inertia of the rod about the pivot point. Thus …

$$I = I_{disc} + I_{rod} = \left[\frac{ma^2}{4} + m\left(\frac{a}{2}\right)^2\right] + \left(\frac{m}{2}\right)\frac{a^2}{3} = \frac{2}{3}ma^2$$

From equation 9.7.10, $\dot{\phi} = \frac{I_s S + \left(I_s^2 S^2 - 4MglI\cos\theta\right)^{\frac{1}{2}}}{2I\cos\theta}$

$$\dot{\phi} = \frac{1}{2\cos\theta}\left\{\frac{I_s}{I}S \pm \left[\left(\frac{I_s}{I}\right)^2 S^2 - \frac{4Mgl\cos\theta}{I}\right]^{\frac{1}{2}}\right\}$$

$$\frac{I_s}{I} = \frac{\frac{ma^2}{2}}{\frac{2ma^2}{3}} = \frac{3}{4}, \quad \cos\theta = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

$$\frac{Ml}{I} = \frac{\left(m + \frac{m}{2}\right)\left(\frac{a}{2}\right)}{\frac{2ma^2}{3}} = \frac{9}{8a} = \frac{9}{80}cm^{-1}$$

$$\dot{\phi} = \frac{1}{\sqrt{2}}\left\{\frac{3}{4}(900\,rpm) \pm \left[\left(\frac{3}{4}\right)^2 (900)^2 - 4\left(\frac{9}{80}cm^{-1}\right)\left(980\frac{cm}{s^2}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{60\,rpm}{2\pi s^{-1}}\right)^2\right]^{\frac{1}{2}}\right\}$$

$\dot{\phi} = 15.1\,rpm$ or $939\,rpm$

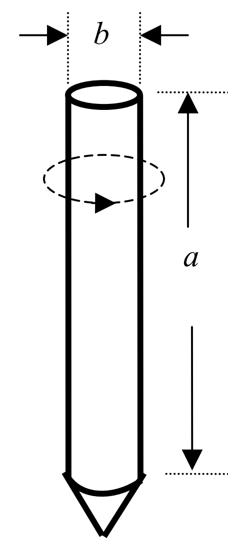
(b) From equation 9.7.12, $I_s^2 S^2 \ge 4MglI$

$$\frac{I}{I_s} = \frac{4}{3}, \quad \frac{Ml}{I_s} = \frac{\left(\frac{3m}{2}\right)\left(\frac{a}{2}\right)}{\frac{ma^2}{2}} = \frac{3}{2a} = \frac{3}{20} cm^{-1}$$

$$S^2 \ge \frac{4Mgl}{I_s} \cdot \frac{I}{I_s} = 4\left(\frac{3}{20} cm^{-1}\right)\left(980\, cm \cdot s^{-2}\right)\left(\frac{4}{3}\right)\left(\frac{60\, rpm}{2\pi\, s^{-1}}\right)^2$$

$$S \ge 267\, rpm$$

9.17 (Neglecting the pencil head) From Table 8.3.1,



$$I_s = \frac{m\left(\frac{b}{2}\right)^2}{2} = \frac{mb^2}{8}$$

The moment of inertia of the pencil about an axis thru its center of mass and ⊥ to its symmetry axis from equation

$$8.3.26 \ldots\ I_c = m\left[\frac{\left(\frac{b}{2}\right)^2}{4} + \frac{a^2}{12}\right] = m\left(\frac{b^2}{16} + \frac{a^2}{12}\right)$$

From the parallel axis theorem, $I = I_c + m\left(\frac{a}{2}\right)^2 = m\left(\frac{b^2}{16} + \frac{a^2}{3}\right)$

From equation 9.7.12, $S^2 \ge \frac{4MglI}{I_s^2}$

$$S^2 \ge \frac{4mg\left(\frac{a}{2}\right)m\left(\frac{b^2}{16} + \frac{a^2}{3}\right)}{\frac{m^2b^4}{64}}$$

$$S \ge \frac{16}{b^2}\left[\frac{ga}{2}\left(\frac{b^2}{16} + \frac{a^2}{3}\right)\right]^{\frac{1}{2}} = \frac{16}{(1)^2}\left[\frac{(980)(20)}{2}\left(\frac{1^2}{16} + \frac{20^2}{3}\right)\right]^{\frac{1}{2}} rad \cdot s^{-1}$$

$$S \ge 18{,}294\, rad \cdot s^{-1} = 2910\, rps$$

9.18 $I_s = I_{rim} + I_{spokes} + I_{hub}$

$$I_{rim} = m_{rim}a^2 = \frac{ma^2}{2}$$

$I_{spokes} = m_{spokes}\frac{a^3}{3} = \frac{ma^2}{12}$, assuming the spokes to be thin rods

$I_{hub} = 0$, assuming its radius is small

From the perpendicular axis theorem, equation 8.3.14, $I = \frac{I_s}{2}$

From equation 9.10.14, $S^2 > \frac{Imga}{I_s\left(I_s + ma^2\right)}$

$$S > \left[\frac{1}{2}\frac{mga}{\left(\frac{7ma^2}{12}\right) + ma^2}\right]^{\frac{1}{2}} = \left(\frac{6g}{19a}\right)^{\frac{1}{2}}$$

For rolling without slipping, $v = aS$

$$v > \left(\frac{6ga}{19}\right)^{\frac{1}{2}} = \left[\frac{6\times 32\times\left(\frac{30}{2}\right)}{19\times 12}\right]^{\frac{1}{2}} ft\cdot s^{-1} = 3.55\ ft\cdot s^{-1}$$

If the spokes and hub are neglected, $I_s = \frac{ma^2}{2}$

$$S > \left[\frac{1}{2}\frac{mga}{\left(\frac{ma^2}{2}\right) + ma^2}\right]^{\frac{1}{2}} = \left(\frac{g}{3a}\right)^{\frac{1}{2}}$$

$$v > \left(\frac{ga}{3}\right)^{\frac{1}{2}} = \left[\frac{32\times\left(\frac{30}{2}\right)}{3\times 12}\right]^{\frac{1}{2}} ft\cdot s^{-1} = 3.65\ ft\cdot s^{-1}$$

9.19 From equations 9.3.5 with $\vec{N} = 0$ …

$$I_1\dot{\omega}_1 + \left(I_3 - I_2\right)\omega_2\omega_3 = 0$$
$$I_2\dot{\omega}_2 + \left(I_1 - I_3\right)\omega_1\omega_3 = 0$$
$$I_3\dot{\omega}_3 + \left(I_2 - I_1\right)\omega_1\omega_2 = 0$$

Differentiating the first equation with respect to t:

$$I_1\ddot{\omega}_1 + \left(I_3 - I_2\right)\left(\omega_2\dot{\omega}_3 + \dot{\omega}_2\omega_3\right) = 0$$

From the second and third equations:

$$\dot{\omega}_2 = \frac{\left(I_3 - I_1\right)}{I_2}\omega_1\omega_3, \text{ and } \dot{\omega}_3 = \frac{\left(I_1 - I_2\right)}{I_3}\omega_1\omega_2$$

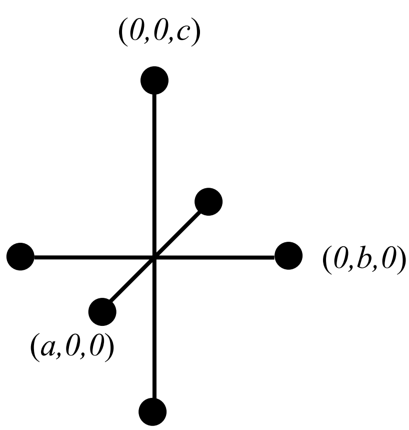
$$I_1\ddot{\omega}_1 + (I_3 - I_2)\left[\frac{(I_1 - I_2)}{I_3}\omega_1\omega_2^2 + \frac{(I_3 - I_1)}{I_2}\omega_1\omega_3^2\right] = 0$$

$$\ddot{\omega}_1 + K_1\omega_1 = 0\,,\ K_1 = -\omega_2^2\left[\frac{(I_3 - I_2)(I_2 - I_1)}{I_1 I_3}\right] + \omega_3^2\left[\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\right]$$

(a) For ω_3 large and $\omega_2 = 0$, $K_1 > 0$ so $\ddot{\omega}_1 + K_1\omega_1 = 0$ is the harmonic oscillator equation. ω_1 oscillates, but remains small. Motion is stable for initial rotation about the 3 axis if the 3 axis is the principal axis having the largest or smallest moment of inertia.

(b) For $\omega_3 = 0$ and ω_2 large, $K_1 < 0$ so $\ddot{\omega}_1 + K_1\omega_1 = 0$ is the differential equation for exponential growth of ω_1 with time: $\omega_1 = Ae^{\sqrt{k_1}t} + Be^{-\sqrt{k_1}t}$. Motion is unstable for the initial rotation mostly about the principal axis having the median moment of inertia.

9.20 $I_{xy} = \sum_i m_i x_i y_i = 0$ since either x_i or y_i is zero for all six particles. Similarly, all the other products of inertia are zero. Therefore the coordinate axes are principle axes.



$$I_{xx} = \sum_i m_i\left(y_i^2 + z_i^2\right) = m\left[0 + 0 + b^2 + (-b)^2 + c^2 + (-c)^2\right]$$

$$I_{xx} = 2m\left(b^2 + c^2\right)$$
$$I_{yy} = 2m\left(a^2 + c^2\right)$$
$$I_{zz} = 2m\left(a^2 + b^2\right)$$

$$\vec{I} = 2m\begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$$

$$\vec{\omega} = \frac{\omega}{\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}\left(a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3\right) = \frac{\omega}{\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

From equation 9.1.28, $\vec{L} = \vec{I}\vec{\omega}$

$$\vec{L} = \frac{2m\omega}{\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}\begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\vec{L} = \frac{2m\omega}{\left(a^2+b^2+c^2\right)^{\frac{1}{2}}}\begin{bmatrix} a\left(b^2+c^2\right) \\ b\left(a^2+c^2\right) \\ c\left(a^2+b^2\right)\end{bmatrix}$$

From equation 9.1.32, $T = \frac{1}{2}\vec{\omega}\cdot\vec{L}$

$$T = \frac{1}{2}\frac{2m\omega^2}{\left(a^2+b^2+c^2\right)}\begin{bmatrix} a & b & c\end{bmatrix}\begin{bmatrix} a\left(b^2+c^2\right) \\ b\left(a^2+c^2\right) \\ c\left(a^2+b^2\right)\end{bmatrix}$$

$$T = \frac{m\omega^2}{a^2+b^2+c^2}\left[a^2\left(b^2+c^2\right)+b^2\left(a^2+c^2\right)+c^2\left(a^2+b^2\right)\right]$$

$$T = \frac{2m\omega^2}{a^2+b^2+c^2}\left(a^2b^2+a^2c^2+b^2c^2\right)$$

9.21 For Problem 9.1:

$$\vec{I} = ma^2\begin{pmatrix} \frac{1}{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{4}{3} & 0 \\ 0 & 0 & \frac{5}{3}\end{pmatrix}, \qquad \vec{\omega} = \frac{\omega}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \\ 0\end{pmatrix}$$

$$\vec{L} = \vec{I}\,\vec{\omega} = \frac{ma^2\omega}{\sqrt{5}}\begin{pmatrix} \frac{1}{3} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{4}{3} & 0 \\ 0 & 0 & \frac{5}{3}\end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0\end{pmatrix}$$

$$= \frac{ma^2\omega}{\sqrt{5}}\begin{pmatrix} \frac{2}{3}-\frac{1}{2} \\ -1+\frac{4}{3} \\ 0\end{pmatrix} = \frac{ma^2\omega}{\sqrt{5}}\begin{pmatrix} \frac{1}{6} \\ \frac{1}{3} \\ 0\end{pmatrix} = \frac{ma^2\omega}{6\sqrt{5}}\begin{pmatrix} 1 \\ 2 \\ 0\end{pmatrix}$$

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\frac{\omega}{\sqrt{5}}\frac{ma^2\omega}{6\sqrt{5}}\begin{pmatrix} 2 & 1 & 0\end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 0\end{pmatrix}$$

$$= \frac{ma^2\omega^2}{60}\left(2+2+0\right) = \frac{ma^2\omega^2}{15}$$

For Problem 9.4:

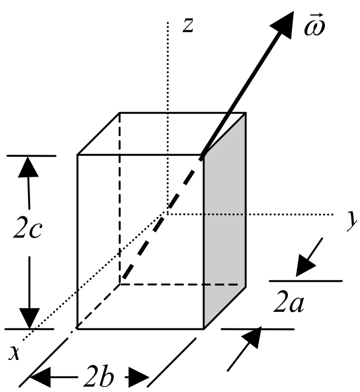
$$\vec{I} = \frac{ma^2}{12}\begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix}, \qquad \vec{\omega} = \frac{\omega}{\sqrt{14}}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$\vec{L} = \vec{I}\vec{\omega} = \frac{ma^2\omega}{12\sqrt{14}}\begin{pmatrix} 13 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{ma^2\omega}{12\sqrt{14}}\begin{pmatrix} 13 \\ 20 \\ 15 \end{pmatrix}$$

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{L} = \frac{1}{2}\frac{\omega}{\sqrt{14}}\frac{ma^2\omega}{12\sqrt{14}}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 13 \\ 20 \\ 15 \end{pmatrix}$$

$$= \frac{ma^2\omega^2}{24(14)}(13+40+45) = \frac{7}{24}ma^2\omega^2$$

9.22 Since the coordinate axes are axes of symmetry, they are principal axes and all products of inertia are zero.



From Table 8.3.1,

$$I_{xx} = \frac{m}{12}\left[(2b)^2 + (2c)^2\right] = \frac{m}{3}\left(b^2 + c^2\right)$$

$$I_{yy} = \frac{m}{3}\left(a^2 + c^2\right), \qquad I_{zz} = \frac{m}{3}\left(a^2 + b^2\right)$$

$$\vec{I} = \frac{m}{3}\begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}$$

$$\vec{\omega} = \frac{\omega}{\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}\left(a\hat{e}_1 + b\hat{e}_2 + c\hat{e}_3\right) = \frac{\omega}{\left(a^2 + b^2 + c^2\right)}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

From equation 9.1.28, $\vec{L} = \vec{I}\vec{\omega}$

$$\vec{L} = \frac{m\omega}{3\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}\begin{bmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\vec{L} = \frac{m\omega}{3\left(a^2 + b^2 + c^2\right)^{\frac{1}{2}}}\begin{bmatrix} a\left(b^2 + c^2\right) \\ b\left(a^2 + c^2\right) \\ c\left(a^2 + b^2\right) \end{bmatrix}$$

From equation 9.1.32, $T = \frac{1}{2}\vec{\omega}^T \cdot \vec{L}$

$$T=\frac{1}{2}\frac{m\omega^2}{3\left(a^2+b^2+c^2\right)}\begin{bmatrix}a & b & c\end{bmatrix}\begin{bmatrix}a\left(b^2+c^2\right)\\ b\left(a^2+c^2\right)\\ c\left(a^2+b^2\right)\end{bmatrix}$$

$$T=\frac{m\omega^2}{6\left(a^2+b^2+c^2\right)}\left[a^2\left(b^2+c^2\right)+b^2\left(a^2+c^2\right)+c^2\left(a^2+b^2\right)\right]$$

$$T=\frac{m\omega^2}{3\left(a^2+b^2+c^2\right)}\left(a^2b^2+a^2c^2+b^2c^2\right)$$

With the origin at one corner, from the parallel axis theorem:

$$I_{xx}=\frac{m\left(b^2+c^2\right)}{3}+m\left(b^2+c^2\right)=\frac{4m}{3}\left(b^2+c^2\right)$$

$$I_{yy}=\frac{4m}{3}\left(a^2+c^2\right),\qquad I_{zz}=\frac{4m}{3}\left(a^2+b^2\right)$$

$$I_{xy}=-\int xy\,dm=-\int xy\rho\,dV$$

$$I_{xy}=-\rho\int_{x=0}^{x=2a}\int_{y=0}^{y=2b}\int_{z=0}^{z=2c}xy\,dx\,dy\,dz=-8\rho a^2b^2c$$

$m=\rho(2a)(2b)(2c)=8\rho abc$, so $I_{xy}=-mab$

$I_{xy}=-mac$, $\qquad I_{yz}=-mbc$

$$\overleftrightarrow{I}=m\begin{bmatrix}\frac{4}{3}\left(b^2+c^2\right) & -ab & -ac\\ -ab & \frac{4}{3}\left(a^2+c^2\right) & -bc\\ -ac & -bc & \frac{4}{3}\left(a^2+b^2\right)\end{bmatrix}$$

9.23 (See Figure 9.7.1)

$$L_z=\left(L_{x'}\right)_z+\left(L_{y'}\right)_z+\left(L_{z'}\right)_z$$

$$L_z=L_{y'}\sin\theta+L_{z'}\cos\theta=\left(I\dot{\phi}\sin\theta\right)\sin\theta+\left(I_sS\right)\cos\theta$$

9.24 (See Figure 9.7.1) If the top precesses without nutation, it must do so at $\theta=\theta_\circ$ where $V\left(\theta_\circ\right)$ is a minimum of $V(\theta)$ …

$$\left.\frac{dV(\theta)}{d\theta}\right|_{\theta=\theta_\circ}=0$$

$$V(\theta)=\frac{\left(L_z-L_{z'}\cos\theta\right)^2}{2I\sin^2\theta}+mgl\cos\theta \qquad \text{(See equation 9.8.7)}$$

$$\left.\frac{dV}{d\theta}\right|_{\theta=\theta_\circ}=\frac{-\cos\theta_\circ\left(L_z-L_{z'}\cos\theta_\circ\right)^2+L_{z'}\sin^2\theta_\circ\left(L_z-L_{z'}\cos\theta_\circ\right)}{I\sin^3\theta_\circ}-mgl\sin\theta_\circ=0$$

let $\gamma=L_z-L_{z'}\cos\theta_\circ$

Then $\gamma^2\cos\theta_\circ-\gamma L_{z'}\sin^2\theta_\circ+mglI\sin^4\theta_\circ=0$ and solving for γ

$$\gamma=\frac{L_{z'}\sin^2\theta_\circ}{2\cos\theta_\circ}\left[1\pm\left(1-\frac{4mglI\cos\theta_\circ}{L_{z'}^2}\right)^{\frac{1}{2}}\right]$$

now $L_{z'}$ is large since $\dot{\psi}$ is large and the precession rate is small, so we can expand the term in square root above and use the (-) solution since γ must be positive …

$$\gamma\approx\frac{L_{z'}\sin^2\theta_\circ}{2\cos\theta_\circ}\left[1-1+\frac{2mglI\cos\theta_\circ}{L_{z'}^2}\right]$$

$$\gamma=L_z-L_{z'}\cos\theta_\circ\approx\frac{mglI\sin^2\theta_\circ}{L_{z'}}$$

From equations 9.7.2, 9.7.5 and 9.7.7 …

$$L_z=I\dot{\phi}^2\sin^2\theta_0+I_s\left(\dot{\phi}\cos\theta_0+\dot{\psi}\right)\cos\theta_0 \qquad \text{and …}$$

$$L_{z'}\cos\theta_0=I_s(\dot{\phi}\cos\theta_0+\dot{\psi})\cos\theta_0 \qquad \text{so …}$$

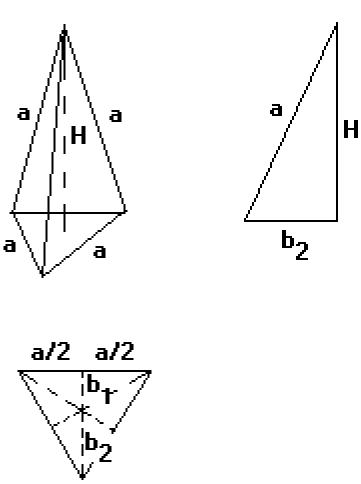
$$\gamma=\left(I\sin^2\theta_\circ+I_s\cos^2\theta_\circ\right)\dot{\phi}+I_s\dot{\psi}\cos\theta_\circ-I_s\dot{\phi}\cos^2\theta_\circ-I_s\dot{\psi}\cos\theta_\circ$$

$$=I\dot{\phi}\sin^2\theta_\circ\approx\frac{mglI\sin^2\theta_\circ}{L_{z'}}=\frac{mglI\sin^2\theta_\circ}{I_s\left(\dot{\psi}+\dot{\phi}\cos\theta_\circ\right)}$$ and since $\dot{\psi}>>0$, we can ignore the $\dot{\phi}$ term in the denominator and we have …

$$\dot{\phi}\approx\frac{mgl}{I_s\dot{\psi}}$$

Hence, if $\dot{\psi}$ large, $\left.\dot{\theta}\right|_{\theta=\theta_1}=0$ and $\left.\dot{\phi}\right|_{\theta=\theta_1}=\frac{mgl}{I_s\dot{\psi}}$ the top will precess without nutation at $\theta_1=\theta_\circ$ the place where $V(\theta)=\min$.

