بسم الله الرحمن الرحیم

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منابع و مراجع :

۱) دینامیک سازه ها (تالیف آقای چوپرا) (طاحونی)

۲) دینامیک سازه ها (تالیف آقای کلاف)

۳) دینامیک سازه ها (تالیف آقای ماریو پاز)

۴) دینامیک سازه ها (تالیف دکتر خسرو برگی)

۵) مرجع طرح سازه ها در برابر زلزله (تالیف فرزاد نعیم)

۶) طرح لرزه ای سازه ها (تالیف دکتر رضائی پژند)

۷) مهندسی زلزله (تالیف آقای دکتر خسرو برگی)

۸) مهندسی زلزله (تالیف آقای دکتر حجت ا... عادلی)

۹) مهندسی زلزله (تالیف آقای دکتر مقدم)

PART I

Single-Degree-of-Freedom Systems

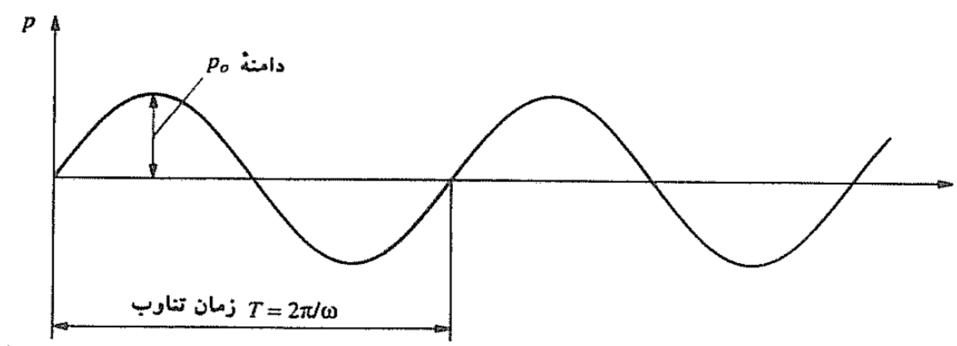
1

Equations of Motion, Problem Statement, and Solution Methods

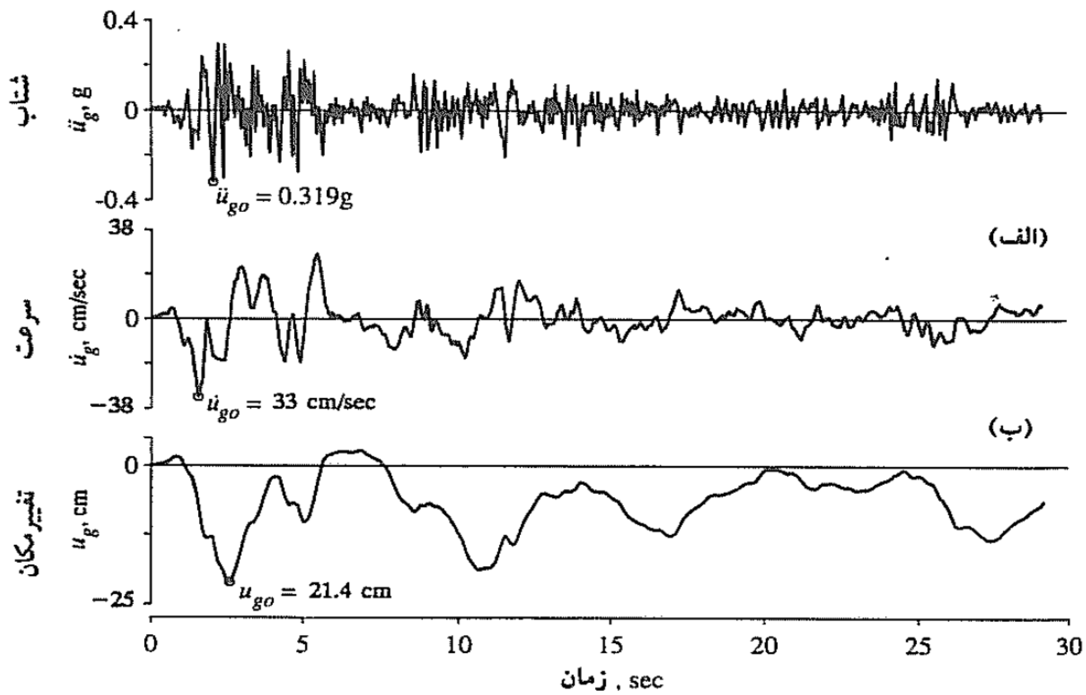
ضرورت ارائه درس دینامیک سازه ها

در رشته مهندسی عمران اکثر سازه ها (ابنیه) تحت اثر نیروهای دینامیکی هستند.
نیروهایی که مقدار (شدت)، جهت و احتمالاً نقطه اثر آنها با زمان تغییر می کنند
و البته نرخ تغییرات فوق به حدی است که پدیده ارتعاش که مشخصه اصلی رفتار دینامیکی
است در سازه بوجود می آید.