9.25



$$b_1+b_2=\sqrt{3}\left(\frac{a}{2}\right)$$

$$\frac{b_1}{b_2}=\sin 30^\circ=\frac{1}{2}$$

$$b_1=\frac{a}{2\sqrt{3}} \qquad b_2=\frac{a}{\sqrt{3}}$$

$$a^2=H^2+b_2^2$$

$$H^2=a^2-b_2^2=a^2-\frac{a^2}{3}=\frac{2a^2}{3}$$

$$H = \sqrt{\frac{2}{3}}a$$

Thus, the coordinates of the 4 atoms are:

Oxygen:	$(0,0,H)$
Hydrogen:	$\left(-b_1,\frac{a}{2},0\right);\left(-b_1,-\frac{a}{2},0\right)$
Carbon:	$(b_2,0,0)$

(a) The axes 1, 2, 3 are principal axes if the products of inertia are zero.

$$-I_{xy} = \sum m_i x_i y_i = 0 + \left(-b_1\frac{a}{2}\right) + \left(-b_1\left(\frac{-a}{2}\right)\right) + 0 \equiv 0$$

$$-I_{yz} = 0+0+0+0 \equiv 0$$

$$-I_{xz} = 0+0+0+0 \equiv 0$$

The 1,2,3 axes are principal axes.

(b) Find principal moments

$$I_1 = I_{xx} = \sum m_i\left(y_i^2+z_i^2\right) = 16H^2 + 1\left(\frac{a}{2}\right)^2 + 1\left(\frac{a}{2}\right)^2 = \frac{67}{6}a^2$$

$$I_2 = I_{yy} = \sum m_i\left(x_i^2+z_i^2\right) = 16H^2 + 1\left(b_1^2\right) + 1\left(b_1^2\right) + 12b_2^2 = \frac{89}{6}a^2$$

$$I_3 = I_{zz} = \sum m_i\left(x_i^2+y_i^2\right) = 0 + \left(b_1^2 + \left(\frac{a}{2}\right)^2\right) + \left(b_1^2 + \left(\frac{a}{2}\right)^2\right) + b_2^2 = \frac{14}{3}a^2$$

$$I = \begin{pmatrix} \frac{67}{6} & 0 & 0 \\ 0 & \frac{89}{6} & 0 \\ 0 & 0 & \frac{14}{3} \end{pmatrix} a^2$$

(c) $I_3 < I_1 < I_2$ therefore rotation about the1-axis is unstable (see discussion Sec. 9.4)

CHAPTER 10
LAGRANGIAN MECHANICS

10.1 Solution … $x(t) = x(0,t) + \alpha\eta(t)$

$$\dot{x}(t) = \dot{x}(0,t) + \alpha\dot{\eta}(t)$$

where $x(0,t) = \sin\omega t$ and $\dot{x}(0,t) = \omega\cos\omega t$

$$T = \frac{1}{2}m\dot{x}^2 \qquad V = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2x^2$$

so: $$J(\alpha) = \frac{m}{2}\int_{t_1}^{t_2}\left(\dot{x}^2 - \omega^2x^2\right)dt$$

$$= \frac{m}{2}\int_{t_1}^{t_2}\left[\left(\omega\cos\omega t + \alpha\dot{\eta}\right)^2 - \omega^2\left(\sin\omega t + \alpha\eta\right)^2\right]dt$$

$$J(\alpha) = \frac{m}{2}\int_{t_1}^{t_2}\omega^2\left(\cos^2\omega t - \sin^2\omega t\right)dt + m\alpha\omega\int_{t_1}^{t_2}\left(\dot{\eta}\cos\omega t - \omega\eta\sin\omega t\right)dt + \frac{\alpha^2 m}{2}\int_{t_1}^{t_2}\left(\dot{\eta}^2 - \omega^2\eta^2\right)dt$$

Examine the term linear in α:

$$\int_{t_1}^{t_2}\left(\dot{\eta}\cos\omega t - \omega\eta\sin\omega t\right)dt = \eta(t)\cos\omega t\Big|_{t_1}^{t_2} + \int_{t_1}^{t_2}\omega\eta\sin\omega t\,dt - \int_{t_1}^{t_2}\omega\eta\sin\omega t\,dt \equiv 0$$

(1^{st} term vanishes at both endpoints: $\eta(t_2) = \eta(t_1) = 0$)

so $$J(\alpha) = \frac{1}{2}m\omega^2\int_{t_1}^{t_2}\cos 2\omega t\,dt + \frac{1}{2}m\alpha^2\int_{t_1}^{t_2}\left(\dot{\eta}^2 - \omega^2\eta^2\right)dt$$

$$= \frac{1}{4}m\omega\left[\sin 2\omega t_2 - \sin 2\omega t_1\right] + \frac{1}{2}m\alpha^2\int_{t_1}^{t_2}\left(\dot{\eta}^2 - \omega^2\eta^2\right)dt$$

which is a <u>minimum</u> at $\alpha = 0$

10.2 $V = mgz$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - mgz$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \qquad \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\ddot{y}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}$$

$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial y} = 0, \qquad \frac{\partial L}{\partial z} = -mg$$

From equations 10.4.5 … $\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0$

$$m\ddot{x} = 0, \qquad m\dot{x} = const$$
$$m\ddot{y} = 0, \qquad m\dot{y} = const$$
$$m\ddot{z} = -mg$$

10.3 Choosing generalized coordinate x as linear displacement down the inclined plane (See Figure 8.6.1), for rolling without slipping …

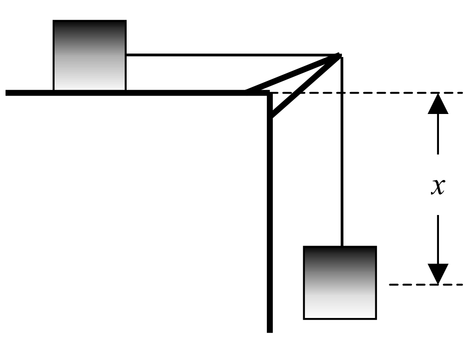
$$\omega = \frac{\dot{x}}{a}$$
$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{2}{5}ma^2\right)\left(\frac{\dot{x}}{a}\right)^2 = \frac{7}{10}m\dot{x}^2$$

For $V = 0$ at the initial position of the sphere,

$$V = -mgx\sin\theta$$
$$L = T - V = \frac{7}{10}m\dot{x}^2 + mgx\sin\theta$$
$$\frac{\partial L}{\partial \dot{x}} = \frac{7}{5}m\dot{x}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{7}{5}m\ddot{x}$$
$$\frac{\partial L}{\partial x} = mg\sin\theta$$
$$\frac{7}{5}m\ddot{x} = mg\sin\theta$$
$$\ddot{x} = \frac{5}{7}g\sin\theta$$

From equations 8.6.11 - 8.6.13 … $\ddot{x}_{cm} = \frac{5}{7}g\sin\theta$

10.4 (a) For x the distance of the hanging block below the edge of the table:



$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{x}^2 = m\dot{x}^2 \quad \text{and} \quad V = -mgx$$
$$L = T - V = m\dot{x}^2 + mgx$$
$$\frac{\partial L}{\partial \dot{x}} = 2m\dot{x}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = 2m\ddot{x}$$
$$\frac{\partial L}{\partial x} = mg$$
$$2m\ddot{x} = mg$$
$$\ddot{x} = \frac{g}{2}$$

(b) $T = m\dot{x}^2 + \frac{1}{2}m'\dot{x}^2 = \left(m + \frac{m'}{2}\right)\dot{x}^2$ and $V = -mgx - \left(\frac{x}{l}m'\right)g\frac{x}{2} = -mgx - \frac{m'g}{2l}x^2$

$$L = T - V = \left(m + \frac{m'}{2}\right)\dot{x}^2 + mgx + \frac{m'g}{2l}x^2$$

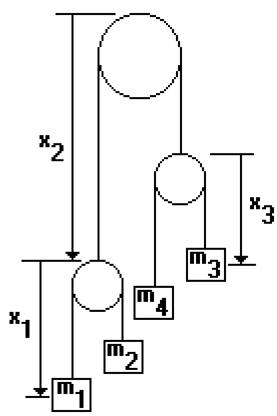
$$\frac{\partial L}{\partial \dot{x}} = (2m + m')\dot{x}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (2m + m')\ddot{x}$$

$$\frac{\partial L}{\partial x} = mg + \frac{m'g}{l}x$$

$$(2m + m')\ddot{x} = mg + \frac{m'g}{l}x$$

$$\ddot{x} = \frac{g}{l}\left(\frac{ml + m'x}{2m + m'}\right)$$

10.5



The four masses have positions:

m1 : $x_1 + x_2$

m2 : $l_1 - x_1 + x_2$

m3 : $l_2 - x_2 + x_3$

m4 : $l_2 - x_2 + l_3 - x_3$

$$V = -g\left[m_1(x_1 + x_2) + m_2(l_1 - x_1 + x_2) + m_3(l_2 - x_2 + x_3) + m_4(l_2 - x_2 + l_3 - x_3)\right]$$

$$T = \frac{1}{2}\left[m_1(\dot{x}_1 + \dot{x}_2)^2 + m_2(-\dot{x}_1 + \dot{x}_2)^2 + m_3(-\dot{x}_2 + \dot{x}_3)^2 + m_4(-\dot{x}_2 - \dot{x}_3)^2\right]$$

$$L = T - V = \frac{1}{2}m_1(\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{2}m_2(-\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{2}m_3(-\dot{x}_2 + \dot{x}_3)^2$$
$$+\frac{1}{2}m_4(-\dot{x}_2 - \dot{x}_3)^2 + gx_1(m_1 - m_2) + gx_2(m_1 + m_2 - m_3 - m_4) + gx_3(m_3 - m_4) + const.$$

$$\frac{\partial L}{\partial \dot{x}_1} = m_1(\dot{x}_1 + \dot{x}_2) - m_2(-\dot{x}_1 + \dot{x}_2)$$
$$= (m_1 + m_2)\dot{x}_1 + (m_1 - m_2)\dot{x}_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = (m_1 + m_2)\ddot{x}_1 + (m_1 - m_2)\ddot{x}_2$$

$$\frac{\partial L}{\partial x_1} = g(m_1 - m_2)$$

$$(m_1 + m_2)\ddot{x}_1 + (m_1 - m_2)\ddot{x}_2 = g(m_1 - m_2)$$

$$\frac{\partial L}{\partial \dot{x}_2} = m_1(\dot{x}_1 + \dot{x}_2) + m_2(-\dot{x}_1 + \dot{x}_2) - m_3(-\dot{x}_2 + \dot{x}_3) - m_4(-\dot{x}_2 - \dot{x}_3)$$

$$\frac{\partial L}{\partial x_2} = g(m_1 + m_2 - m_3 - m_4)$$

$$(m_1 - m_2)\ddot{x}_1 + (m_1 + m_2 + m_3 + m_4)\ddot{x}_2 + (m_4 - m_3)\ddot{x}_3 = g(m_1 + m_2 - m_3 - m_4)$$

$$\frac{\partial L}{\partial \dot{x}_3} = m_3(-\dot{x}_2 + \dot{x}_3) - m_4(-\dot{x}_2 - \dot{x}_3)$$

$$\frac{\partial L}{\partial x_3} = g(m_3 - m_4)$$

$$(m_4 - m_3)\ddot{x}_2 + (m_3 + m_4)\ddot{x}_3 = g(m_3 - m_4)$$

For $m_1 = m$, $m_2 = 4m$, $m_3 = 2m$, and $m_4 = m$:

$$5m\ddot{x}_1 - 3m\ddot{x}_2 = -3mg, \qquad \ddot{x}_1 = \frac{3}{5}(\ddot{x}_2 - g)$$

$$-3m\ddot{x}_1 + 8m\ddot{x}_2 - m\ddot{x}_3 = 2mg$$

$$-m\ddot{x}_2 + 3m\ddot{x}_3 = mg, \qquad \ddot{x}_3 = \frac{1}{3}(\ddot{x}_2 + g)$$

Substituting into the second equation:

$$-\frac{9}{5}\ddot{x}_2 + \frac{9}{5}g + 8\ddot{x}_2 - \frac{1}{3}\ddot{x}_2 - \frac{1}{3}g = 2g$$

$$\frac{88}{15}\ddot{x}_2 = \frac{8}{15}g, \qquad \ddot{x}_2 = \frac{g}{11}$$

$$\ddot{x}_1 = \frac{3}{5}\left(-\frac{10}{11}g\right) = -\frac{6}{11}g$$

$$\ddot{x}_3 = \frac{1}{3}\left(\frac{12}{11}g\right) = \frac{4}{11}g$$

Accelerations:

$$m_1: \quad \ddot{x}_1 + \ddot{x}_2 = -\frac{5}{11}g$$

$$m_2: \quad -\ddot{x}_1 + \ddot{x}_2 = \frac{7}{11}g$$

$$m_3: \quad -\ddot{x}_2 + \ddot{x}_3 = \frac{3}{11}g$$

$$m_4: \quad -\ddot{x}_2 - \ddot{x}_3 = -\frac{5}{11}g$$

10.6 See figure 10.5.3, replacing the block with a ball. The square of the speed of the ball is calculated in the same way as for the block in Example 10.5.6.

$$v^2 = \dot{x}^2 + \dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta$$

The ball also rotates with angular velocity ω so …

$$T = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 + \frac{1}{2}M\dot{x}^2$$

For rolling without slipping, $\omega = \frac{\dot{x}'}{a}$. $\quad I = \frac{2}{5}ma^2$

$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta\right) + \frac{1}{5}m\dot{x}'^2 + \frac{1}{2}M\dot{x}^2$$

$V = -mgx'\sin\theta$, for $V = 0$ at the initial position of the ball.

$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \frac{7}{5}\dot{x}'^2 + 2\dot{x}\dot{x}'\cos\theta\right) + \frac{1}{2}M\dot{x}^2 + mgx'\sin\theta$$

$$\frac{\partial L}{\partial \dot{x}'} = \frac{7}{5}m\dot{x}' + m\dot{x}\cos\theta, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}'}\right) = \frac{7}{5}m\ddot{x}' + m\ddot{x}\cos\theta$$

$$\frac{\partial L}{\partial x'} = mg\sin\theta$$

$$\frac{7}{5}m\ddot{x}' + m\ddot{x}\cos\theta = mg\sin\theta$$

$$\ddot{x}' = \frac{5}{7}\left(g\sin\theta - \ddot{x}\cos\theta\right)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + m\dot{x}'\cos\theta + M\dot{x}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(m + M\right)\ddot{x} + m\ddot{x}'\cos\theta$$

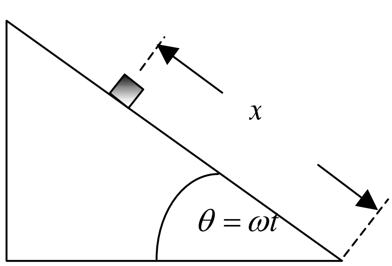
$$\frac{\partial L}{\partial x} = 0$$

$$\left(m + M\right)\ddot{x} + m\ddot{x}'\cos\theta = 0$$

$$\left(m + M\right)\ddot{x} + \frac{5}{7}mg\sin\theta\cos\theta - \frac{5}{7}m\ddot{x}\cos^2\theta = 0$$

$$\ddot{x} = \frac{5mg\sin\theta\cos\theta}{5m\cos^2\theta - 7\left(m + M\right)}$$

10.7 Let x be the slant height of the particle …



$$v^2 = \dot{x}^2 + x^2\omega^2$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{x}^2 + x^2\omega^2\right)$$

$$V = mgx\sin\theta = mgx\sin\omega t$$

$$L = T - V = \frac{1}{2}m\left(\dot{x} + x^2\omega^2\right) - mgx\sin\omega t$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x}$$

$$\frac{\partial L}{\partial x} = mx\omega^2 - mg\sin\omega t$$

$$m\ddot{x} = mx\omega^2 - mg\sin\omega t$$

$$\ddot{x} - \omega^2 x = -g\sin\omega t$$

The solution to the homogeneous equation

$$\ddot{x}-\omega^2 x=0\text{, is } x=Ae^{\omega t}+Be^{-\omega t}$$

Assuming a particular solution to have the form $x_p = C\sin\omega t$,

$$-\omega^2 C\sin\omega t-\omega^2 C\sin\omega t=-g\sin\omega t$$

$$C=\frac{g}{2\omega^2}$$

$$x=Ae^{\omega t}+Be^{-\omega t}+\frac{g}{2\omega^2}\sin\omega t$$

At time $t=0$, $x=x_\circ$ and $\dot{x}=0$

$$x_\circ = A+B$$

$$0=\omega A-\omega B+\frac{g}{2\omega}$$

$$A=\frac{1}{2}\left(x_\circ-\frac{g}{2\omega^2}\right)$$

$$B=\frac{1}{2}\left(x_\circ+\frac{g}{2\omega^2}\right)$$

$$x=x_\circ\left[\frac{1}{2}\left(e^{\omega t}+e^{-\omega t}\right)\right]-\frac{g}{2\omega^2}\left[\frac{1}{2}\left(e^{\omega t}-e^{-\omega t}\right)\right]+\frac{g}{2\omega^2\sin\omega t}$$

From Appendix B, we use the identities for hyperbolic sine and cosine to obtain

$$x=x_\circ\cosh\omega t-\frac{g}{2\omega^2}\sinh\omega t+\frac{g}{2\omega^2}\sin\omega t$$

10.8 In order that a particle continues to move in a plane in a rotating coordinate system, it is necessary that the axis of rotation be perpendicular to the plane of motion.
For motion in the xy plane, $\vec{\omega}=\hat{k}\omega$.

$$\vec{v}=\vec{v}'+\vec{\omega}\times\vec{r}'=\left(\hat{i}\dot{x}+\hat{j}\dot{y}\right)+\hat{k}\omega\times\left(\hat{i}x+\hat{j}y\right)$$

$$\vec{v}=\hat{i}\left(\dot{x}-\omega y\right)+\hat{j}\left(\dot{y}+\omega x\right)$$

$$T=\frac{1}{2}m\vec{v}\cdot\vec{v}=\frac{m}{2}\left(\dot{x}^2-2\dot{x}\omega y+\omega^2 y^2+y^2+2\dot{y}\omega x+\omega^2 x^2\right)$$

$$L=T-V$$

$$\frac{\partial L}{\partial \dot{x}}=m\left(\dot{x}-\omega y\right),\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)=m\left(\ddot{x}-\omega\dot{y}\right),$$

$$\frac{\partial L}{\partial x}=m\left(\dot{y}\omega+\omega^2 x\right)-\frac{\partial V}{\partial x}$$

$$m\left(\ddot{x}-\omega\dot{y}\right)=m\left(\dot{y}\omega+\omega^2 x\right)-\frac{\partial V}{\partial x}$$

$$F_x=m\left(\ddot{x}-2\omega\dot{y}-\omega^2 x\right)$$

$$\frac{\partial L}{\partial \dot{y}}=m\left(\dot{y}+\omega x\right),\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)=m\left(\ddot{y}+\omega\dot{x}\right),$$

$$\frac{\partial L}{\partial y} = m\left(-\dot{x}\omega + \omega^2 y\right) - \frac{\partial V}{\partial y}$$

$$m\left(\ddot{y} + \omega\dot{x}\right) = m\left(-\dot{x}\omega + \omega^2 y\right) - \frac{\partial V}{\partial y}$$

$$F_y = m\left(\ddot{y} + 2\omega\dot{x} - \omega^2 y\right)$$

For comparison, from equation 5.3.2 ($\vec{A}_0 = 0$ and $\dot{\vec{\omega}} = 0$)

$$\vec{F} = m\vec{a}' + 2m\vec{\omega}\times\vec{v}' + m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}'\right)$$

$$\vec{F} = m\left(\hat{i}\ddot{x} + \hat{j}\ddot{y}\right) + 2m\,\omega\hat{k}\times\left(\hat{i}\dot{x} + \hat{j}\dot{y}\right) + m\,\omega\hat{k}\times\left[\omega\hat{k}\times\left(\hat{i}x + \hat{j}y\right)\right]$$

$$F_x = m\left(\ddot{x} - 2\omega\dot{y} - \omega^2 x\right)$$

$$F_y = m\left(\ddot{y} + 2\omega\dot{x} - \omega^2 y\right)$$

10.9 Choosing the axis of rotation as the z axis …

$$\vec{v} = \vec{v}' + \vec{\omega}\times\vec{r}' = \left(\hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z}\right) + \omega\hat{k}\times\left(\hat{i}x + \hat{j}y + \hat{k}z\right)$$

$$\vec{v} = \hat{i}\left(\dot{x} - \omega y\right) + \hat{j}\left(\dot{y} + \omega x\right) + \hat{k}\dot{z}$$

$$T = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{m}{2}\left(\dot{x}^2 - 2\dot{x}\omega y + \omega^2 y^2 + \dot{y}^2 + 2\dot{y}\omega x + \omega^2 x^2 + \dot{z}^2\right)$$

$$L = T - V$$

$$\frac{\partial L}{\partial \dot{x}} = m\left(\dot{x} - \omega y\right) \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\left(\ddot{x} - \omega\dot{y}\right),$$

$$\frac{\partial L}{\partial x} = m\left(\dot{y}\omega + \omega^2 x\right) - \frac{\partial V}{\partial x}$$

$$m\left(\ddot{x} - \omega\dot{y}\right) = m\left(\dot{y}\omega + \omega^2 x\right) - \frac{\partial V}{\partial x}$$

$$F_x = m\left(\ddot{x} - 2\omega\dot{y} - \omega^2 x\right)$$

$$\frac{\partial L}{\partial \dot{y}} = m\left(\dot{y} + \omega x\right), \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) = m\left(\ddot{y} + \omega\dot{x}\right)$$

$$\frac{\partial L}{\partial y} = m\left(-\dot{x}\omega + \omega^2 y\right) - \frac{\partial V}{\partial y}$$

$$m\left(\ddot{y} + \omega\dot{x}\right) = m\left(-\dot{x}\omega + \omega^2 y\right) - \frac{\partial V}{\partial y}$$

$$F_y = m\left(\ddot{y} + 2\omega\dot{x} - \omega^2 y\right)$$

$$\frac{\partial L}{\partial \dot{z}} = m\dot{z}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}, \qquad \frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z}$$

$$m\ddot{z} = -\frac{\partial V}{\partial z} = F_z$$

From equation 5.3.2 ($\vec{A}_0 = 0$ and $\dot{\vec{\omega}} = 0$) …

$$\vec{F} = m\vec{a}' + 2m\vec{\omega}\times\vec{v}' + m\vec{\omega}\times\left(\vec{\omega}\times\vec{r}'\right)$$

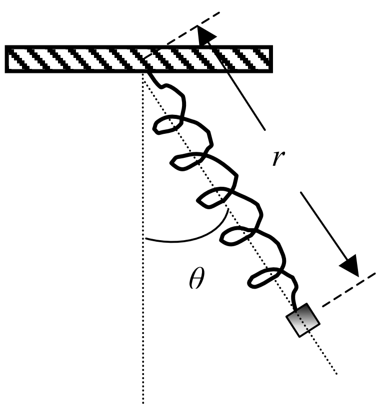
$$\vec{F} = m\left(\hat{i}\ddot{x} + \hat{j}\ddot{y} + \hat{k}\ddot{z}\right) + 2m\omega\hat{k}\times\left(\hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z}\right) + m\omega\hat{k}\times\left[\omega\hat{k}\times\left(\hat{i}x + \hat{j}y + \hat{k}z\right)\right]$$

$$F_x = m\left(\ddot{x} - 2\omega\dot{y} - \omega^2 x\right)$$

$$F_y = m\left(\ddot{y} - 2\omega\dot{x} - \omega^2 y\right)$$

$$F_z = m\ddot{z}$$

10.10



$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$V = \frac{1}{2}k\left(r - l_\circ\right)^2 - mgr\cos\theta$$

$$L = T - V = \frac{m}{2}\left(r^2 + r^2\dot{\theta}^2\right) - \frac{k}{2}\left(r - l_\circ\right)^2 + mgr\cos\theta$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}\,, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 - k\left(r - l_\circ\right) + mg\cos\theta$$

$$m\ddot{r} = m\dot{r}^2 - k\left(r - l_\circ\right) + mg\cos\theta$$

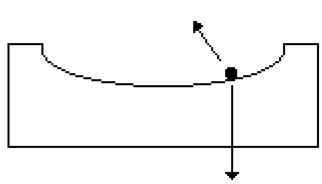
$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgr\sin\theta$$

$$\frac{d}{dt}\left(mr^2\dot{\theta}\right) = -mgr\sin\theta$$

10.11 (See Example 4.6.2)

$$x = \frac{a}{4}\left(2\theta + \sin 2\theta\right) \qquad y = \frac{a}{4}\left(1 - \cos 2\theta\right)$$

at $\theta = 0$ $\qquad x \,\&\, y = 0$



$$T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) \qquad V = mgy$$

$$\dot{x} = \frac{a}{2}\left(\dot{\theta} + \dot{\theta}\cos 2\theta\right) = \frac{a\dot{\theta}}{2}\left(1 + \cos 2\theta\right)$$

$$\dot{y} = \frac{a}{2}\dot{\theta}\sin 2\theta$$

$$L = T - V = \frac{ma^2\dot{\theta}^2}{8}\left[\left(1 + \cos 2\theta\right)^2 + \sin^2 2\theta\right] - \frac{mga}{4}\left(1 - \cos 2\theta\right)$$

$$= \frac{ma^2\dot{\theta}^2}{8}\left[1 + 2\cos 2\theta + 1\right] - \frac{mga}{4}\left(1 - \cos 2\theta\right)$$

$$= \frac{ma^2\dot{\theta}^2}{2}\left[\cos^2\theta\right] - \frac{mga}{2}\left[\sin^2\theta\right]$$

where we used the trigonometric identies … $2\cos^2\theta = 1 + \cos 2\theta$ and $2\sin^2\theta = 1 - \cos 2\theta$

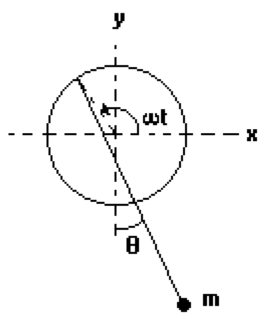
Let $s = a\sin\theta$ so $\dot{s} = a\dot{\theta}\cos\theta$

$$L = \frac{m\dot{s}^2}{2} - \frac{mg}{2a}s^2 = \frac{1}{2}m\dot{s}^2 - \frac{1}{2}ks^2 \text{ where } k = \frac{mg}{a}$$

The equation of motion is thus

$$\ddot{s} + \frac{k}{m}s = 0 \text{ or } \ddot{s} + \frac{g}{a}s = 0$$ -a simple harmonic oscillator

10.12 Coordinates:



$$x = a\cos\omega t + b\sin\theta$$
$$y = a\sin\omega t - b\cos\theta$$

$$\dot{x} = -a\omega\sin\omega t + b\dot{\theta}\cos\theta$$
$$\dot{y} = a\omega\cos\omega t + b\dot{\theta}\sin\theta$$

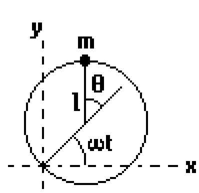
$$L = T - V = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) - mgy$$
$$= \frac{m}{2}\left[a^2\omega^2 + b^2\dot{\theta}^2 + 2b\dot{\theta}a\omega\sin\left(\theta - \omega t\right)\right] - mg\left(a\sin\omega t - b\cos\theta\right)$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = mb^2\ddot{\theta} + mba\omega\left(\dot{\theta} - \omega\right)\cos\left(\theta - \omega t\right)$$
$$\frac{\partial L}{\partial\theta} = mb\dot{\theta}a\omega\cos\left(\theta - \omega t\right) - mgb\sin\theta$$

The equation of motion $\frac{\partial L}{\partial\theta} - \frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = 0$ is

$$\ddot{\theta} - \omega\frac{a}{b}\cos\left(\theta - \omega t\right) + \frac{g}{b}\sin\theta = 0$$

Note – the equation reduces to equation of simple pendulum if $\omega \to 0$.

10.13 Coordinates:



$$x = l\cos\omega t + l\cos\left(\theta + \omega t\right)$$
$$y = l\sin\omega t + l\sin\left(\theta + \omega t\right)$$
$$\dot{x} = -\omega l\sin\omega t - \left(\dot{\theta} + \omega\right)l\sin\left(\theta + \omega t\right)$$
$$\dot{y} = \omega l\cos\omega t + \left(\dot{\theta} + \omega\right)l\cos\left(\theta + \omega t\right)$$

$$L = T = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right) = \frac{1}{2}ml^2\left[\omega^2 + \left(\dot{\theta} + \omega\right)^2\right] + \ldots$$
$$\ldots ml^2\omega\left(\dot{\theta} + \omega\right)\left[\sin\omega t\left(\sin\theta\cos\omega t + \sin\omega t\cos\theta\right) + \cos\omega t\left(\cos\theta\cos\omega t - \sin\omega t\sin\theta\right)\right]$$

$$= \frac{1}{2}ml^2\left[\omega^2 + \left(\dot{\theta}+\omega\right)^2\right] + ml^2\omega\left(\dot{\theta}+\omega\right)\cos\theta$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = 0$$

$$ml^2\left(\ddot{\theta} - \omega\dot{\theta}\sin\theta\right) + ml^2\omega\left(\dot{\theta}+\omega\right)\sin\theta = 0$$

(a) $\ddot{\theta} + \omega^2 \sin\theta = 0$

(b) The bead executes simple harmonic motion $\left(\theta \gtrsim 0\right)$ about a point diametrically opposite the point of attachment.

(c) The effective length is $"l" = \frac{g}{\omega^2}$

10.14 $\vec{v} = \hat{j}at + l\dot{\theta}\left(\hat{i}\cos\theta + \hat{j}\sin\theta\right)$

$$= \hat{i}l\theta\cos\theta + \hat{j}\left(at + l\dot{\theta}\sin\theta\right)$$

$$T = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{m}{2}\left(l^2\dot{\theta}^2\cos^2\theta + a^2t^2 + 2atl\dot{\theta}\sin\theta + l^2\dot{\theta}^2\sin^2\theta\right)$$

$$V = mg\left(\frac{1}{2}at^2 - l\cos\theta\right)$$

$$L = T - V = \frac{m}{2}\left(l^2\dot{\theta}^2 + a^2t^2 + 2atl\dot{\theta}\sin\theta\right) - mg\left(\frac{at^2}{2} - l\cos\theta\right)$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} + matl\sin\theta, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} + mal\sin\theta + matl\dot{\theta}\cos\theta$$

$$\frac{\partial L}{\partial \theta} = matl\dot{\theta}\cos\theta - mgl\sin\theta$$

$$ml^2\ddot{\theta} + mal\sin\theta + matl\dot{\theta}\cos\theta = matl\dot{\theta}\cos\theta - mgl\sin\theta$$

$$\ddot{\theta} + \frac{a+g}{l}\sin\theta = 0$$

For small oscillations, $\sin\theta \approx \theta$

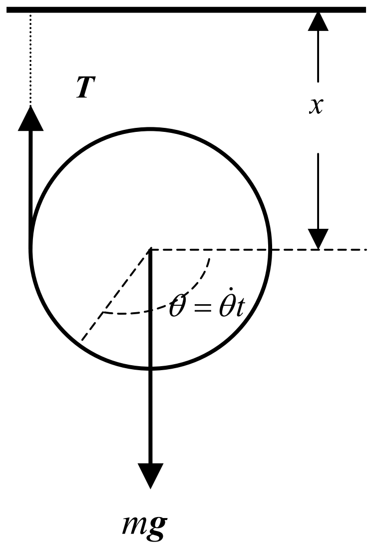
$$\ddot{\theta} + \frac{a+g}{l}\theta = 0$$

$$T_{\circ} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{a+g}}$$

10.15 (a) $T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\theta}^2$ and $I = \frac{2}{5}ma^2$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\left(\frac{2}{5}ma^2\right)\dot{\theta}^2$$

$$V = -mgx$$



$$L = T - V = \frac{1}{2}m\dot{x}^2 + \frac{1}{5}ma^2\dot{\theta}^2 + mgx$$

The equation of constraint is …

$$f(x,\theta) = x - a\theta = 0$$

The 2 Lagrange equations with multipliers are …

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} + \lambda\frac{\partial f}{\partial x} = 0$$

$$\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} \qquad \frac{\partial L}{\partial x} = mg \qquad \lambda\frac{\partial f}{\partial x} = \lambda$$

$mg - m\ddot{x} + \lambda = 0$ and from the θ-equation …

$$-\frac{2}{5}ma^2\ddot{\theta} - \lambda a = 0$$

Differentiating the equation of constraint … $\ddot{\theta} = \dfrac{\ddot{x}}{a}$ and substituting into the above …

$$\ddot{x} = \frac{5}{7}g \quad \text{and} \quad \lambda = -\frac{2}{7}mg$$

(b) The generalized force that is equivalent to the tension T is …

$$Q_x = \lambda\frac{\partial f}{\partial x} = \lambda = -\frac{2}{7}mg$$

10.16 For $\vec{v}'$ the velocity of a differential mass element, dm, of the spring at a distance x' below the support …

$$\vec{v}' = \frac{x'}{x}\vec{v}, \qquad dm = \frac{m'}{x}dx'$$

$$T = \frac{1}{2}mv^2 + \int_0^{m'}\frac{1}{2}(v')^2\,dm = \frac{1}{2}m\dot{x}^2 + \int_0^x \frac{1}{2}\left(\frac{x'}{x}\dot{x}\right)^2\frac{m'}{x}dx'$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{m'}{3}\dot{x}^2$$

$$V = \frac{1}{2}k(x-l)^2 - mgx - \int_0^{m'} gx'dm'$$

$$= \frac{1}{2}k(x-l)^2 - mgx - \int_0^x gx'\frac{m'}{x}dx'$$

$$V = \frac{1}{2}k(x-l)^2 - mgx - \frac{m'}{2}gx$$

$$L = T - V = \frac{1}{2}\left(m + \frac{m'}{3}\right)\dot{x}^2 - \frac{1}{2}k(x-l)^2 + \left(m + \frac{m'}{2}\right)gx$$

$$\frac{\partial L}{\partial \dot{x}} = \left(m + \frac{m'}{3}\right)\dot{x}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \left(m + \frac{m'}{3}\right)\ddot{x},$$

$$\frac{\partial L}{\partial x} = -k(x-l) + \left(m + \frac{m'}{2}\right)g$$

$$\left(m + \frac{m'}{3}\right)\ddot{x} = -k(x-l) + \left(m + \frac{m'}{2}\right)g$$

For $\quad y = x - l - \frac{g}{k}\left(m + \frac{m'}{2}\right),$

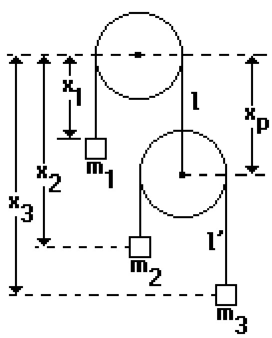
$$\left(m + \frac{m'}{3}\right)\ddot{y} + ky = 0$$

The block oscillates about the point $x = l + \frac{g}{k}\left(m + \frac{m'}{2}\right)$ …

with a period …

$$T_\circ = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{\left(m + \frac{m'}{3}\right)}{k}}$$

10.17 Note: 4 objects move – their coordinates are labeled x_i: The coordinate of the movable, massless pulley is labeled x_p.



Two equations of constraint:

$$f_1(x_1, x_p) = x_p - (l - x_1) = 0$$

$$f_2(x_2, x_3, x_p) = (x_2 + x_3) - (2x_p + l') = 0$$

$$L = T - V = \frac{1}{2}m_1\dot{x}_1^2 - m_1 g x_1 + \frac{1}{2}m\dot{x}_2^2 - m_2 g x_2 + \frac{1}{2}m_3\dot{x}_3^2 - m_3 g x_3$$

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial q_i} + \sum_j \lambda_j \frac{\partial f_j}{\partial q_i} = 0$$

Thus: (1) $\quad -m_1 g - m_1\ddot{x}_1 + \lambda_1 = 0$

(2) $\quad -m_2 g - m_2\ddot{x}_2 + \lambda_2 = 0$

(3) $\quad -m_3 g - m_3\ddot{x}_3 + \lambda_2 = 0$

Now – apply Lagrange's equations to the movable pulley – note $m_p \to 0$

So:

(4) $\quad \lambda_1 \frac{\partial f_1}{\partial x_p} + \lambda_2 \frac{\partial f_2}{\partial x_p} = 0$ <u>or</u> $\lambda_1 - 2\lambda_2 = 0$

Now - x_p can be eliminated between f_1 and f_2

$$f_2 = 2f_1 = x_2 + x_3 + 2x_1 - (2l + l') = 0$$

(5) $\quad$ Thus $\ddot{x}_2 + \ddot{x}_3 + 2\ddot{x}_1 = 0$

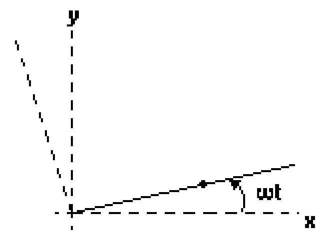
With a little algebra – we can solve (1) –(5) for the 5 unknowns $\ddot{x}_1, \ddot{x}_2, \ddot{x}_3, \lambda_1$, and λ_2

$$\lambda_1 = \frac{2g}{\left[\frac{1}{m_1} + \frac{1}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\right]} \qquad \lambda_2 = \frac{g}{\left[\frac{1}{m_1} + \frac{1}{4}\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\right]}$$

As the check, let $m_2 = m_3 = m$ and $m_1 = 2m$.

Thus, there is no acceleration and $\lambda_1 = 2mg$ ✔

10.18 (See Example 5.3.3)



(a) $\bar{r} = r\hat{e}_r$

$\dot{\bar{r}} = \dot{r}\hat{e}_r + r\dot{\theta}\hat{e}_\theta$

Constraint: $f(\theta) = \theta - \omega t = 0$

so … $\dot{\theta} = \omega$ and $\ddot{\theta} = 0$

$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) = L$

$\frac{\partial L}{\partial r} - \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = 0$ and $\frac{\partial L}{\partial \theta} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} + \lambda\frac{\partial f}{\partial \theta} = 0$

$\ddot{r} = \omega^2 r$ $\qquad -2mr\dot{r}\dot{\theta} - mr^2\ddot{\theta} + \lambda = 0$

$r = Ae^{\omega t} + Be^{-\omega t}$ $\qquad r(0) = 0$ $\qquad$ apply constraint

$\dot{r} = \omega Ae^{\omega t} - \omega Be^{-\omega t}$ $\qquad \dot{r}(0) = \omega l$

so … $A + B = 0$ $\qquad \underline{\lambda = 2mr\dot{r}\omega}$

$\omega A - \omega B = \omega l$

$2A = l \qquad A = \frac{l}{2} \qquad B = -\frac{l}{2}$

thus .. $r = \frac{l}{2}\left(e^{\omega t} - e^{-\omega t}\right)$

$\left.\begin{array}{l} r = l\sinh\omega t \\ \dot{r} = \omega l\cosh\omega t \end{array}\right\}$ $\qquad \underline{\lambda = 2m\omega^2 l^2 \sinh\omega t\cosh\omega t}$

now $r = l$ at $t = T$ so $\qquad T = \frac{1}{\omega}\sinh^{-1}1 = \frac{0.88}{\omega}$

(b) There are 2 ways to calculate F …

(i) See Example 5.3.3 …

$F = 2m\omega\dot{x}'$

$x' = \frac{l}{2}\left(e^{\omega t} - e^{-\omega t}\right)$ and $\dot{x}' = \frac{\omega l}{2}\left(e^{\omega t} + e^{-\omega t}\right)$

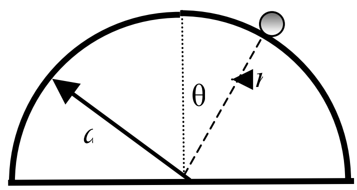
$F = 2m\omega^2 l\cosh\omega t$

(ii) $\lambda\frac{\partial f}{\partial \theta}$ is the generalized force, in this case – a *torque T*, acting on the bead

$T = rF = \lambda\frac{\partial f}{\partial \theta}$

$$F=\frac{\lambda}{r}\frac{\partial f}{\partial \theta}=\frac{2m\omega^2 l^2 \sinh\omega t\cosh\omega t}{l\sinh\omega t}$$
$$=2m\omega^2 l\cosh\omega t$$

10.19 (See Example 4.6.1) The equation of constraint is $f(r,\theta)=r-a=0$



$$T=\frac{m}{2}\left(\dot r^2+r^2\dot\theta^2\right) \qquad V=mgr\cos\theta$$
$$L=\frac{m}{2}\left(\dot r^2+r^2\dot\theta^2\right)-mgr\cos\theta$$
$$\frac{\partial L}{\partial r}-\frac{d}{dt}\frac{\partial L}{\partial \dot r}+\lambda\frac{\partial f}{\partial r}=0$$
$$\frac{\partial L}{\partial \theta}-\frac{d}{dt}\frac{\partial L}{\partial \dot\theta}+\lambda\frac{\partial f}{\partial \theta}=0 \qquad \frac{\partial f}{\partial r}=1 \qquad \frac{\partial f}{\partial \theta}=0$$

Thus $mr\dot\theta^2-mg\cos\theta-m\ddot r+\lambda=0$
$$mgr\sin\theta-mr^2\ddot\theta-2mr\dot r\dot\theta=0$$

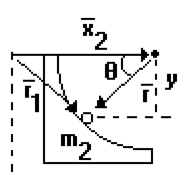
Now $r=a$, $\dot r=\ddot r=0$ so
$$ma\dot\theta^2-mg\cos\theta+\lambda=0$$
$$mga\sin\theta-ma^2\ddot\theta=0$$
$$\ddot\theta=\frac{g}{a}\sin\theta \qquad \text{and} \qquad \ddot\theta=\dot\theta\frac{d\dot\theta}{d\theta}$$

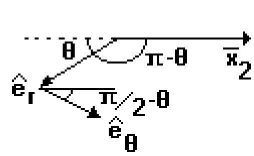
so $\int\dot\theta d\dot\theta=\frac{g}{a}\int\sin\theta d\theta$ or $\frac{\dot\theta^2}{2}=-\frac{g}{a}\cos\theta+\frac{g}{a}$

hence, $\lambda=mg\left(3\cos\theta-2\right)$

and when $\lambda\to 0$ particle falls off hemisphere at
$$\theta_\circ=\cos^{-1}\left(\frac{2}{3}\right)$$

10.20 Let $\bar x_2$ mark the location of the center of curvature of the movable surface relative to a fixed origin. This point defines the position of m_2. $\bar r_1$ marks the position of the particle of mass m_1. $\bar r$ is the position of the particle relative to the movable center of curvature of m_2.