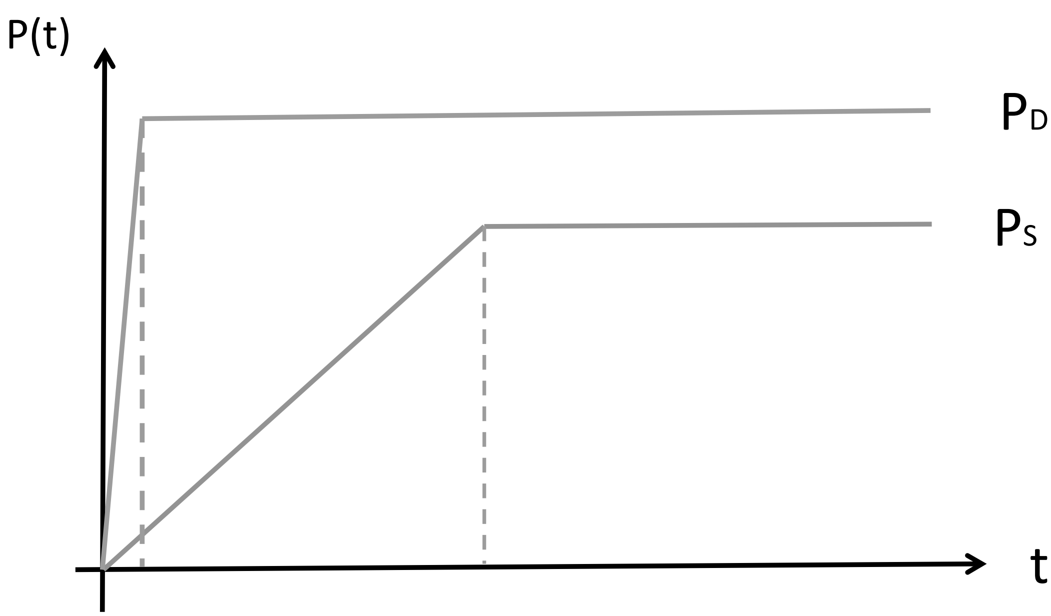
$P(t)$



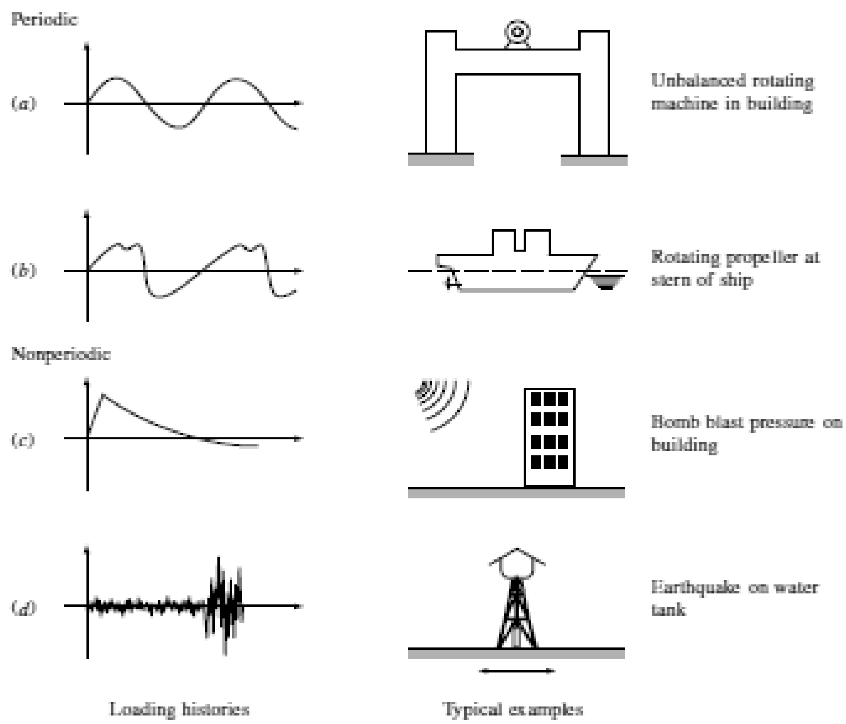
$$p(t) = p_o \sin \omega t$$



0.4
0
-0.4
شتاب
$\ddot{u}_g$, g
$\ddot{u}_{go} = 0.319$g
(الف)
38
0
−38
سرعت
$\dot{u}_g$, cm/sec
$\dot{u}_{go} = 33$ cm/sec
(ب)
0
−25
تغییرمکان
u_g, cm
$u_{go} = 21.4$ cm
0
5
10
15
20
25
30
زمان , sec



P(t)
P_D
P_S
t



انواع سدها - زلزله، هیدرودینامیک (پدیده اندرکنش سازه - خاک - آب)