Kinetic energy $T=\frac{1}{2}m_2\dot{\bar x}_2^2+\frac{1}{2}m_1\dot{\bar r}_1^2$
$$\dot{\bar r}_1\cdot\dot{\bar r}_1=\left(\dot{\bar x}_2+\dot{\bar r}\right)\cdot\left(\dot{\bar x}_2+\dot{\bar r}\right)=\dot{\bar x}_2^2+\dot{\bar r}^2+2\dot{\bar x}_2\cdot\dot{\bar r}$$

but $\dot{\bar r}=\dot r\hat e_r+r\dot\theta\hat e_\theta$
$$\therefore T=\frac{1}{2}m_2\dot x_2^2+\frac{1}{2}m_1\left[\dot x_2^2+\dot r^2+r^2\dot\theta^2+2\dot{\bar x}_2\cdot\left(\dot r\hat e_r+r\dot\theta\hat e_\theta\right)\right]$$
$$=\frac{1}{2}m_2\dot x_2^2+\frac{1}{2}m_1\left[\dot x_2^2+\dot r^2+r^2\dot\theta^2+2\dot x_2 r\dot\theta\sin\theta-2\dot x_2\dot r\cos\theta\right]$$

Potential energy $V = -mgy = -mgr\sin\theta$

$L = T - V$

Equation of constraint $f(r,\theta) = r - a = 0 \quad \therefore \frac{\partial f}{\partial r} = 1 \quad \frac{\partial f}{\partial \theta} = 0 \quad \frac{\partial f}{\partial x_2} = 1$

Lagrange's equations $\frac{\partial L}{\partial x_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) + \lambda \frac{\partial f}{\partial x_i} = 0$

*x*2) $-m_2\ddot{x}_2 - m_1\ddot{x}_1 - m_1\frac{d}{dt}\left[r\dot{\theta}\sin\theta - \dot{r}\cos\theta\right] = 0$

$$m_2\ddot{x}_2 - m_1\ddot{x}_1 - m_1\frac{d}{dt}\left[\dot{r}\dot{\theta}\sin\theta + r\ddot{\theta}\sin\theta + r\dot{\theta}^2\cos\theta - \ddot{r}\cos\theta + \dot{r}\dot{\theta}\sin\theta\right] = 0$$

using constraint $r = a = constant$; $\dot{r} = \ddot{r} = 0$

$$\ddot{x}_2 = -\frac{m_1}{m_1 + m_2}a\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \qquad (1)$$

θ) $-r^2\ddot{\theta} - 2\dot{r}r\dot{\theta} - \ddot{x}_2 r\sin\theta - \dot{x}_2\dot{r}\sin\theta - \dot{x}_2 r\dot{\theta}\cos\theta + \dot{x}\dot{r}\sin\theta + \dot{x}_2 r\dot{\theta}\cos\theta + gr\cos\theta = 0$

using constraint $r = a = constant$; $\dot{r} = \ddot{r} = 0$)

$$-\ddot{\theta} - \frac{\ddot{x}_2}{a}\sin\theta + \frac{g}{a}\cos\theta = 0 \qquad (2)$$

r) $-\ddot{r} + \ddot{x}_2\cos\theta - \dot{x}_2\dot{\theta}\sin\theta + r\dot{\theta}^2 + \dot{x}_2\dot{\theta}\sin\theta + g\sin\theta + \frac{\lambda}{m_1} = 0$

Apply constraint…

$$\ddot{x}_2\cos\theta + a\dot{\theta}^2 + g\sin\theta + \frac{\lambda}{m_1} = 0 \qquad (3)$$

Now, plug solution for $\ddot{x}_2$ (1) into equation (2):

$$\ddot{\theta} - \frac{m_1}{m_1 + m_2}\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right)\sin\theta - \frac{g}{a}\cos\theta = 0$$

$$\ddot{\theta}\left(1 - f_{m_1}\sin^2\theta\right) - \dot{\theta}^2 f_{m_1}\sin\theta\cos\theta - \frac{g}{a}\cos\theta = 0$$

where $f_{m_1} = \frac{m_1}{m_1 + m_2}$

Now $\ddot{\theta} = \frac{d\dot{\theta}}{dt} = \frac{d\dot{\theta}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\frac{d\dot{\theta}}{d\theta} = \frac{d}{d\theta}\left\{\frac{1}{2}\dot{\theta}^2\right\}$

So – let $x = \dot{\theta}^2$

$$\frac{dx}{d\theta}\left[1 - f_{m_1}\sin^{-2}\theta\right] - 2xf_{m_1}\sin\theta\cos\theta - \frac{2g}{a}\cos\theta = 0$$

Let $y = 1 - f_{m_1}\sin^2\theta$

$$y\frac{dx}{d\theta} + x\frac{dy}{d\theta} = \frac{2g}{a}\frac{d(\sin\theta)}{d\theta}$$

Hence: $d(yx) = \frac{2g}{a} d(\sin\theta)$ and $\int_{yx(\theta=0^\circ)}^{yx(\theta=\theta)} d(yx) = \frac{2g}{a} \int_0^{\sin\theta} d(\sin\theta)$

but $x = \dot{\theta}^2 = 0$ at $\theta = 0^\circ$ so

$$yx(\theta) = \frac{2g}{a}\sin\theta$$

$$x(\theta) = \dot{\theta}^2 = \frac{2g}{a}\frac{\sin\theta}{\left[1 - f_{m_1}\sin^2\theta\right]}$$

Now we can solve for $\ddot{\theta}$ and plug $\dot{\theta}^2$, $\ddot{\theta}$ into (3) to obtain $\lambda(\theta)$

$$\ddot{\theta} = \frac{d}{d\theta}\left\{\frac{1}{2}\dot{\theta}^2\right\} = \frac{g}{a}\frac{d}{d\theta}\left[\frac{\sin\theta}{1 - f_{m_1}\sin^2\theta}\right]$$

After some algebra – yields

$$\ddot{\theta} = \frac{g}{a}\cos\theta\frac{\left(1 + f_{m_1}\sin^2\theta\right)}{\left(1 - f_{m_1}\sin^2\theta\right)^2}$$

The solution for $\lambda(\theta)$ is thus determined from equation (3)

$$\dot{\theta}^2 - f_{m_1}\left[\ddot{\theta}\sin\theta\cos\theta + \dot{\theta}^2\cos\theta\right] + \frac{g}{a}\sin\theta = -\frac{\lambda}{m_1 a}$$

collecting terms:

$$\dot{\theta}^2\left(1 - f_{m_1}\cos^2\theta\right) - \ddot{\theta} f_{m_1}\sin\theta\cos\theta + \frac{g}{a}\sin\theta = -\frac{\lambda}{m_1 a}$$

Plug $\ddot{\theta}$, $\dot{\theta}$ into the above --- plus --- a lot of algebra yields …

$$-\frac{\lambda(\theta)}{m_1 g} = \frac{2f_{m_2}\sin\theta - f_{m_1}\sin\theta\cos^2\theta\left[1 - f_{m_1}\sin^2\theta\right]}{\left[1 - f_{m_1}\sin^2\theta\right]^2} + \sin\theta$$

where $f_{m_2} = \frac{m_2}{m_1 + m_2}$

As a check, let $m_2 \to \infty$ so it is immoveable … then $f_{m_2} \to 1$ and $f_{m_1} \to 0$ and we have

$$-\frac{\lambda(\theta)}{m_1 g} \to 3\sin\theta \qquad \text{… which checks}$$

10.21 (a) $T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2\right)$

$$L = T - V = \frac{m}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + \dot{z}^2\right) - V$$

$$\frac{\partial L}{\partial \dot{R}} = m\dot{R}, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{R}}\right) = m\ddot{R}, \qquad \frac{\partial L}{\partial R} = mR\dot{\phi}^2 - \frac{\partial V}{\partial R}$$

$$m\ddot{R} = mR\dot{\phi}^2 - \frac{\partial V}{\partial R}$$

$$m\ddot{R} - mR\dot{\phi}^2 = Q_R$$

$$\frac{\partial L}{\partial \dot{\phi}} = mR^2\dot{\phi}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = mR^2\ddot{\phi} + 2mR\dot{R}\dot{\phi}, \quad \frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial \phi}$$

$$mR^2\ddot{\phi} + 2mR\dot{R}\dot{\phi} = -\frac{\partial V}{\partial \phi} = Q_\phi$$

$$\frac{\partial L}{\partial z} = m\dot{z}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) = m\ddot{z}, \quad \frac{\partial L}{\partial z} = -\frac{\partial V}{\partial z}$$

$$m\ddot{z} = -\frac{\partial V}{\partial z} = Q_z$$

For $\vec{F} = m\vec{a}$, using the components of $\vec{a}$ from equation 1.12.3:

$$F_R = m\left(\ddot{R} - R\dot{\phi}^2\right), \quad F_\phi = m\left(2\dot{R}\dot{\phi} + R\ddot{\phi}\right), \quad F_z = m\ddot{z}$$

From Section 10.2, since R and z are distances, Q_R and Q_z are forces. However, since ϕ is an angle, Q_ϕ is a torque. Since F_ϕ is coplanar with and perpendicular to R,

$Q_\phi = RF_\phi$ … and all equations agree.

(b) $v^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta$

$$L = T - V = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\dot{\phi}^2 \sin^2\theta\right) - V$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}, \quad \frac{\partial L}{\partial r} = mr\theta^2 + mr\dot{\phi}^2 \sin^2\theta - \frac{\partial V}{\partial r}$$

$$m\ddot{r} - mr\dot{\theta}^2 - mr\dot{\phi}^2 \sin^2\theta = -\frac{\partial V}{\partial r} = Q_r$$

$$\frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta}, \quad \frac{\partial L}{\partial \theta} = mr^2\dot{\phi}^2 \sin\theta\cos\theta - \frac{\partial V}{\partial \theta}$$

$$mr^2\ddot{\theta} + 2mr\dot{r}\dot{\theta} - mr^2\dot{\phi}^2 \sin\theta\cos\theta = -\frac{\partial V}{\partial \theta} = Q_\theta$$

$$\frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi}\sin^2\theta, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = mr^2\ddot{\phi}\sin^2\theta + 2mr\dot{r}\dot{\phi}\sin^2\theta + 2mr^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta,$$

$$\frac{\partial L}{\partial \phi} = -\frac{\partial V}{\partial \phi}$$

$$mr^2\ddot{\phi}\sin^2\theta + 2mr\dot{r}\dot{\phi}\sin^2\theta + 2mr^2\dot{\theta}\dot{\phi}\sin\theta\cos\theta = -\frac{\partial V}{\partial \phi} = Q_\phi$$

Q_r is a force F_r. Q_θ and Q_ϕ are torques.

Since ϕ is in the xy plane, the moment arm for ϕ is $r\sin\theta$; i.e. $Q_\phi = r\sin\theta F_\phi$.

$Q_\theta = rF_\theta$.

From equation 1.12.14 …

$$m\left(\ddot{r}-r\dot{\phi}^2\sin^2\theta-r\dot{\theta}^2\right)=F_r$$
$$m\left(r\ddot{\theta}+2\dot{r}\dot{\theta}-r\dot{\phi}^2\sin\theta\cos\theta\right)=F_\theta$$
$$m\left(r\ddot{\phi}\sin\theta+2\dot{r}\dot{\phi}\sin\theta+2r\dot{\theta}\dot{\phi}\cos\theta\right)=F_\phi$$

The equations agree.

10.22 For a central field, $V=V(r)$, so $\frac{\partial V}{\partial\theta}=\frac{\partial V}{\partial\phi}=0$ and $\frac{\partial V}{\partial r}=-F$.

For spherical coordinates, using equation 1.12.12:

$$v^2=\dot{r}^2+r^2\dot{\phi}^2\sin^2\theta+r^2\dot{\theta}^2$$
$$L=T-V=\frac{m}{2}\left(\dot{r}^2+r^2\dot{\phi}^2\sin^2\theta+r^2\dot{\theta}^2\right)-V(r)$$
$$\frac{\partial L}{\partial\dot{r}}=m\dot{r}\,,\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right)=m\ddot{r}\,,\qquad \frac{\partial L}{\partial r}=mr\dot{\phi}^2\sin^2\theta+mr\dot{\theta}^2-\frac{\partial V}{\partial r}$$
$$m\ddot{r}-mr\dot{\phi}^2\sin^2\theta-mr\dot{\theta}^2=F_r$$
$$\frac{\partial L}{\partial\dot{\theta}}=mr^2\dot{\theta}\,,\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=mr^2\ddot{\theta}+2mr\dot{r}\dot{\theta}\,,\quad \frac{\partial L}{\partial\theta}=mr^2\dot{\phi}^2\sin\theta\cos\theta$$
$$mr^2\ddot{\theta}+2mr\dot{r}\dot{\theta}-mr^2\dot{\phi}^2\sin\theta\cos\theta=0$$
$$\frac{\partial L}{\partial\dot{\phi}}=mr^2\dot{\phi}\sin^2\theta\,,$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)=mr^2\ddot{\phi}\sin^2\theta+2mr\dot{r}\dot{\phi}\sin^2\theta+2mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta$$
$$\frac{\partial L}{\partial\phi}=0$$
$$mr^2\ddot{\phi}\sin^2\theta+2mr\dot{r}\dot{\phi}\sin^2\theta+2mr^2\dot{\phi}\dot{\theta}\sin\theta\cos\theta=0$$

10.23 Since $\theta=\alpha=$ constant, there are two degrees of freedom, r and θ.

$$v_r=\dot{r}\,,\ v_\phi=r\dot{\phi}\sin\alpha$$
$$T=\frac{1}{2}mv^2=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\phi}^2\sin\alpha\right)$$
$$V=mgr\cos\alpha$$
$$L=T-V=\frac{m}{2}\left(\dot{r}^2+r^2\dot{\phi}^2\sin^2\alpha\right)-mgr\cos\alpha$$
$$\frac{\partial L}{\partial\dot{r}}=m\dot{r}\,,\qquad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right)=m\ddot{r}\,,\qquad \frac{\partial L}{\partial r}=mr\dot{\phi}^2\sin^2\alpha-mg\cos\alpha$$
$$m\ddot{r}=mr\dot{\phi}^2\sin^2\alpha-mg\cos\alpha$$
$$\frac{\partial L}{\partial\dot{\phi}}=mr^2\dot{\phi}\sin^2\alpha\,,\qquad \frac{\partial L}{\partial\phi}=0$$

$$\frac{d}{dt}\left(mr^2\dot{\phi}\sin\alpha\right)=0$$

Say $\quad mr^2\dot{\phi}\sin\alpha=\ell=\text{constant}$

$$m\ddot{r}=\frac{\ell^2}{mr^3}-mg\cos\alpha$$

$$\ddot{r}=\frac{d}{dt}\dot{r}=\frac{d\dot{r}}{dr}\dot{r}=\frac{1}{2}\frac{d}{dr}\dot{r}^2$$

$$\frac{m}{2}d\left(\dot{r}^2\right)=\frac{\ell^2}{m}\frac{dr}{r^3}-mg\cos\alpha\, dr$$

$$\frac{m\dot{r}^2}{2}=-\frac{\ell^2}{2mr^2}-mgr\cos\alpha+C$$

The constant of integration C is the total energy of the particle: kinetic
energy due to the component of motion in the radial direction, kinetic energy due to the component of motion in the angular direction, and the potential energy.

$$U\left(r\right)=\frac{\ell^2}{2mr^2}+mgr\cos\alpha$$

For $\quad \dot{\phi}\neq 0$, $\ell\neq 0$, and turning points occur at $\dot{r}=0$

Then $\quad 0=-\frac{\ell^2}{2mr^2}-mgr\cos\alpha+C$

$$\left(mg\cos\alpha\right)r^3-Cr^2+\frac{\ell^2}{2m}=0$$

The above equation is quadratic in r ($\ell^2\propto r^4$) and has two roots.

10.24 Note that the relation obtained in Problem 10.23, $\left(mg\cos\alpha\right)r^3-Cr^2+\frac{l^2}{2m}=0$, defines the turning points. For the particle to remain on a single horizontal circle, there must be two roots with $r=r_\circ$. Thus $\left(r-r_\circ\right)^2$ divided into the above expression leaves a term that is linear in r.

$$\begin{array}{l}
\qquad\qquad\qquad r+\left(2r_\circ-c\right) \\
r^2-2rr_\circ+r_\circ^2\,\overline{\left)\,r^3-Cr^2 \qquad\qquad\qquad + \quad \ell^2/2m\right.} \\
\qquad\qquad\qquad \underline{r^3-2r^2r_\circ+rr_\circ^2} \\
\qquad\qquad\qquad \left(2r_\circ-C\right)r^2-r_\circ^2 r \\
\qquad\qquad\qquad \underline{\left(2r_\circ-C\right)r^2-2r_\circ\left(2r_\circ-C\right)r+r_\circ^2\left(2r_\circ-C\right)} \\
\qquad\qquad\qquad\qquad \left[2r_\circ\left(2r_\circ-C\right)-r_\circ^2\right]+ \quad \ell^2/2m-r_\circ^2\left(2r_\circ-C\right)
\end{array}$$

For the remainder to vanish, both terms must equal zero.

$$4r_\circ^2-2r_\circ C-r_\circ^2=0$$

$$C=\frac{3}{2}r_\circ$$

$$\frac{\ell^2}{2m} - 2r_\circ^3 + Cr_\circ^2 = 0$$

$$\frac{\ell^2}{2mr_\circ^2} = \frac{r_\circ}{2}$$

And with $\ell = mr_\circ^2 \dot{\phi}_\circ \sin\alpha$

$$\dot{\phi}_\circ = \left(\frac{1}{mr_\circ \sin^2\alpha} \right)^{\frac{1}{2}}$$

From Prob. 10.23, $m\ddot{r} = \dfrac{\ell^2}{mr^3} - mg\cos\alpha$

For small oscillations about $r_\circ$, $r = r_\circ + \varepsilon$.

$$\ddot{r} = \ddot{\varepsilon}$$

$$\frac{1}{r^3} = \frac{1}{r_\circ^3}\left(1 + \frac{\varepsilon}{r_\circ}\right)^{-3} \approx \frac{1}{r_\circ^3}\left(1 - \frac{3\varepsilon}{r_\circ}\right)$$

$$m\ddot{\varepsilon} = \frac{\ell^2}{mr_\circ^3}\left(1 - \frac{3\varepsilon}{r_\circ}\right) - mg\cos\alpha$$

$$m\ddot{\varepsilon} + \frac{3\ell^2}{mr_\circ^4}\varepsilon = \frac{\ell^2}{mr_\circ^3} - mg\cos\alpha$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m^2 r_\circ^4}{3\ell^2}} = \frac{2\pi}{\dot{\phi}_\circ \sin\alpha}\sqrt{\frac{1}{3}}$$

10.25 $L = \dfrac{1}{2}mv^2 + q\vec{v}\cdot\vec{A} = \dfrac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + q\left(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z\right)$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + qA_x, \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} + q\frac{dA_x}{dt}$$

Using the hint, $\dfrac{dA_x}{dt} = \dot{x}\dfrac{\partial A_x}{\partial x} + \dot{y}\dfrac{\partial A_x}{\partial y} + \dot{z}\dfrac{\partial A_x}{\partial z}$

$$\frac{\partial L}{\partial x} = q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right)$$

$$m\ddot{x} + q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_x}{\partial y} + \dot{z}\frac{\partial A_x}{\partial z}\right) = q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right)$$

$$m\ddot{x} = q\left[\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \dot{z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]$$

$$= q\left[\dot{y}\left(\vec{\nabla}\times\vec{A}\right)_z - \dot{z}\left(\vec{\nabla}\times\vec{A}\right)_y\right]$$

$$= q\left[\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right)\right]_x$$

$$m\ddot{x} = q\left(\vec{v}\times\vec{B}\right)_x$$

Due to the cyclic nature of the Cartesian coordinates, i.e., $\hat{i}\times\hat{j}=\hat{k}$ …

$$m\ddot{y} = q\left(\vec{v}\times\vec{B}\right)_y$$

$$m\ddot{z} = q\left(\vec{v}\times\vec{B}\right)_z$$

Altogether, $m\ddot{\vec{r}} = q\left(\vec{v}\times\vec{B}\right)$

10.26 $V = mgz \qquad T = \frac{1}{2}m\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)$

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x}, \qquad \dot{x} = \frac{p_x}{m}$$ similarly, $\dot{y} = \frac{p_y}{m}$ and $\dot{z} = \frac{p_z}{m}$

$$H = T + V = \frac{1}{2m}\left(p_x^2+p_y^2+p_z^2\right) + mgz$$

$$\frac{\partial H}{\partial p_x} = \frac{p_x}{m} = \dot{x}$$

$$\frac{\partial H}{\partial x} = 0 = -\dot{p}_x, \qquad p_x = constant$$

$$\dot{p}_x = \frac{d}{dt}\left(m\dot{x}\right) = m\ddot{x} = 0$$ similarly, $p_y = constant$, or $m\ddot{y} = 0$

$$\frac{\partial H}{\partial p_y} = \frac{p_y}{m} = \dot{y}$$

$$\frac{\partial H}{\partial p_z} = \frac{p_z}{m} = \dot{z}$$

$$\frac{\partial H}{\partial z} = mg = -\dot{p}_z, \qquad \dot{p}_z = \frac{d}{dt}\left(m\dot{z}\right) = m\ddot{z} = -mg$$

These agree with the differential equations for projectile motion in Section 4.3.

10.27 (a) Simple pendulum …

$$V = -mgl\cos\theta \qquad T = \frac{1}{2}ml^2\dot{\theta}^2$$

$$p_\theta = \frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta}, \qquad \dot{\theta} = \frac{p_\theta}{ml^2}$$

$$H = T + V = \frac{p_\theta^2}{2ml^2} - mgl\cos\theta$$

$$\frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2} = \dot{\theta}$$

$$\frac{\partial H}{\partial \theta} = mgl\sin\theta = -\dot{p}_\theta$$

(b) Atwood's machine …

$$V = -m_1 g x - m_2 g\left(l - x\right) \qquad T = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 + \frac{1}{2}I\frac{\dot{x}^2}{a^2}$$ [Includes the pulley]

$$p = \frac{\partial T}{\partial x} = \left(m_1 + m_2 + \frac{I}{a^2}\right)\dot{x}\,, \quad \dot{x} = \frac{p}{\left(m_1 + m_2 + \frac{I}{a^2}\right)}$$

$$H = T + V = \frac{p^2}{2\left(m_1 + m_2 + \frac{I}{a^2}\right)} - \left(m_1 - m_2\right)gx - m_2 g l$$

$$\frac{\partial H}{\partial p} = \frac{p}{\left(m_1 + m_2 + \frac{I}{a^2}\right)} = \dot{x}$$

$$\frac{\partial H}{\partial x} = -\left(m_1 - m_2\right)g = -\dot{p}\,, \qquad \dot{p} = \left(m_1 - m_2\right)g$$

(c) Particle sliding down a smooth inclined plane …

$$V = -mgx\sin\theta \qquad T = \frac{1}{2}m\dot{x}^2$$

$$p = \frac{\partial T}{\partial \dot{x}} = m\dot{x}\,, \qquad \dot{x} = \frac{p}{m}$$

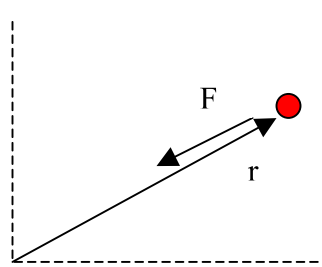
$$H = T + V = \frac{p^2}{2m} - mgx\sin\theta$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x}$$

$$\frac{\partial H}{\partial x} = -mg\sin\theta = -\dot{p}\,, \qquad \dot{p} = mg\sin\theta$$

10.28

(a)



$$L = T(q_i, \dot{q}_i) - T(q_i, t)$$ …note, potential energy is time dependent.

$$F = -\frac{\partial V}{\partial r} \text{ so } \qquad V = \int F dr = -\frac{k}{r}e^{-\beta t}$$

$$T = \frac{1}{2}m\vec{v}\cdot\vec{v} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

$$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r} \qquad \dot{r} = \frac{p_r}{m}$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \qquad \dot{\theta} = \frac{p_\theta}{mr^2}$$

$$H = \sum \dot{p}_i\, \dot{q}_i - L = p_r\dot{r} + p_\theta\dot{\theta} - \frac{p_r^{\ 2}}{2m} - \frac{p_\theta^{\ 2}}{2mr^2} - \frac{k}{r}e^{-\beta t}$$

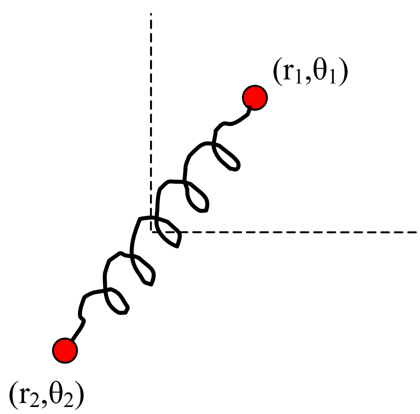
Substituting for $\dot{r}$ and $\dot{\theta}$ …

$$H = \frac{p_r^{\ 2}}{2m} + \frac{p_\theta^{\ 2}}{2mr^2} - \frac{k}{r}e^{-\beta t}$$

(b) H = T + V … which is time-dependent

(c) E is not conserved

10.29 Locate center of coordinate system at C.M. The potential is independent of the center of mass coordinates. Therefore, they are ignorable.



$$L = T - V$$
$$= \frac{1}{2}m_1\left(\dot{r}_1^2 + r_1^2\dot{\theta}_1^{\ 2}\right) + \frac{1}{2}m_2\left(\dot{r}_2^{\ 2} + r_2^{\ 2}\dot{\theta}_2^{\ 2}\right) - \frac{1}{2}k\left(r_1 + r_2 - l\right)^2$$

where l is the length of the relaxed spring and k is the spring constant.

$$p_{1r} = \frac{\partial L}{\partial \dot{r}_1} = m\dot{r}_1 \qquad \dot{r}_1 = \frac{p_{1r}}{m_1}$$

$$p_{1\theta} = \frac{\partial L}{\partial \dot{\theta}_1} = mr_1^{\ 2}\dot{\theta}_1 \qquad \dot{\theta}_1 = \frac{p_{1\theta}}{m_1 r_1^{\ 2}}$$

… and similarly for m_2

$$H = \sum \dot{p}_i\, \dot{q}_i - L$$

$$H = \frac{p_{1r}^{\ 2}}{2m_1} + \frac{p_{1\theta}^{\ 2}}{2m_1 r_1^{\ 2}} + \frac{p_{2r}^{\ 2}}{2m_2} + \frac{p_{2\theta}^{\ 2}}{2m_2 r_2^{\ 2}} - \frac{1}{2}k(r_1 + r_2 - l)^2$$

Equations of motion …

First, θ_1, θ_2 are ignorable coordinates, so $p_{1\theta}$, $p_{2\theta}$ are each conserved.

$$\frac{\partial H}{\partial r_1} = -\dot{p}_{1r} = k\left(r_1 + r_2 - l\right) = -m_1\ddot{r}_1$$

$$\frac{\partial H}{\partial r_2} = -\dot{p}_{2r} = k\left(r_1 + r_2 - l\right) = -m_2\ddot{r}_2$$

$$\dot{p}_{1r} = \dot{p}_{2r}$$

$$\Delta\dot{p}_r = 0 \qquad \Delta p_r = \text{constant.}$$

The radial momenta are equal and opposite.

10.30 $L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

$$0 = \delta\int_{t_1}^{t_2} Ldt = \int_{t_1}^{t_2} \delta Ldt = \int_{t_1}^{t_2} \delta\left(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2\right)dt$$

$$0 = \int_{t_1}^{t_2}\left(m\dot{x}\delta\dot{x} - kx\delta x\right)dt$$

$$\delta\dot{x} = \frac{d}{dt}\delta x$$

$$\int_{t_1}^{t_2} m\dot{x}\delta\dot{x}dt = \int_{t_1}^{t_2} m\dot{x}\frac{d}{dt}\left(\delta x\right)dt = \int_{t_1}^{t_2} m\dot{x}d\left(\delta x\right)$$

Integrating by parts:

$$\int_{t_1}^{t_2} m\dot{x}\delta\dot{x}dt = m\dot{x}\delta x\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2}\delta x\, d\left(m\dot{x}\right)$$

$\delta x = 0$ at t_1 and t_2

$$d\left(m\dot{x}\right) = \frac{d}{dt}\left(m\dot{x}\right)dt = m\ddot{x}\,dt$$

$$\int_{t_1}^{t_2} m\dot{x}\delta\dot{x}\,dx = -\int_{t_1}^{t_2}\delta x\, m\ddot{x}\,dt$$

$$0 = \int_{t_1}^{t_2}\left(-m\ddot{x}\delta x - kx\delta x\right)dt$$

$$m\ddot{x} + kx = 0$$

10.31 (a) $L = -m_0c^2\sqrt{1-\frac{v^2}{c^2}} - V$

Let $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

$\frac{\partial L}{\partial \dot{x}} = \gamma m_0\dot{x} \equiv p_x$ This is the generalized momentum for part (b).

Thus, Lagrange's equations for the x-component are …

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt}p_x - \frac{\partial V}{\partial x} = 0$$

… and so on for the y and z components.

(b) $H = \sum_i v_i p_i - L$

but … $v_i = \frac{p_i}{\gamma m_0}$

So … $H = \sum_i \frac{p_i^2c^2}{\gamma m_0c^2} + \frac{m_0c^2}{\gamma} + V$

$$H = \frac{p^2c^2}{\gamma m_0c^2} + \frac{m_0c^2}{\gamma} + V$$

$$H = \frac{1}{\gamma m_0 c^2}\left(p^2c^2 + m_0{}^2c^4\right) + V$$

(c) Now, if $T = \gamma m_0 c^2$ then we have …

$$p^2c^2 + m_0{}^2c^4 = m_0{}^2c^4\left(1 + \frac{p^2c^2}{m_0{}^2c^4}\right)$$

$$= m_0{}^2c^4\left(1 + \frac{\gamma^2 m_0{}^2 v^2 c^2}{m_0{}^2c^4}\right) = m_0{}^2c^4\left(1 + \gamma^2\, v^2/c^2\right)$$

$$= m_0{}^2c^4\left(1 + \frac{v^2/c^2}{1 - v^2/c^2}\right) = m_0{}^2c^4\left(\frac{1}{1 - v^2/c^2}\right) = \gamma^2 m_0{}^2c^4$$

Thus … $H = \gamma m_0 c^2 + V = T + V$

(d) $$T = \frac{m_0 c^2}{1 - v^2/c^2} \approx m_0 c^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots\right)$$

$$T \approx \frac{1}{2} m_0 v^2 + m_0 c^2$$

Chapter 11

11.1

(a) $V(x)=\frac{k}{2}x^2+\frac{k^2}{x}$

$V'=kx-\frac{k^2}{x^2}$ At equilibrium, $V'=0$

$x=k^{1/3}$

$V''=k+\frac{2k^2}{x^3}$

$V''\big|_{x=k^{1/3}}=k+2k=3k>0$ **Stable**

(b) $V(x)=kxe^{-bx}$

$V'=ke^{-bx}-bkxe^{-bx}$

At equilibrium $ke^{-bx}-bkxe^{-bx}=0$

$x=\frac{1}{b}$

$V''=-bke^{-bx}-bkxe^{-bx}+b^2kxe^{-bx}$

$V''\big|_{x=1/b}=-2bke^{-1}+bke^{-1}=-bke^{-1}<0$ **Unstable**

(c) $V(x)=k\left(x^4-b^2x^2\right)$

$V''=k\left(4x^3-2b^2x\right)$

At equilibrium $k\left(4x^3-2b^2x\right)=0$

$x=0,\quad \pm b/\sqrt{2}$

$V''=k\left(12x^2-2b^2\right)$

$V''\big|_{x=0}=-2kb^2<0$ **Unstable**

$V''\big|_{x=\pm b/\sqrt{2}}=k\left(6b^2-2b^2\right)=4kb^2>0$ **Stable**

(d) for case (a) $\omega^2=\frac{3k}{m}$ $T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{3k}}=\frac{2\pi}{\sqrt{3}}s$

for case (c) at $x=\pm b/\sqrt{2}$ $\omega^2=\frac{4kb^2}{m}$ $T=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{m}{4kb^2}}=\pi\, s$

11.2

$V(x,y)=k\left(x^2+y^2-2bx-4by\right)$

$$\frac{\partial V}{\partial x} = 2k(x-b) \qquad \frac{\partial V}{\partial y} = 2k(y-2b)$$

at equilibrium

$$x = b \qquad and \qquad y = 2b$$

$$\frac{\partial^2 V}{\partial x^2} = 2k \qquad \frac{\partial^2 V}{\partial x \partial y} = 0 \qquad \frac{\partial^2 V}{\partial y^2} = 2k$$

$$k_{11} = 2k > 0 \qquad k_{12} = k_{21} = 0 \qquad k_{22} = 2k$$

$$\begin{vmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{vmatrix} = 4k^2 - 0 > 0$$

The equilibrium is stabe.

11.3

$$V(x) = -\frac{1}{2}kx^2$$

$$F(x) = -\frac{dV(x)}{dx} = kx = m\ddot{x} = m\dot{x}\frac{d\dot{x}}{dx}$$

$$kx\,dx = m\dot{x}\,d\dot{x}$$

$$\int_{x_0}^{x} kx dx = \int_0^v m\dot{x}\,d\dot{x} \qquad \frac{k}{2}\left(x^2 - x_0^2\right) = m\frac{v^2}{2}$$

$$v = \frac{dx}{dt} = \sqrt{k/m}\left(x^2 - x_0^2\right)^{1/2}$$

$$\int_{x_0}^{x} \frac{dx}{\left(x^2 - x_0^2\right)^{1/2}} = \int_0^t \alpha\,dt; \qquad where\ \alpha = \sqrt{k/m}$$

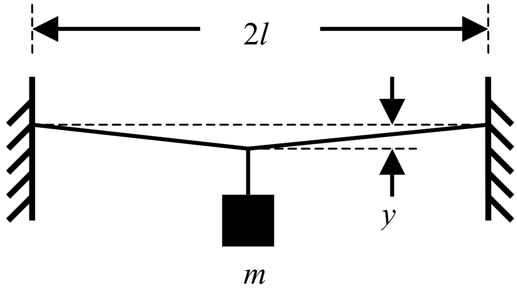
$$\ln\left(x + \sqrt{x^2 - x_0^2}\right) - \ln x_0 = \alpha t$$

$$x + \sqrt{x^2 - x_0^2} = x_0 e^{\alpha t}$$

$$x^2 - x_0^2 = x_0^2 e^{2\alpha t} - 2xx_0 e^{\alpha t} + x^2$$

$$x = x_0 \frac{e^{\alpha t} + e^{-\alpha t}}{2} = x_0 \cosh \alpha t$$

11.4



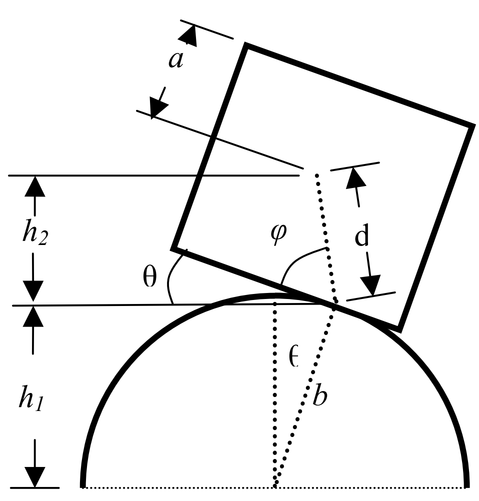
Let the length of the unstretched, elastic cord be d. Then

$$d = 2\sqrt{l^2 + y^2}$$

$$V = \frac{1}{2}k\left(d - 2l\right)^2 - mgy$$

$$V=\frac{k}{2}\left(4l^2+4y^2-8l\sqrt{l^2+y^2}+4l^2\right)-mgy$$
$$V=2k\left(2l^2+y^2-2l\sqrt{l^2+y^2}\right)-mgy$$
The first term, $4kl^2$, is an additive constant to the potential energy, so with appropriate adjustment of the zero reference point …
$$V(y)=2k\left(y^2-2l\sqrt{l^2+y^2}\right)-mgy$$
$$\frac{dV(y)}{dy}=2k\left[2y-2yl\left(l^2+y^2\right)^{-1/2}\right]-mg$$
At equilibrium, the above expression is zero, so …
$$4ky-\frac{4kly}{\sqrt{l^2+y^2}}-mg=0$$
$$4ky-mg=\frac{4kly}{\sqrt{l^2+y^2}}$$
$$16k^2y^2-8kmgy+m^2g^2=\frac{16k^2l^2y^2}{l^2+y^2}$$
$$16k^2y^4-8kmgy^3+\left(16k^2l^2+m^2g^2-16k^2l^2\right)y^2-8kl^2mgy+l^2m^2g^2=0$$
$$\frac{y^4}{l^4}-\frac{mg}{2kl^4}y^3+\frac{m^2g^2}{16k^2l^4}y^2-\frac{mg}{2kl^2}y+\frac{m^2g^2}{16k^2l^2}=0$$
letting $u=\frac{y}{l}$ and $a=\frac{mg}{4kl}$
$$u^4-2au^3+a^2u^2-2au+a^2=0$$

11.5



$$V=mg\left(h_1+h_2\right)$$
$$h_1=b\cos\theta$$
$$h_2=d\sin\left(\theta+\varphi\right)$$
$$h_2=d\left(\sin\theta\cos\varphi+\cos\theta\sin\varphi\right)$$
$$\cos\varphi=\frac{b\theta}{d}\qquad\sin\varphi=\frac{a}{d}$$
$$h_2=b\theta\sin\theta+a\cos\varphi$$

$$V=mg\left[\left(a+b\right)\cos\theta+b\theta\sin\theta\right]$$
$$V'=mg\left[\left(a+b\right)\left(-\sin\theta\right)+b\sin\theta+b\theta\cos\theta\right]$$

$V' = mg\left[-a\sin\theta + b\theta\cos\theta\right]$
$V'' = mg\left[b\cos\theta - b\theta\sin\theta - a\cos\theta\right]$
$V_0'' = mg(b-a)$

Equilibrium …

Stable	Unstable
$a<b$	$a>b$

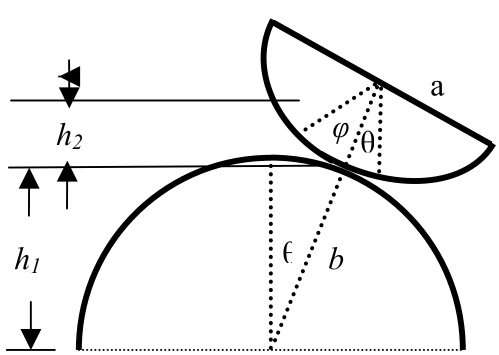
11.6

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots$$
$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots$$
$$V = mg\left[(a+b)\left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \ldots\right) + b\theta\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \ldots\right)\right]$$
$$V = mg\left[a+b - \frac{a-b}{2}\theta^2 + \frac{a-3b}{24}\theta^4 - \ldots\right]$$

For a = b, $V = mg\left[2a - \frac{a}{12}\theta^4 + \ldots\right]$

$V' = -\frac{mga}{3}\theta^3 + \ldots$ terms in higher order of θ
$V'' = -mga\theta^2 + \ldots$
$V''' = -2mga\theta + \ldots$
$V'''' = -2mga < 0$ $\quad\therefore$ Equilibrium is unstable

11.7



The center of mass (CM) of the hemisphere is $\frac{3}{8}a$ from the flat side (see Equation 8.1.8). The height of CM) above the point of contact between the two hemispheres is designated by h_2 in the figure. h_1 is the height of the point of contact above the ground.

$$V = mg(h_1 + h_2) = mg\left[b\cos\theta + a\cos\theta - \frac{3}{8}a\cos(\theta+\varphi)\right] \quad \text{and} \quad a\varphi = b\theta$$

$$V = mg\left(h_1 + h_2\right) = mg\left[\left(a+b\right)\cos\theta - \frac{3}{8}a\cos\left(\theta + \frac{b\theta}{a}\right)\right]$$

$$V' = mg\left[-\left(a+b\right)\sin\theta + \frac{3a}{8}\left(\frac{a+b}{a}\right)\sin\left(\left(\frac{a+b}{a}\right)\theta\right)\right]$$ Equilibrium occurs at $\theta = 0^o$

$$V'' = mg\left[-\left(a+b\right)\cos\theta + \frac{3a}{8}\left(\frac{a+b}{a}\right)^2\cos\left(\left(\frac{a+b}{a}\right)\theta\right)\right]$$

$$V_0'' = mg\left[-\left(a+b\right) + \frac{3a}{8}\left(\frac{a+b}{a}\right)^2\right] = \frac{mg}{8a}\left(a+b\right)\left(3b-5a\right)$$

$$V_0'' > 0 \quad for \;\; 3b > 5a$$

Therefore, the equilibrium is stable for $a < \frac{3b}{5}$

11.8

From Problem 11.4, we have

$$V(y) = 2k\left(y^2 - 2l\sqrt{l^2 + y^2}\right) - mgy = 2k\left(y^2 - 2l^2\left(1 + \frac{y^2}{l^2}\right)^{\frac{1}{2}}\right) - mgy$$