ارتعاش تجهیزات و ماشین های نیروگاه

انواع پل ها - زلزله، ترافیک، ترمز، باد، ضربه، جریان رودخانه

سیلوها - ویدانژ (تخلیه سریع مواد ذخیره شده)، زلزله، حرکت تسمه و

برج آب - زلزله، هیدرودینامیک، باد

اسکله و موج‌شکن‌ها - امواج دریا، زلزله، برخورد کشتی، جریان های دریایی،

دکل و دودکش و برج های خنک کننده - زلزله، باد

استحکامات و پناهگاه ها - انفجار (دور - نزدیک - مجاور - برخورد مستقیم)، زلزله

تأسیسات هسته ای - انفجار هسته ای، زلزله، برخورد هواپیما

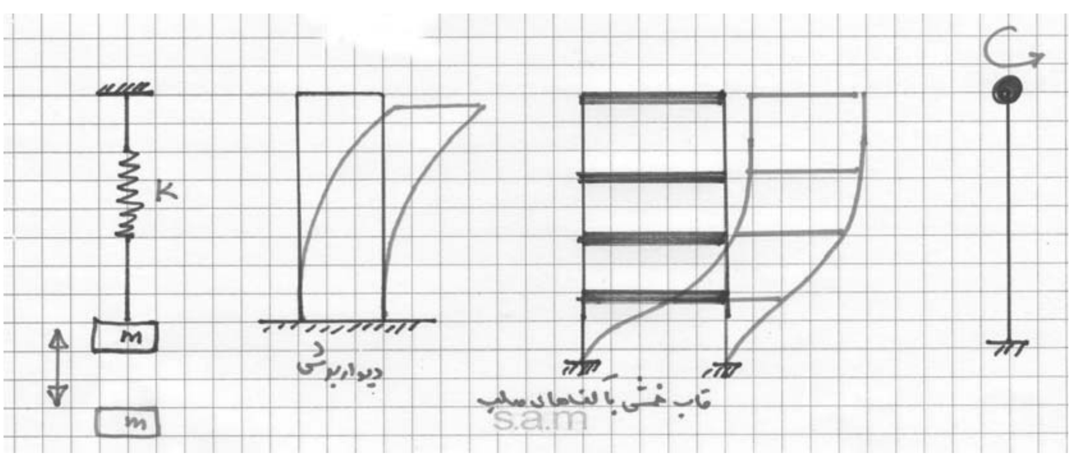
ورزشگاه ها - تشویق تماشاچیان، زلزله