Expanding the square root for small $\frac{y}{l}$ … $\left(1 + \frac{y^2}{l^2}\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\frac{y^2}{l^2} - \frac{1}{8}\frac{y^4}{l^4} + ...$

$$V(y) \approx 2k\left[y^2 - 2l^2 - y^2 + \frac{y^4}{4l^2}\right] - mgy$$

$$V' \approx \frac{2k}{l^2}y^3 - mg$$

at equilibrium, $V' = 0 \quad \Rightarrow \quad y = \left(\frac{mgl^2}{2k}\right)^{\frac{1}{3}}$

$$V'' = \frac{6k}{l^2}y^2$$

$$V''\Big|_{y=\left(mgl^2/2k\right)^{\frac{1}{3}}} = \frac{6k}{l^2}\left(\frac{mgl^2}{2k}\right)^{\frac{2}{3}} = 6k^{\frac{1}{3}}\left(\frac{mg}{2l}\right)^{\frac{2}{3}}$$

$$\omega = \sqrt{\frac{V''}{m}} = \sqrt{6\left(\frac{k}{m}\right)^{\frac{1}{3}}\left(\frac{g}{2l}\right)^{\frac{2}{3}}} = \sqrt{6}\left(\frac{g}{2l}\right)^{\frac{1}{3}}\left(\frac{k}{m}\right)^{\frac{1}{6}}$$

11.9

From Problem 11.5, $V_0'' = mg(b-a)$

$$\omega = \sqrt{\frac{V_0''}{m}} = \sqrt{g(b-a)}$$

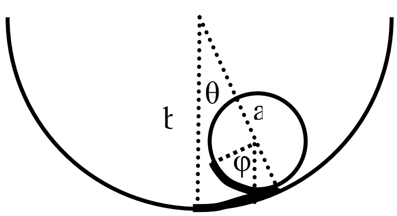
$$T_0 = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{g(b-a)}}$$

11.10

From Problem 11.7, $V_0'' = \dfrac{mg}{8a}(a+b)(3b-5a)$

$$T_0 = 2\pi\sqrt{\frac{m}{V_0''}} = 4\pi\sqrt{\frac{2a}{g(a+b)(3b-5a)}}$$

11.11



$$T = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I\dot{\varphi}^2$$

$$v_{cm} = (b-a)\dot{\theta}$$

$$I = \frac{2}{5}ma^2$$

The relationship between the angles, θ *and* φ (see Figure), can be determined from the condition that there is no slipping as the ball rolls in the hemisphere, so the length of roll measured along the ball, $a(\theta+\varphi)$, must equal the length of roll measured along the hemisphere, $b\theta$ … so we have …

$$b\theta = a(\theta+\varphi) \quad and \quad \therefore \quad b\dot{\theta} = a(\dot{\theta}+\dot{\varphi})$$

and … $\dot{\varphi} = \left(\dfrac{b-a}{a}\right)\dot{\theta}$

$$T = \frac{1}{2}m(b-a)^2\dot{\theta}^2 + \frac{1}{2}\frac{2}{5}ma^2\frac{(b-a)^2\dot{\theta}^2}{a^2} = \frac{1}{2}m(b-a)^2\left(1+\frac{2}{5}\right)\dot{\theta}^2$$

$$V = -mg(b-a)\cos\theta$$

$$V' = mg(b-a)\sin\theta \quad equilibrium \ @ \ \theta = 0^o$$

$$V''=mg(b-a)\cos\theta \quad \therefore V_0''=mg(b-a)$$

$$\omega=\sqrt{\frac{V_0''}{M}}=\sqrt{\frac{mg(b-a)}{\frac{7}{5}m(b-a)^2}}=\sqrt{\frac{5g}{7(b-a)}}$$

$$T_0=\frac{2\pi}{\omega}=2\pi\sqrt{\frac{7(b-a)}{5g}}$$

11.12

The potential energy of the satellite shaped like a "thin rod" is (See Example 11.2.2, Figure 11.2.1) …

$$V=\int -G\frac{M_e\,dm}{r} \quad \text{but } dm=\frac{m}{2a}dx \text{ where } 2a \text{ is the length of the rod.}$$

$$V=\int_{-a}^{a}-\frac{GM_e}{r}\frac{m}{2a}dx=-\frac{GM_e m}{2a}\int_{-a}^{a}\frac{dx}{r}$$

$$r=\left(r_0^2+x^2+2r_0\,x\cos\phi\right)^{\frac{1}{2}}$$

$$r=\left(r_0^2+x^2\right)^{\frac{1}{2}}\left(1+\varepsilon\cos\phi\right)^{\frac{1}{2}} \quad \text{where } \varepsilon=\frac{2xr_0}{r_0^2+x^2}$$

$$V=-\frac{GM_e m}{2a}\int_{-a}^{a}\frac{\left(1+\varepsilon\cos\phi\right)^{-\frac{1}{2}}dx}{\left(r_0^2+x^2\right)^{\frac{1}{2}}}$$

For $r_0>>x,\quad r_0^2+x^2\approx r_0^2,\quad \varepsilon\approx\frac{2x}{r_0}\quad and \quad r\approx r_0\left(1+\varepsilon\cos\phi\right)^{\frac{1}{2}}$

Thus, for small ε, the expression for the potential energy, V, can be approximated …

$$V\approx-\frac{GM_e m}{2ar_0}\int_{-a}^{a}\left(1-\frac{1}{2}\left(\frac{2x}{r_0}\right)\cos\phi+\frac{3}{8}\left(\frac{2x}{r_0}\right)^2\cos^2\phi\right)dx$$

$$V\approx-\frac{GM_e}{2ar_0}\left[2a+0+\frac{3\cos^2\phi}{2r_0^2}\frac{2a^3}{3}\right]$$

$$V\approx-\frac{GM_e}{r_0}\left[1+\frac{a^2\cos^2\phi}{2r_0^2}\right]$$

$$V'\approx\frac{GM_e}{r_0}\left(\frac{a^2}{2r_0^2}\right)2\cos\phi\sin\phi=\frac{GM_e a^2}{2r_0^3}\sin 2\phi$$

Equilibrium @ $\phi = 0$

$$V'' \approx \frac{GM_e a^2}{r_0^3}\cos 2\phi \quad \text{and} \quad V_0' \approx \frac{GM_e a^2}{r_0^3}$$

$$M = I = \frac{1}{3}ma^2$$

$$\omega = \sqrt{\frac{V_0''}{M}} = \sqrt{\frac{3GM_e}{r_0^3}}$$

$$T_0 = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{r_0^3}{3GM_e}}$$

11.13

The amplitude of the symmetric component is A_1 and the amplitude of the anti-symmetric component is A_2 (See Equations 11.3.19a through 11.3.20b).

$$A_1^2 = \frac{1}{4}\left[x_1(0) + x_2(0)\right]^2$$

$$A_1 = \frac{1}{2}\left[x_1(0) + x_2(0)\right] = \frac{A_0}{2}$$

$$A_2^2 = \frac{1}{4}\left[x_2(0) - x_1(0)\right]^2$$

$$A_2 = \frac{1}{2}\left[x_2(0) - x_1(0)\right] = \frac{A_0}{2} \qquad \therefore\ A_1 = A_2$$

From Equation 11.3.18 the solution for x_1 is …

$$x_1(t) = \frac{A_0}{2}\left(\cos\omega_1 t + \cos\omega_2 t\right) \text{ (The phase } \delta_2 \text{ is } 180^{\circ}\text{, which insures that } x_1(0) = A_0\text{)}$$

$$x_1(t) = \frac{A_0}{2}\left(2\cos\frac{(\omega_1+\omega_2)t}{2}\cos\frac{(\omega_1-\omega_2)t}{2}\right)$$

Letting … $\quad \bar{\omega} = \dfrac{\omega_1+\omega_2}{2} \quad \textit{and} \quad \Delta = \dfrac{\omega_1-\omega_2}{2}$

$$x_1(t) = A_0\left(\cos\bar{\omega}t\ \cos\Delta t\right)$$

From Equation 11.3.18, the solution for x_2 is …

$$x_2(t) = \frac{A_0}{2}\left(\cos\omega_1 t - \cos\omega_2 t\right)$$

$$x_2(t) = \frac{A_0}{2}\left(2\sin\frac{(\omega_1+\omega_2)t}{2}\sin\frac{(\omega_1-\omega_2)t}{2}\right)$$

$$x_2(t) = A_0\left(\sin\bar{\omega}t\sin\Delta t\right)$$

11.14

At time $t = 0$ and short times thereafter … $\cos\Delta t \approx 1$ and $\sin\Delta t \approx 0$. Thus, …
$x_1 \approx A_0 \cos\bar{\omega}t \quad and \quad x_2 \approx 0$. This situation occurs again when $\Delta t = 2\pi$.

$$\Delta = \frac{\omega_2 - \omega_1}{2} = \frac{1}{2}\left[\left(\frac{k+2k'}{m}\right)^{\frac{1}{2}} - \left(\frac{k}{m}\right)^{\frac{1}{2}}\right]$$

$$\Delta = \frac{1}{2}\left(\frac{k}{m}\right)^{\frac{1}{2}}\left[\left(1+\frac{2k'}{k}\right)^{\frac{1}{2}} - 1\right]$$

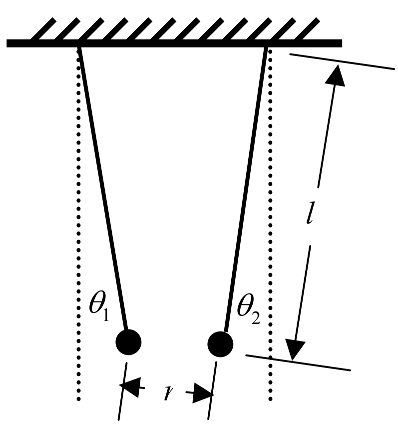
for $k' << k$, $\left(1+\frac{2k'}{k}\right)^{\frac{1}{2}} - 1 \approx 1 + \frac{1}{2}\left(\frac{2k'}{k}\right) + ... - 1 = \frac{k'}{k}$

$$\Delta = \frac{1}{2}\left(\frac{k}{m}\right)^{\frac{1}{2}}\left(\frac{k'}{k}\right)$$

$$T = \frac{2\pi}{\Delta} = 2\pi\left(\frac{m}{k}\right)^{\frac{1}{2}}\frac{2k}{k'} = \frac{2\pi}{\omega_1}\left(\frac{2k}{k'}\right)$$

$$T = T_1\left(\frac{2k}{k'}\right)$$

11.15



$$T = \frac{1}{2}ml^2\left(\dot{\theta}_1^2 + \dot{\theta}_2^2\right)$$

$$V = mgl\left[\left(1-\cos\theta_1\right) + \left(1-\cos\theta_2\right)\right] - \frac{k}{r}$$

$$r = r_0 + l\sin\theta_2 - l\sin\theta_1$$

$$\frac{\partial V}{\partial\theta_1} = mgl\sin\theta_1 - \frac{kl\cos\theta_1}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^2}$$

$$\frac{\partial^2 V}{\partial\theta_1^2} = mgl\cos\theta_1 - \frac{kl\sin\theta_1}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^2} - \frac{2kl^2\cos^2\theta_1}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^3}$$

$$k_{11} = \left.\frac{\partial^2 V}{\partial\theta_1^2}\right|_{\theta_1=\theta_2=0} = mgl - \frac{2kl^2}{r_0^3}$$

$$\frac{\partial^2 V}{\partial\theta_2\partial\theta_1} = \frac{2kl^2\cos\theta_1\cos\theta_2}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^3}$$

$$k_{12} = \left.\frac{\partial^2 V}{\partial\theta_1 \partial\theta_2}\right|_{\theta_1=\theta_2=0} = \left.\frac{\partial^2 V}{\partial\theta_2 \partial\theta_1}\right|_{\theta_1=\theta_2=0} = \frac{2kl^2}{r_0^{\ 3}}$$

$$\frac{\partial V}{\partial\theta_2} = mgl\sin\theta_2 + \frac{kl\cos\theta_2}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^2}$$

$$\frac{\partial^2 V}{\partial\theta_2^{\ 2}} = mgl\cos\theta_2 - \frac{kl\sin\theta_2}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^2} - \frac{2kl^2\cos^2\theta_2}{\left(r_0 + l\sin\theta_2 - l\sin\theta_1\right)^3}$$

$$k_{22} = \left.\frac{\partial^2 V}{\partial\theta_2^{\ 2}}\right|_{\theta_1=\theta_2=0} = mgl - \frac{2kl^2}{r_0^{\ 3}}$$

Thus, from Equation 11.3.37a or b, we have

$$V = \frac{1}{2}\left[k_{11}\theta_1^{\ 2} + 2k_{12}\theta_1\theta_2 + k_{22}\theta_2^{\ 2}\right]$$

But from Equation 11.3.9, for the coupled oscillator, we have

$$V = \frac{1}{2}kx_1^{\ 2} + \frac{1}{2}k'\left(x_1^{\ 2} - 2x_1x_2 + x_2^{\ 2}\right) + \frac{1}{2}kx_1^{\ 2}$$

The forms of the potential energy function are similar with …

$$k_{11} = k + k' = k_{22} \quad and \quad k_{12} = -k'$$

In the case here …

$$k = mgl \quad and \quad k' = -\frac{2kl^2}{r_0^{\ 3}}$$

Chapter 11 (continued)

11.16

$$T = \frac{1}{2}m_1\dot{x}_1^{\,2} + \frac{1}{2}m_2\dot{x}_2^{\,2}$$
$$V = \frac{1}{2}k_1x_1^{\,2} + \frac{1}{2}k'\left(x_2 - x_1\right)^2 + \frac{1}{2}k_2x_2^{\,2}$$
$$L = T - V$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = m_1\ddot{x}_1 \qquad \frac{\partial L}{\partial x_1} = -k_1x_1 + k'\left(x_2 - x_1\right)$$
$$m_1\ddot{x}_1 + k_1x_1 - k'\left(x_2 - x_1\right) = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = m_2\ddot{x}_2 \qquad \frac{\partial L}{\partial x_2} = -k_2x_2 - k'\left(x_2 - x_1\right)$$
$$m_2\ddot{x}_2 + k_2x_2 + k'\left(x_2 - x_1\right)$$

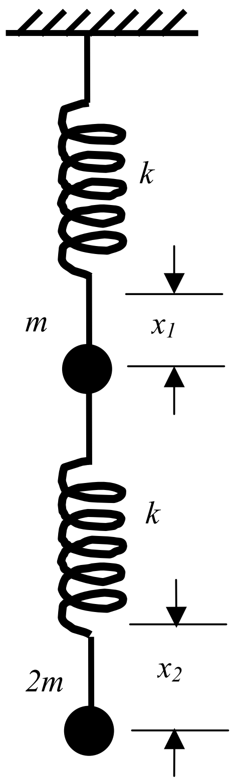
$$\begin{vmatrix} -m_1\omega^2 + k_1 + k' & -k' \\ -k' & -m_2\omega^2 + k_2 + k' \end{vmatrix} = 0$$

$$m_1m_2\omega^4 - \left[m_2\left(k_1 + k'\right) + m_1\left(k_2 + k'\right)\right]\omega^2 + \left(k_1 + k'\right)\left(k_2 + k'\right) - k'^2 = 0$$
$$m_1m_2\omega^4 - \left[m_2\left(k_1 + k'\right) + m_1\left(k_2 + k'\right)\right]\omega^2 + k_1k_2\left(k_1 + k_2\right)k' = 0$$
$$\omega^2 = \frac{m_2\left(k_1 + k'\right) + m_1\left(k_2 + k'\right) \pm \sqrt{\left[m_2\left(k_1 + k'\right) + m_1\left(k_2 + k'\right)\right]^2 - 4m_1m_2\left[k_1k_2 + \left(k_1 + k_2\right)k'\right]}}{2m_1m_2} = 0$$

For $\quad m_1 = m,\ m_2 = 2m,\ k_1 = k,\ k_2 = 2k,\ k' = 2k$

$$\omega^2 = \frac{2m\left(3k\right) + m\left(4k\right) \pm \sqrt{\left(6mk + 4mk\right)^2 - 4\left(2m^2\right)\left(2k^2 + 6k^2\right)}}{2\left(2m^2\right)}$$
$$\omega^2 = \frac{10}{4}\frac{k}{m} \pm \frac{6}{4}\frac{k}{m}$$
$$\omega = 2\omega_0 \ \ and \ \ \omega_0 \ \ where \ \ \omega_0 = \sqrt{\frac{k}{m}}$$

11.17



Note: As discussed in Section 3.2, the effect of any constant external force on a harmonic oscillator is to shift the equilibrium position. x_1 and x_2 are the positions of the harmonic oscillator masses away from their respective "shifted" equilibrium positions.

$$T = \frac{1}{2} m\dot{x}_1^{\,2} + \frac{1}{2}(2m)\dot{x}_2^{\,2}$$

$$V = \frac{1}{2} kx_1^{\,2} + \frac{1}{2} k(x_2 - x_1)^2$$

$$L = T - V$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = m\ddot{x}_1\,, \quad \frac{\partial L}{\partial x_1} = -kx_1 + k(x_2 - x_1)$$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = 2m\ddot{x}_2\,, \quad \frac{\partial L}{\partial x_2} = -k(x_2 - x_1)$$

$$2m\ddot{x}_2 + kx_2 - kx_1 = 0$$

The secular equation (11.4.12) is thus

$$\begin{vmatrix} -m\omega^2 + 2k & -k \\ -k & -2m\omega^2 + k \end{vmatrix} = 0$$

$$2m^2\omega^4 - 5mk\omega^2 + 2k^2 + k^2 = 0$$

The eigenfrequencies are thus …

$$\omega^2 = \frac{5 \pm \sqrt{17}}{4}\left(\frac{k}{m}\right)$$

The homogeneous equations (Equations 11.4.10) for the two components of the j[th] eigenvector are …

$$\begin{pmatrix} -m\omega^2 + 2k & -k \\ -k & -2m\omega^2 + k \end{pmatrix}\begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix} = 0$$

For the first eigenvector (the anti-symmetric mode, j = 1) …

Inserting $\omega_1^{\,2} = \frac{5 + \sqrt{17}}{4}\left(\frac{k}{m}\right)$ into the first of the two homogeneous equations yields

$$\left[-\frac{5 + \sqrt{17}}{4}k + 2k\right]a_{11} = ka_{21}$$

$$a_{21} = \frac{3 - \sqrt{17}}{4}a_{11}$$

Letting $a_{11} = 1$, then $a_{21} = -0.281$ (Thus, in the anti-symmetric normal mode, the amplitude of the vibration of the second mass is 0.281 that of the first mass and 180° out of phase with it.)

For the second eigenvector (the symmetric mode, j = 2) …

Inserting $\omega_2^{\,2} = \frac{5-\sqrt{17}}{4}\left(\frac{k}{m}\right)$ into the first of the two homogeneous equations yields

$$\left[-\frac{5-\sqrt{17}}{4}k + 2k\right]a_{12} = ka_{22}$$

$$a_{22} = \frac{3+\sqrt{17}}{4}a_{12}$$

Letting $a_{12} = 1$, then $a_{22} = 1.781$ (Thus, in the symmetric normal mode, the amplitude of the vibration of the second mass is 1.781 that of the first mass and in phase with it.)

The two eigenvectors (Equation 11.4.13 and see accompanying Table) are …

$$Q_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix}\cos\left(\omega_1 t - \delta_1\right) = \begin{pmatrix} 1 \\ -0.281 \end{pmatrix}\cos\left(\omega_1 t - \delta_1\right)$$

$$Q_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}\cos\left(\omega_2 t - \delta_2\right) = \begin{pmatrix} 1 \\ 1.781 \end{pmatrix}\cos\left(\omega_2 t - \delta_2\right)$$

11.18

$$T = \frac{1}{2}ml_1^{\,2}\dot{\theta}^2 + \frac{1}{2}m\left(l_1\dot{\theta} + l_2\dot{\phi}\right)^2$$

$$V = -mgl_1\cos\theta - mg\left(l_1\cos\theta + l_2\cos\phi\right)$$

For small angular displacements …

$$L = T - V \approx \frac{m}{2}l_1^{\,2}\dot{\theta}^2 + \frac{m}{2}\left(l_1\dot{\theta} + l_2\dot{\phi}\right)^2 + 2mgl_1\left(1 - \frac{\theta^2}{2}\right) + mgl_2\left(1 - \frac{\phi^2}{2}\right)$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = ml_1^{\,2}\ddot{\theta} + ml_1\left(l_1\ddot{\theta} + l_2\ddot{\phi}\right), \quad \frac{\partial L}{\partial\theta} = -2mgl_1\theta$$

$$2l_1\ddot{\theta} + l_2\ddot{\phi} + 2g\theta = 0$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\phi}} = ml_2\left(l_1\ddot{\theta} + l_2\ddot{\phi}\right), \quad \frac{\partial L}{\partial\phi} = -mgl_2\phi$$

$$l_1\ddot{\theta} + l_2\ddot{\phi} + g\phi = 0$$

The secular equation (Equation 11.4.12) is …

$$\begin{vmatrix} -2l_1\omega^2 + 2g & -l_2\omega^2 \\ -l_1\omega^2 & -l_2\omega^2 + g \end{vmatrix} = 0$$

$$2l_1l_2\omega^4 - 2g\left(l_1 + l_2\right)\omega^2 + 2g^2 - l_1l_2\omega^4 = 0$$

Solving for the eigenfrequencies ω^2 …

$$\omega^2 = \frac{2g\left(l_1 + l_2\right) \pm \sqrt{4g^2\left(l_1 + l_2\right)^2 - 8l_1l_2g^2}}{2l_1l_2} = \frac{g}{l_1l_2}\left(l_1 + l_2 + \sqrt{l_1^{\,2} + l_2^{\,2}}\right)$$

The homogeneous equations (Equations 11.4.10) for the two components of the j[th] eigenvector are …

$$\begin{pmatrix} -2l_1\omega^2+2g & -l_2\omega^2 \\ -l_1\omega^2 & -l_2\omega^2+g \end{pmatrix}\begin{pmatrix} a_{1j} \\ a_{2j} \end{pmatrix}=0$$

Inserting the larger eigenfrequency (the (+) solution for ω^2 above) into the upper homogeneous equation yields the solution for the components of the 1st eigenvector …

$$\left(-2l_1\omega_1{}^2+2g\right)a_{11}-l_2\omega_1{}^2a_{21}=0$$

$$2ga_{11}\left[1-\frac{l_1+l_2+\sqrt{l_1^2+l_2^2}}{l_2}\right]=ga_{21}\frac{l_1+l_2+\sqrt{l_1^2+l_2^2}}{l_1}$$

$$a_{21}=a_{11}\frac{l_2-l_1-\sqrt{l_1^2+l_2^2}}{l_2}$$ for the higher frequency, anti-symmetric mode.