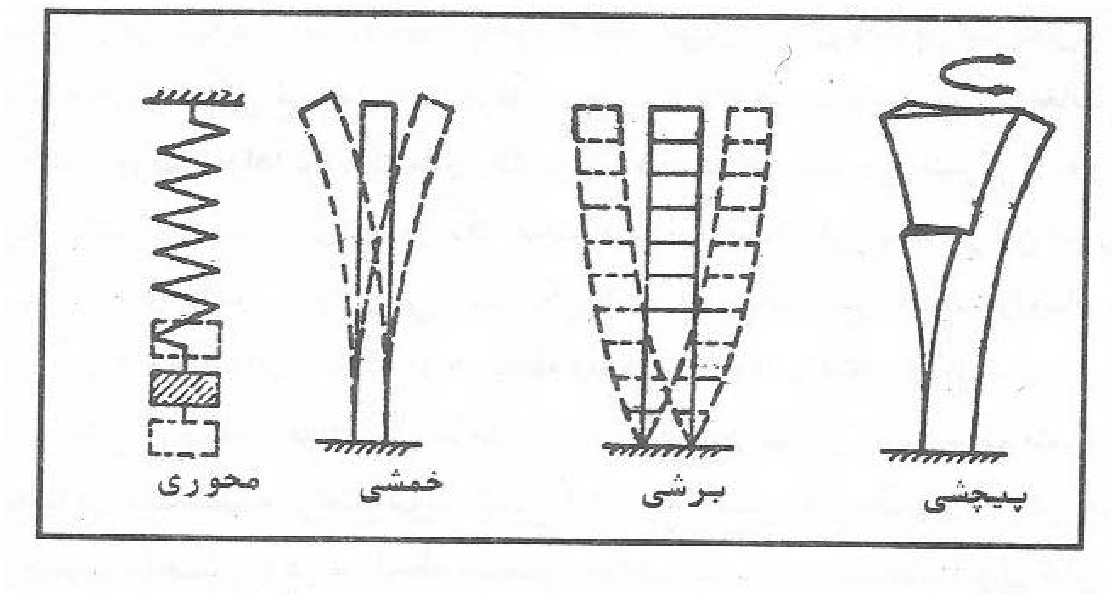
لوله ها - زلزله، عبور سیال

برج های ساختمانی - زلزله، باد

تونل ها - عبور ترافیک، قطار، زلزله

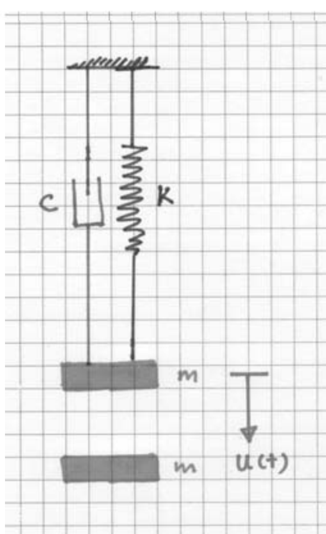
انواع ارتعاشات : هر سازه ای به هر دلیلی مرتعش شود به یکی از چهار فرم زیر ارتعاش خواهد کر د



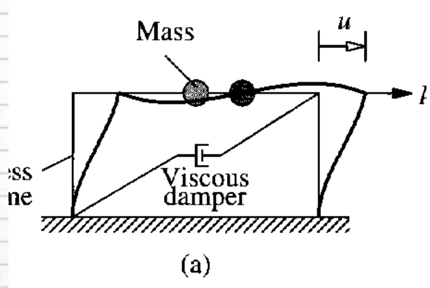


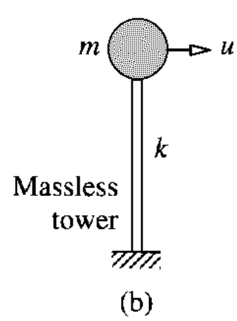
تعریف درجه آزادی دینامیکی :

تعداد مولفه های تغییر مکان مستقل مورد نیاز جهت مشخص کردن موقعیت جرم در حال ارتعاش در هر لحظه از زمان را درجـــه آزادی دینامیکی گویند .