Inserting the smaller eigenfrequency (the (-) solution for ω^2) into the upper homogeneous equation yields the solution for the components of the 2nd eigenvector …

$$\left(-2l_1\omega_2{}^2+2g\right)a_{12}-l_2\omega_2{}^2a_{22}=0$$

$$a_{22}=a_{12}\frac{l_2-l_1+\sqrt{l_1^2+l_2^2}}{l_2}$$ for the lower frequency, symmetric mode.

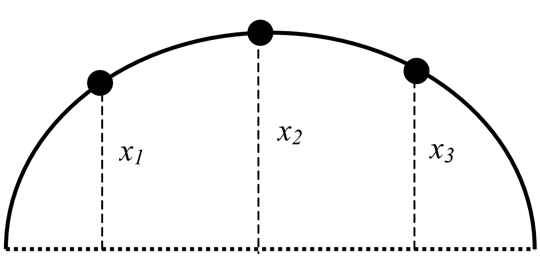
Again, we let a_{11} = 1 and a_{21} = 1, since only ratios of the components of a given eigen vector can be determined. The two eigenvectors are thus (Equation 11.4.12 and accompanying table)

$$Q_1=\begin{pmatrix} 1 \\ \dfrac{l_2-l_1-\sqrt{l_1^2+l_2^2}}{l_2} \end{pmatrix}\cos\left(\omega_1t-\delta_1\right)$$ … anti-symmetric

$$Q_2=\begin{pmatrix} 1 \\ \dfrac{l_2-l_1+\sqrt{l_1^2+l_2^2}}{l_2} \end{pmatrix}\cos\left(\omega_2t-\delta_2\right)$$ … symmetric

As a check, set $l_1=l_2=l$ and compare with the solution for Example 11.3.1.

11.19



$$T=\frac{1}{2}m\left(\dot{x}_1{}^2+\dot{x}_2{}^2+\dot{x}_3{}^2\right)$$

$$V=\frac{1}{2}k\left[x_1{}^2+\left(x_2-x_1\right)^2+\left(x_3-x_2\right)^2+x_3{}^2\right]$$

$$L=T-V$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}_i}-\frac{\partial L}{\partial x_i}=0$$

$$m\ddot{x}_1 + kx_1 - k\left(x_2 - x_1\right) = 0$$
$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 + k\left(x_2 - x_1\right) - k\left(x_3 - x_2\right) = 0$$
$$-kx_1 + m\ddot{x}_2 + 2kx_2 - kx_3 = 0$$
$$m\ddot{x}_3 + k\left(x_3 - x_2\right) + kx_3 = 0$$
$$-kx_2 + m\ddot{x}_3 + 2kx_3 = 0$$

The secular equation (Equation 11.4.12) is …

$$\begin{vmatrix} -m\omega^2 + 2k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + 2k \end{vmatrix} = 0$$

$$\left(-m\omega^2 + 2k\right)^3 - 2k^2\left(-m\omega^2 + 2k\right) = 0$$
$$-m\omega^2 + 2k = 0, \quad or \quad \left(-m\omega^2 + 2k\right)^2 - 2k^2 = 0$$
$$\omega^2 = \frac{2k}{m} = 2\,\omega_0{}^2$$
$$-m\omega^2 + 2k = \pm\sqrt{2}\,k$$
$$\omega^2 = \left(2 \pm \sqrt{2}\right)\frac{k}{m} = \left(2 \pm \sqrt{2}\right)\omega_0{}^2$$

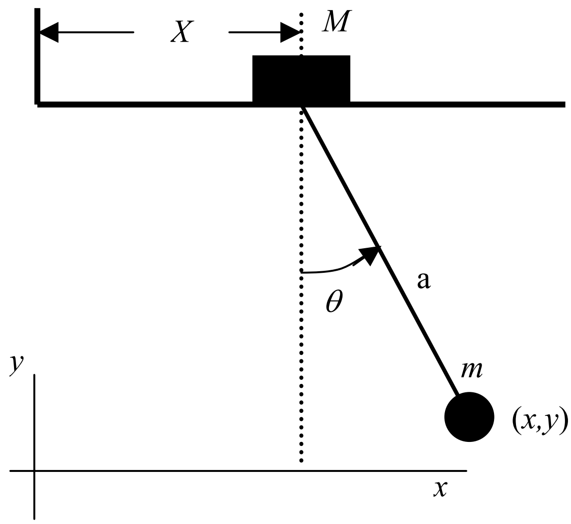
From Equation 11.5.17 ($N = 1$ and $n = 3$)

$$\omega_1 = 2\omega_0 \sin\frac{\pi}{8}$$

Because $\sin\frac{\theta}{2} = \sqrt{\frac{1-\cos\theta}{2}}$

$$\omega_1 = 2\omega_0\sqrt{\frac{1-\cos\pi/4}{2}} = \sqrt{2}\omega_0\sqrt{1-\sqrt{2}/2}$$
$$\omega_1 = \omega_0\sqrt{2-\sqrt{2}}$$
$$\omega_2 = 2\omega_0 \sin\frac{2\pi}{8} = 2\omega_0\frac{\sqrt{2}}{2} = \sqrt{2}\omega_0$$
$$\omega_3 = 2\omega_0 \sin\frac{3\pi}{8} = 2\omega_0\sqrt{\frac{1-\cos 3\pi/4}{2}}$$
$$\omega_3 = \sqrt{2}\omega_0\sqrt{1+\sqrt{2}/2} = \omega_0\sqrt{2+\sqrt{2}}$$

11.20



Generalized coordinates: $X,\ s\left(=a\theta\right)$

and

$$x = X + a\sin\theta \qquad y = a(1-\cos\theta)$$
$$\dot{x} = \dot{X} + a\dot{\theta}\cos\theta \qquad \dot{y} = a\dot{\theta}\sin\theta$$

$$T = \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$
$$= \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left[\left(\dot{X} + a\dot{\theta}\cos\theta\right)^2 + \left(a\dot{\theta}\sin\theta\right)^2\right]$$
$$V = mgy = mga\left(1-\cos\theta\right)$$

For small oscillations, in terms of the generalized coordinates X and s …

$$T \approx \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{X} + \dot{s}\right)^2 \qquad V \approx \frac{1}{2a}mgs^2$$
$$L = T - V \approx \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{X} + \dot{s}\right)^2 - \frac{1}{2a}mgs^2$$

Lagrange's equations of motion yield …

$$\ddot{X} + \ddot{s} + \frac{g}{a}s = 0 \qquad \left(M+m\right)\ddot{X} + m\ddot{s} = 0$$

Assuming …

$$X = Ae^{i\omega t} \qquad s = Be^{i\omega t}$$

we obtain the matrix equation …

$$\begin{pmatrix} \omega^2 & \omega^2 - g/a \\ \left(M+m\right)\omega^2 & m\omega^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0$$

Setting the determinant of the above matrix equal to zero yields…

$$\omega_1^2 = 0\,, \qquad \omega_2^2 = \frac{g\left(m+M\right)}{M}$$

The mode corresponding to ω_1 is …

$$\begin{pmatrix} 0 & -g/a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \qquad \text{implies that} \quad A = B = 0$$

Thus, mode 1 exhibits no oscillation! It is a pure translation with …

$\theta = 0$ and $X = A_1 t + A_2$

The mode corresponding to ω_2 is …

$$\omega_2^2 A + \left(\omega_2^2 - g/a\right)B = 0$$

or… $\frac{B}{A} = \frac{\omega_2^{\,2}}{\omega_2^{\,2} - g/a} = -\frac{m+M}{m}$

Setting $A = 1$, we have for the 2nd mode …

$X = e^{i\omega_2 t}$ and $s = a\theta = -\frac{m+M}{m} e^{i\omega_2 t}$

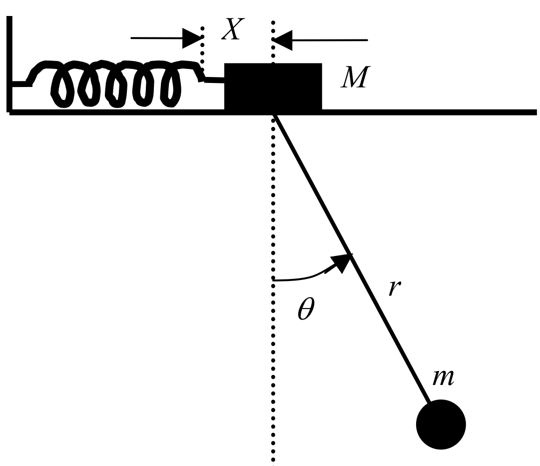
This mode corresponds to an oscillation about the CM where …

$(m+M)X = -ms\,(= -ma\theta)$

The normal mode vectors are …

$$Q_1 = \begin{pmatrix} A_1 t + A_2 \\ 0 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 1 \\ -\frac{m+M}{m} \end{pmatrix} e^{i\omega_2 t}$$

11.21



(a) We can solve for the normal modes using Equation 11.4.9 …

$$\left(\boldsymbol{K} - \omega^2 \boldsymbol{M}\right)\boldsymbol{a} = 0$$

K and ***M*** are the potential energy and kinetic energy matrices respectively. ***a*** is a two-component vector whose elements are the amplitudes of the coordinates ***q***. The kinetic energy in Example 11.3.2, assuming small displacements from equilibrium, are …

$$T \approx \frac{1}{2} m\dot{X}^2 + \frac{1}{2} m\left(\dot{X}^2 + \left(r\dot{\theta}\right)^2 + 2\dot{X}r\dot{\theta}\right)$$

$$T \approx \frac{1}{2} 2m\dot{X}^2 + \frac{1}{2} m\left(2\dot{X}r\dot{\theta}\right) + \frac{1}{2} m\left(r\dot{\theta}\right)^2$$

or, in matrix form

$$T = \frac{1}{2}\tilde{\boldsymbol{q}}\boldsymbol{M}\boldsymbol{q} \quad \text{where } \boldsymbol{q} = \begin{pmatrix} X \\ r\theta \end{pmatrix}$$

Thus, $\boldsymbol{M} = m\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

The potential energy is …

$$V \approx \frac{1}{2} k\dot{X}^2 + m\frac{g}{2r}\left(r\theta\right)^2$$

$$V \approx \frac{1}{2}\frac{k}{2m}\left[2m\dot{X}^2 + \frac{2mg}{kr} m\left(r\theta\right)^2\right] = \frac{1}{2} m\omega_0^{\,2}\left[2\dot{X}^2 + \left(r\theta\right)^2\right]$$

where $\omega_0^{\,2} = \frac{k}{M+m} = \frac{k}{2m}$

Thus $\boldsymbol{K} = m\omega_0^{\ 2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ The above matrix equation is thus …

$$\left[m\omega_0^{\ 2}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} - m\omega^2\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right]\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 2\left(\omega_0^{\ 2} - \omega^2\right) & -\omega^2 \\ -\omega^2 & \omega_0^{\ 2} - \omega^2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

From the top equation in the matrix equation above, we get …

$$2\left(\omega_0^{\ 2} - \omega^2\right)a_1 - \omega^2 a_2 = 0$$

$$\frac{a_2}{a_1} = \frac{2\left(\omega_0^{\ 2} - \omega^2\right)}{\omega^2}$$

The amplitudes for the two normal modes can be found by setting $\omega^2 = \omega_1^{\ 2}$ *or* $\omega_2^{\ 2}$.
In each case we set $a_1 = 1$, which we are free to do since only ratios of the amplitudes can be determined.

For $\omega^2 = \omega_1^{\ 2} = \left(2 - \sqrt{2}\right)\omega_0^{\ 2}$ we obtain …

$$a_2 = \frac{2\left(\omega_0^{\ 2} - \omega_1^{\ 2}\right)}{\omega_1^{\ 2}} = \frac{2\left[1 - \left(2 - \sqrt{2}\right)\right]}{\left(2 - \sqrt{2}\right)} = \sqrt{2}$$

Thus, for mode 1, the lower frequency, symmetric mode …

$$\boldsymbol{q}_1 = \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix} e^{i\omega_1 t}$$

For $\omega^2 = \omega_2^{\ 2} = \left(2 + \sqrt{2}\right)\omega_0^{\ 2}$ we obtain …

$$a_2 = \frac{2\left(\omega_0^{\ 2} - \omega_2^{\ 2}\right)}{\omega_1^{\ 2}} = \frac{2\left[1 - \left(2 + \sqrt{2}\right)\right]}{\left(2 + \sqrt{2}\right)} = -\sqrt{2}$$

Thus, for mode 2, the higher frequency, anti-symmetric mode …

$$\boldsymbol{q}_2 = \begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} e^{i\omega_2 t}$$

(b) In this case, $\frac{m}{m+M} << 1$ *or* $M >> m$. We also assume that the spring is "slack", i.e., $\frac{k}{M+m} << \frac{g}{r}$, an assumption not stated in the problem (which needs to be rectified in the next edition, I suppose)

Also, $2mg \neq kr$, so we let $\omega_0^{\ 2} = \frac{k}{M+m} \approx \frac{k}{M}$ *and* $\Omega_0^{\ 2} = \frac{g}{r}$

The kinetic energy is

$$T \approx \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(\dot{X}^2 + \left(r\dot{\theta}\right)^2 + 2\dot{X}r\dot{\theta}\right)$$

$$T \approx \frac{1}{2}(M+m)\dot{X}^2 + \frac{1}{2}m\left(\left(r\dot{\theta}\right)^2 + 2\dot{X}r\dot{\theta}\right) \approx \frac{1}{2}M\dot{X}^2 + \frac{1}{2}m\left(2\dot{X}r\dot{\theta}\right) + \frac{1}{2}m\left(r\dot{\theta}\right)^2$$

and the ***M***-matrix is

$$\boldsymbol{M} \approx \begin{pmatrix} M & m \\ m & m \end{pmatrix}$$

The potential energy is

$$V \approx \frac{1}{2}k\dot{X}^2 + \frac{1}{2}m\frac{g}{r}(r\theta)^2 = \frac{1}{2}M\omega_0^{\ 2} + \frac{1}{2}m\Omega_0^{\ 2}$$

The ***K***-matrix is thus

$$\boldsymbol{K} \approx \begin{pmatrix} \omega_0^{\ 2}M & 0 \\ 0 & \Omega_0^{\ 2}m \end{pmatrix}$$

To solve for ω^2, we set $\left|\boldsymbol{K} - \omega^2\boldsymbol{M}\right| = 0$

$$\begin{vmatrix} \left(\omega_0^{\ 2} - \omega^2\right)M & -\omega^2 m \\ -\omega^2 m & \left(\Omega_0^{\ 2} - \omega^2\right)m \end{vmatrix} = 0$$

which yields …

$$\left[\omega_0^{\ 2}\Omega_0^{\ 2} + \omega^4 - \omega^2\Omega_0^{\ 2} - \omega^2\omega_0^{\ 2}\right]mM - \omega^4 m^2 = 0$$

$$\omega^4(M-m) - \omega^2\left(\omega_0^{\ 2} + \Omega_0^{\ 2}\right)M + \omega_0^{\ 2}\Omega_0^{\ 2}M = 0$$

Neglecting m with respect to M and simplifying yields …

$$\omega^4 - \omega^2\left(\omega_0^{\ 2} + \Omega_0^{\ 2}\right) + \omega_0^{\ 2}\Omega_0^{\ 2} = 0$$

Solving for ω^2 …

$$\omega^2 = \frac{\left(\omega_0^{\ 2} + \Omega_0^{\ 2}\right)}{2} \pm \frac{\sqrt{\left(\omega_0^{\ 2} + \Omega_0^{\ 2}\right)^2 - 4\omega_0^{\ 2}\Omega_0^{\ 2}}}{2}$$

$$\omega^2 = \frac{\left(\omega_0^{\ 2} + \Omega_0^{\ 2}\right)}{2} \pm \frac{\left(\omega_0^{\ 2} - \Omega_0^{\ 2}\right)}{2}$$ Thus, we get …

$$\omega_1^{\ 2} = \omega_0^{\ 2} \quad and \quad \omega_2^{\ 2} = \Omega_0^{\ 2}$$

Now, we solve for the amplitudes ***a*** of the normal mode vectors …

$$\left(\boldsymbol{K} - \omega^2\boldsymbol{M}\right)\boldsymbol{a} = 0$$

$$\begin{pmatrix} \left(\omega_0^{\ 2} - \omega^2\right)M & -\omega^2 m \\ -\omega^2 m & \left(\Omega_0^{\ 2} - \omega^2\right)m \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = 0$$

Using the first of the above matrix equations for $\omega^2 = \omega_1^{\ 2} = \omega_0^{\ 2}$ gives …

$$a_2 = 0 \quad and \quad a_1 = 1$$

Using the second equation for $\omega^2 = \omega_2^{\ 2} = \Omega_0^{\ 2}$ gives …

$a_1 = 0 \;\; and \;\; a_2 = 1$

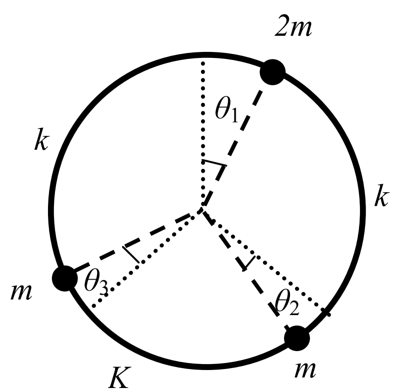
Thus, the normal modes are approximately …

$$\boldsymbol{Q}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i\omega_0 t} \quad and \quad \boldsymbol{Q}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\Omega_0 t}$$

Note that we could have guessed this almost immediately. The above assumption is tantamount to omitting the cross elements $-\omega^2 m$. This completely eliminates the small coupling between the two oscillators, which reduces the matrix $\boldsymbol{K} - \omega^2 \boldsymbol{M}$ to the purely diagonal terms …

$$\begin{pmatrix} \left(\omega_0^{\,2} - \omega^2\right)M & 0 \\ 0 & \left(\Omega_0^{\,2} - \omega^2\right)m \end{pmatrix}$$, which leads directly to the above solution.

22.



We "scale" the force constants and masses to 1 unit, namely,

$m = 1 \;\; and \;\; k = 1$.

Let $k' = K \;\; and \;\; l = 1$

such that *3l* = circumference

So: $T = \frac{1}{2}\left(2\dot{\theta}_1^{\,2} + \dot{\theta}_2^{\,2} + \dot{\theta}_3^{\,2}\right)$

$$V = \frac{1}{2}(\theta_1 - \theta_2)^2 + \frac{1}{2}(\theta_2 - \theta_1)^2 + \frac{1}{2}(\theta_1 - \theta_3)^2 + \frac{1}{2}(\theta_3 - \theta_1)^2 + \frac{1}{2}K(\theta_2 - \theta_3)^2 + \frac{1}{2}K(\theta_3 - \theta_2)^2$$

Collecting terms …

$$V = \frac{1}{2}\left[4\theta_1^2 + 2(1+K)\theta_2^{\,2} + 2(1+K)\theta_3^{\,2} - 4\theta_1\theta_2 - 4\theta_1\theta_3 - 4K\theta_2\theta_3\right]$$

K **- Matrix**

$$\boldsymbol{K} = \begin{pmatrix} 4 & -2 & -2 \\ -2 & 2+2K & -2K \\ -2 & -2K & 2+2K \end{pmatrix}$$

M **- Matrix**

$$\boldsymbol{M} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The ttansformation matrix A that diagonalizes these matrices is made up of the three eigenvectors $\boldsymbol{Q}_i$ whose amplitudes are $\boldsymbol{a}_i$ … we guess that …

1. Uniform rotation

$$\theta_1 = \theta_2 = \theta_3 \;\; or \;\; \boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

2. Anti-symmetric oscillation of 2&3, while 1 remains fixed

$$\theta_1 = 0;\ \theta_3 = -\theta_2 \ \ or \ \ \boldsymbol{a}_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

3. Anti-symmetric oscillation of 1&2 together with respect to 3

$$\theta_1 = 1;\ \theta_3 = \theta_2 \ \ or \ \ \boldsymbol{a}_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$$

Thus, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$ *and* $\tilde{\boldsymbol{A}} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix}$

We can now diagonalize $\boldsymbol{K}$and $\boldsymbol{M}$ …

$$\tilde{\boldsymbol{A}}\boldsymbol{K}\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 2+2K & -2K \\ -2 & -2K & 2+2K \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{pmatrix} = \boldsymbol{K}_{diag}$$

$$\boldsymbol{K}_{diag} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4+8K & \\ 0 & & 16 \end{pmatrix} \ \ Likewise \ \ \boldsymbol{M}_{diag} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

The eigenfrequencies are $\omega_i^2 = \left[\boldsymbol{K}_{diag} / \boldsymbol{M}_{diag}\right]_i$

$\omega_2^2 = 0;\ \omega_2^2 = 2 + 4K;\ \omega_3^2 = 4$

The general solution can be generated from the following table …

	$\underline{Q_1}$	$\underline{Q_2}$	$\underline{Q_3}$
θ_1	a_1	0	a_3
θ_2	a_1	a_2	$-a_3$
θ_3	a_1	$-a_2$	$-a_3$

where $\boldsymbol{Q}_1 = a_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cos\left(\omega_1 t - \delta_1\right)$, $\boldsymbol{Q}_2 = a_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \cos\left(\omega_2 t - \delta_2\right)$, $\boldsymbol{Q}_3 = a_3 \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cos\left(\omega_3 t - \delta_3\right)$

Thus,
$$\theta_1 = a_1 \cos\left(\omega_2 t - \delta_2\right) \qquad\qquad + a_3 \cos\left(\omega_3 t - \delta_3\right)$$
$$\theta_1 = a_1 \cos\left(\omega_2 t - \delta_2\right) + a_2 \cos\left(\omega_2 t - \delta_2\right) - a_3 \cos\left(\omega_3 t - \delta_3\right)$$
$$\theta_3 = a_1 \cos\left(\omega_2 t - \delta_2\right) - a_2 \cos\left(\omega_2 t - \delta_2\right) - a_3 \cos\left(\omega_3 t - \delta_3\right)$$

Initial conditions are: $\theta_1 = \theta_0 = 10^o$; $\theta_2 = \theta_3 = 0$; $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = 0$

The conditions generate 6 equations with 6 unknowns and solving gives …

$$\theta_1 = \frac{\theta_0}{2}\cos\omega_1 t + \frac{\theta_0}{2}\cos\omega_3 t$$

$$\theta_2 = \frac{\theta_0}{2}\cos\omega_1 t - \frac{\theta_0}{2}\cos\omega_3 t$$

$$\theta_2 = \frac{\theta_0}{2}\cos\omega_1 t - \frac{\theta_0}{2}\cos\omega_3 t$$

11.23 See Ex. 11.4.1, page 497-498

The amplitudes of the eigenvectors are …

$$\boldsymbol{a}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \boldsymbol{a}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \quad \boldsymbol{a}_3 = \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$$

<u>***K*** - Matrix</u> <u>***M*** - Matrix</u>

$$\boldsymbol{K} = \begin{pmatrix} K & \text{-}K & 0 \\ \text{-}K & 2K & \text{-}K \\ 0 & \text{-}K & K \end{pmatrix} \qquad \boldsymbol{M} = \begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}$$

$$\boldsymbol{K}_{diag} = \tilde{\boldsymbol{A}}\boldsymbol{K}\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2m/M & 1 \end{pmatrix}\begin{pmatrix} K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & K \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2m/M \\ 1 & -1 & 1 \end{pmatrix}$$

$$\boldsymbol{K}_{diag} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2K & 0 \\ 0 & 2K + 8K\,m/M & 8K\,m^2/M^2 \end{pmatrix}$$

$$\boldsymbol{M}_{diag} = \tilde{\boldsymbol{A}}\boldsymbol{M}\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -2m/M & 1 \end{pmatrix}\begin{pmatrix} m & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & m \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -2m/M \\ 1 & -1 & 1 \end{pmatrix}$$

$$\boldsymbol{M}_{diag} = \begin{pmatrix} 2m + M & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 2m + 4m^2/M \end{pmatrix}$$

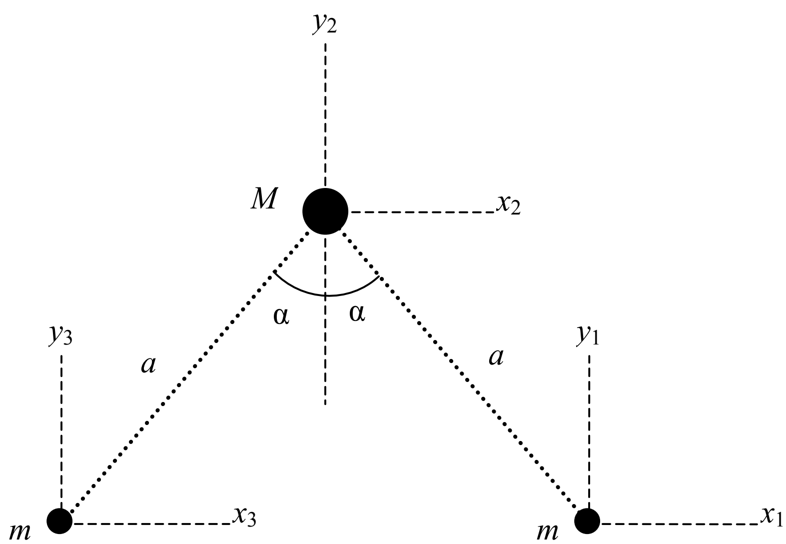
So, we have …

$\omega_1^{\,2} = 0$, $\omega_2^{\,2} = \dfrac{2K}{2m} = \dfrac{K}{m}$, and …

$$\omega_3^{\,2} = \frac{2K + 8K\,m/M + 8K\,m^2/M^2}{2m + 4\,m^2/M} = \frac{K\left(1 + 4m/M + 4m^2/M^2\right)}{m\left(1 + 2m/M\right)}$$

$$\omega_3^{\ 2} = \frac{K}{m}\frac{\left(1+2\,m/M\right)^2}{\left(1+2\,m/M\right)} = \frac{K}{m}\left(1+2\,m/M\right)$$

11.24



Select coordinates (x_i , y_i) as shown, then …

$$T = \frac{1}{2}m\left(\dot{x}_1^{\ 2} + \dot{y}_1^{\ 2} + \dot{x}_3^{\ 2} + \dot{y}_3^{\ 2}\right) + \frac{1}{2}M\left(\dot{x}_2^{\ 2} + \dot{y}_2^{\ 2}\right)$$

The potential energy depends only on the compression (or stretching) of the two springs connecting each m to M (hydrogen to sulfur). Let δa_1 and δa_2 be incremental changes in the distances a_1 or $HS(1 \to 2)$ and a_2 or $HS(3 \to 2)$. We have …

$$\delta a_1 = \left(x_1 - x_2\right)\sin\alpha + \left(y_2 - y_1\right)\cos\alpha = \frac{1}{\sqrt{2}}\left(x_1 - x_2 + y_2 - y_1\right)$$

$$\delta a_2 = \left(x_2 - x_3\right)\sin\alpha + \left(y_2 - y_3\right)\cos\alpha = \frac{1}{\sqrt{2}}\left(x_2 - x_3 + y_2 - y_3\right)$$

$$V = \frac{1}{2}k\left[\left(\delta a_1\right)^2 + \left(\delta a_2\right)^2\right]$$

We can reduce the degrees of freedom from 6 to 3 by ignoring the two translational modes and the rotational mode. Thus we consider only vibrational modes. The coordinates must obey the following constraints …

No center of mass motion:

$$m\left(y_1 + y_3\right) + My_2 = 0 \quad and \quad m\left(x_1 + x_3\right) + Mx_2 = 0$$

No angular momentum about any point. We choose that point to be the sulfur atom (M)…

$$my_3 a\sin\alpha - my_1 a\sin\alpha - mx_3 a\sin\alpha - mx_1 a\sin\alpha = 0$$

$$y_3 - y_1 - x_3 - x_1 = 0$$

We introduce three generalized coordinates Q, q_1 and q_2 that should be close to what we guess would be normal modes …

$$Q = x_1 + x_3\,;\quad q_1 = x_1 - x_3\,;\quad q_2 = y_1 + y_3$$

Solve for x_i , y_i in terms of Q, q_1 and q_2 using the above 3 equations of constraint …

$$x_1 = \frac{1}{2}(Q + q_1);\quad x_2 = -\frac{m}{M}Q;\quad = x_1 - x_3\,;\quad x_3 = \frac{1}{2}(Q - q_1)$$

$$y_1 = \frac{1}{2}(q_2 - Q)\,;\quad y_2 = -\frac{m}{M}q_2\,;\quad y_3 = \frac{1}{2}(Q + q_2)$$

Thus, the kinetic energy, in terms of these generalized coordinates, is …

$$T = \frac{1}{2}m\left(1 + \frac{m}{M}\right)\dot{Q}^2 + \frac{1}{4}m\dot{q}_1^{\,2} + \frac{1}{4}m\frac{\mu}{M}\dot{q}_2^{\,2}$$

where $\mu = M + 2m$ is the mass of the H_2S molecule

The potential energy is …

$$V = \frac{1}{2}k\left(1 + \frac{m}{M}\right)^2 Q^2 + \frac{1}{8}kq_1^{\,2} + \frac{1}{8}k\frac{\mu^2}{M^2}q_2^{\,2} - \frac{1}{4}k\frac{\mu}{M}q_1 q_2$$

Note, that in the Lagrangian $L = T - V$, the only cross term is one involving $q_1 q_2$. therefore, Q is a normal mode with eigenfrequency given by the ratio …

$$\omega_Q^{\,2} = K_Q / M_Q = \frac{k}{m}\left(1 + \frac{m}{M}\right)$$

Constructing the residual 2x2 ***K*** and ***M*** matrices involving only q_1 and q_2 terms, which we will call ***Kq*** and ***Mq*** gives …

$$\boldsymbol{Kq} = \frac{1}{4}\begin{pmatrix} 1 & -\mu/M \\ -\mu/M & \mu^2/M^2 \end{pmatrix} \quad and \quad \boldsymbol{Mq} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & \mu/M \end{pmatrix}$$

We have omitted the factors k and m, which we'll replace in the final solution, remembering that the eigenfrequencies that we find as a solution to $\left|\boldsymbol{Kq} - \omega^2 \boldsymbol{Mq}\right| = 0$ will be multiples of k / m.

$$\left|\boldsymbol{Kq} - \omega^2 \boldsymbol{Mq}\right| = \frac{1}{4}\begin{vmatrix} 1 - 2\omega^2 & -\mu/M \\ -\mu/M & \mu/M\left(\mu/M - 2\omega^2\right) \end{vmatrix} = 0$$

which reduces to …

$2\omega^2\left[2\omega^2 - (1 + \mu/M)\right] = 0$ which has the non-trivial solution …

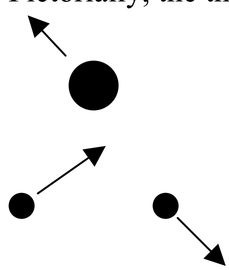
$\omega_{1,2}^{\;2} = \frac{1}{2}(1 + \mu/M)\frac{k}{m}$ in which, we have put back the factor k / m. These two modes are "degenerate."

Plugging ω^2 back into the matrix equation $\left(\boldsymbol{Kq} - \omega^2 \boldsymbol{Mq}\right)\boldsymbol{q} = 0$ gives …

$q_1 = -q_2 \quad or \quad -x_1 + x_3 = y_1 + y_3$

which can be satisfied in a variety of ways, for example, with $x_1 = x_3 \quad and \quad y_1 = -y_3$, etc

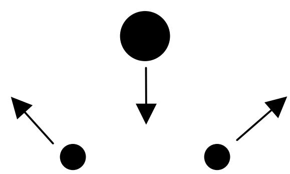
Pictorially, the three normal modes are …



Q-mode: ant-symmetric about y-axis:

$x_1 = x_3 \quad and \quad y_1 = -y_3$

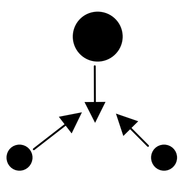
$$\omega_Q^{\ 2} = \frac{k}{m}\left(1 + \frac{m}{M}\right)$$



Breathing mode: symmetric about y-axis:

$x_1 = -x_3 \quad and \quad y_1 = y_3$

$$\omega_1^{\ 2} = \frac{k}{2m}\left(1 + \mu/M\right)$$



Stretching mode: symmetric about y-axis:

$x_1 = -x_3 \quad and \quad y_1 = y_3$

$$\omega_2^{\ 2} = \frac{k}{2m}\left(1 + \mu/M\right)$$

11.25

(a) Plug each of these functions into the wave equation and it is satisfied!

(b)
$$Q = q_1 + q_2 = e^{i\omega t}\, e^{-ikx} + e^{i(\omega+\Delta\omega)t}\, e^{-i(k+\Delta k)x}$$
$$= e^{i(\omega+\Delta\omega/2)t}\, e^{-i(k+\Delta k/2)x}\left[e^{-i[(\Delta\omega)t-(\Delta k)x]/2} + e^{+i[(\Delta\omega)t-(\Delta k)x]/2}\right]$$

The real part of the above is …

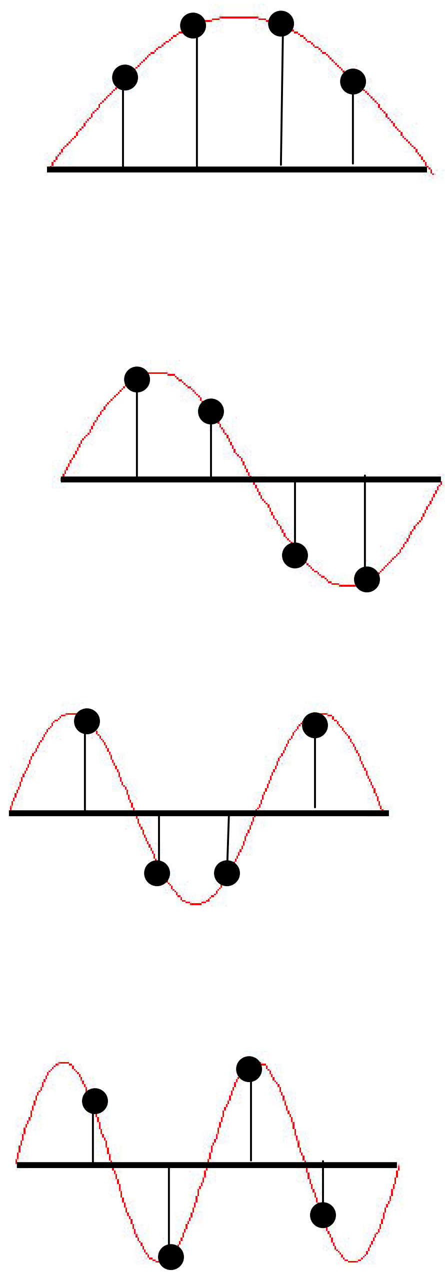
$$Q = 2\cos\left[\frac{(\Delta\omega)t - (\Delta k)x}{2}\right]\cos\left[\left(\omega + \frac{\Delta\omega}{2}\right)t - \left(k + \frac{\Delta k}{2}\right)x\right]$$
$$\approx 2\cos\left[\frac{(\Delta\omega)t - (\Delta k)x}{2}\right]\cos\left(\omega t - kx\right)$$

(c) The group speed is …

$$u_g = \frac{dx}{dt}$$ (the phase of the amplitude remains the same)

$$u_g = \frac{\Delta\omega}{\Delta k}$$

11.26



From Equation 11.5.17 ...

$$\frac{\omega_N}{\omega_1} = \frac{\sin\frac{N\pi}{10}}{\sin\frac{\pi}{10}}, \quad \frac{\omega_2}{\omega_1} = \frac{\sin\frac{2\pi}{10}}{\sin\frac{\pi}{10}} = 1.28, \quad \frac{\omega_3}{\omega_1} = \frac{\sin\frac{3\pi}{10}}{\sin\frac{\pi}{10}} = 2.62, \quad \frac{\omega_4}{\omega_1} = \frac{\sin\frac{4\pi}{10}}{\sin\frac{\pi}{10}} = 3.08$$

11.27

$F = k\,\Delta l$ is the tension in the cord

$d = \dfrac{l+\Delta l}{n+1}$ is its stretched length

From the equation following Equation 11.5.7…

$$K = \frac{F}{d}$$

From Equation 11.6.4c …

$$v^2 = \frac{k}{m}\left(l+\Delta l\right)\Delta l$$

$$v_{trans} = \left(\frac{k}{m}\right)^{\frac{1}{2}}\left[\left(l+\Delta l\right)\Delta l\right]^{\frac{1}{2}} \quad \text{for transverse waves}$$

For longitudinal vibrations, we use Y, the tension in the cord per unit stretched length

$$Y = \frac{k\,\Delta l}{\Delta l/\left(l+\Delta l\right)} = k\left(l+\Delta l\right)$$

$$K = \frac{Y}{d} \qquad \text{(Equation 11.6.8)}$$

$$v^2 = \frac{kd^2}{m/n} = \frac{k\left(l+\Delta l\right)nd}{m} \qquad \text{(Equation 11.6.9a)}$$

$$nd \approx \left(n+1\right)d = l+\Delta l$$

$$v_{long} = \left(\frac{k}{m}\right)^{\frac{1}{2}}\left(l+\Delta l\right)$$

11.28

From Equation 11.6.7b …

$$v = \left(\frac{F}{\mu}\right)^{\frac{1}{2}}$$

As in problem 11.27, $F = k\,\Delta l$

$$v_{trans} = \left(\frac{k\,\Delta l}{\mu}\right)^{\frac{1}{2}}$$

From Equation 11.6.9b …

$v = \left(\dfrac{Y}{\mu}\right)^{\frac{1}{2}}$ and from problem 11.27 … $Y = k\left(l+\Delta l\right)$ so … $v_{long} = \left(\dfrac{k\left(l+\Delta l\right)}{\mu}\right)^{\frac{1}{2}}$

11.29

The general solution to the wave equation (Equation 11.6.10) that yields a standing wave of any arbitrary shape can be obtained as a linear combination of standing sine waves of a form given by Equation 11.6.14, i.e.,

$$y(x,t)=\sum_{n=1}^{\infty}\left(A_n\sin\omega_n t+B_n\cos\omega_n t\right)\sin\left(\frac{2\pi x}{\lambda_n}\right)$$

where $\omega_n=\dfrac{2\pi}{T_n}$

since the speed of a wave is …

$$v=\frac{\lambda_n}{T_n}=\frac{\omega_n\lambda_n}{2\pi}=\left(\frac{F_0}{\mu}\right)^{\frac{1}{2}}$$

we have …

$$\omega_n=\left(\frac{F_0}{\mu}\right)^{\frac{1}{2}}\frac{2\pi}{\lambda_n}$$

But the wavelength of the standing wave is constrained by the fixed endpoints of the string, i.e. …

$\lambda_n=\dfrac{2l}{n}$, so …

$$\omega_n=\left(\frac{F_0}{\mu}\right)^{\frac{1}{2}}\frac{n\pi}{l}$$

Now, at t = 0, the wave starts from rest in the configuration specified, so…

$$y(x,0)=\sum B_n\sin\frac{n\pi x}{l}$$

From the discussion of Fourier analysis in Appendix G or in Section 3.9, the Fourier coefficients are given by …

$$B_n=\frac{2}{l}\int_0^l y(x,0)\sin\frac{n\pi x}{l}dx$$

Since the string starts from rest, we have …

$$\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0}=\sum\omega_n A_n\sin\frac{n\pi x}{l}\equiv 0$$

Therefore, $A_n=0$

The initial configuration is shown in the Figure P11.29. Thus …

$$y(x,0)=\frac{2a}{l}x\qquad 0<x<\frac{l}{2}$$

$$y(x,0) = \frac{2a}{l}(l-x) \qquad \frac{l}{2} < x < l$$

We can now determine B_n …

$$B_n = \frac{2}{l}\left[\frac{2a}{l}\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \frac{2a}{l}\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx\right]$$

Using integration tables, we obtain the result …

$$B_n = \left[(-1)^{\frac{n-1}{2}}\right]\frac{8a}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad n = 1,3,5,...$$

$$B_n = 0 \qquad n = 2,4,6,...$$

Thus, the general solution is …

$$y(x,t) = \frac{8a}{\pi^2}\left(\cos\frac{\pi vt}{l}\sin\frac{\pi x}{l} - \frac{1}{9}\cos\frac{3\pi vt}{l}\sin\frac{3\pi x}{l} + \frac{1}{25}\cos\frac{5\pi vt}{l}\sin\frac{5\pi x}{l} - ...\right)$$

None of the harmonics which have a node at the midpoint have been stimulated … only the odd harmonics have been excited.

11.30

By analogy with the generation of the traveling sine wave of Equation 11.6.14 from Equation 11.6.13, we get …

$$y(x,t) = \frac{4a}{\pi^2}\left[\left(\sin\frac{\pi(x+vt)}{l} + \sin\frac{\pi(x-vt)}{l}\right) - \frac{1}{9}\left(\sin\frac{3\pi(x+vt)}{l} + \sin\frac{3\pi(x-vt)}{l}\right) + \frac{1}{25}\left(\sin\frac{5\pi(x+vt)}{l} + \sin\frac{5\pi(x-vt)}{l}\right)...\right]$$