سیستم یک درجه آزادی SDF

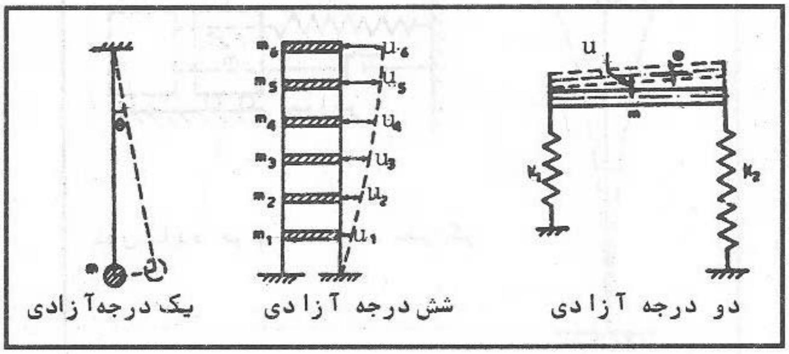




سیستم یک درجه آزادی SDF

c : ضریب میرایی

k : سختی خمشی جانبی قاب



شکل ۹۲ ـ درجات آزادی سیستمهای مختلف

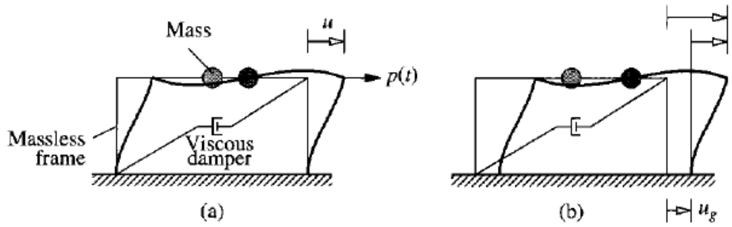
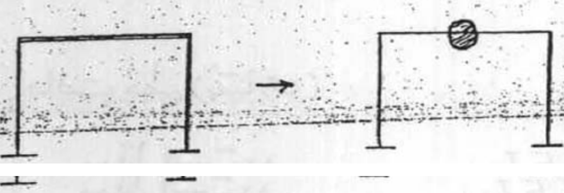


Figure 1.2.1 Single-degree-of-freedom system: (a) applied force $p(t)$; (b) earthquake-induced ground motion.

درجه آزادی و روش های کاهش آن متناسب با امکانات در دسترس

الف ـ روش تمرکز جرم Lumped-mass Procedure

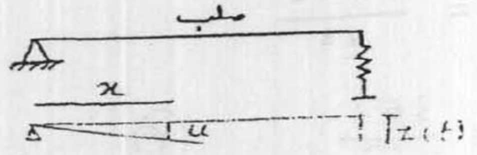


ب ـ روش تغییرمکان های تعمیم داده شده Generalized Displacement

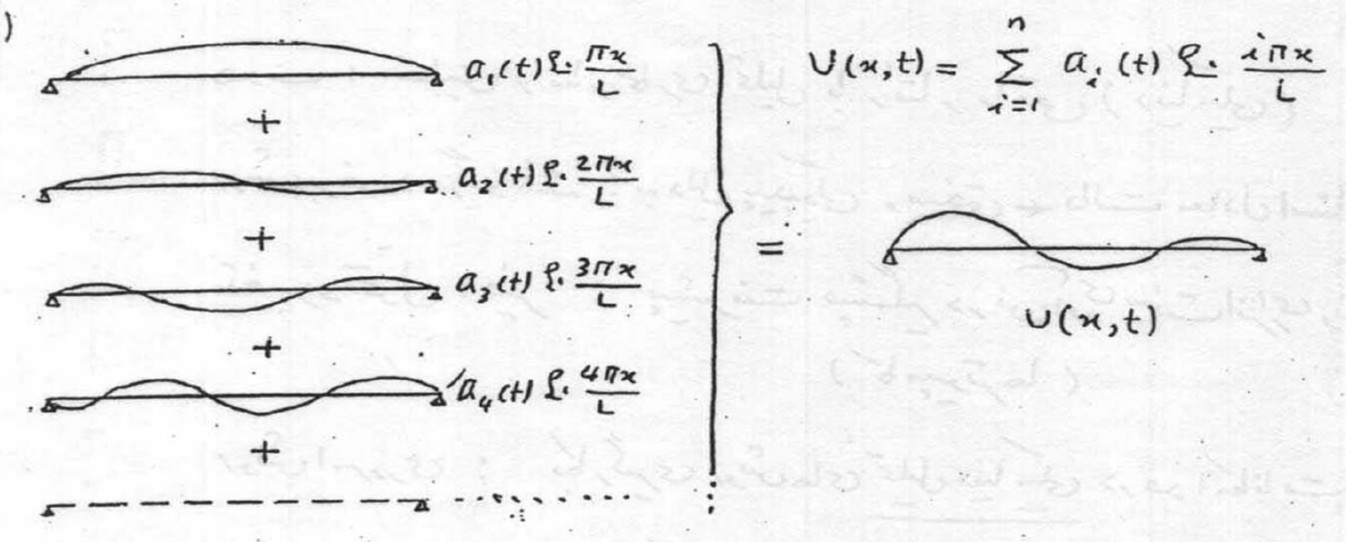
$$U(x,t) = \sum_{i=1}^{n} a_i(t)\ \phi_i(x)$$

$a_i(t)$ تابع زمانی

$\phi_i(x)$ تابع مکانی (شرایط حدی و سازگاری)



تشابه مثلث ← $u(x,t) = \frac{x}{l} \cdot z(t)$



ج - روش اجزاء محدود :Finite Element Concept

- مبانی خاص خود در تمرکز خواص رفتاری سازه در تعداد محدود گره (درجه آزادی)

د - روش اجزاء مرزی

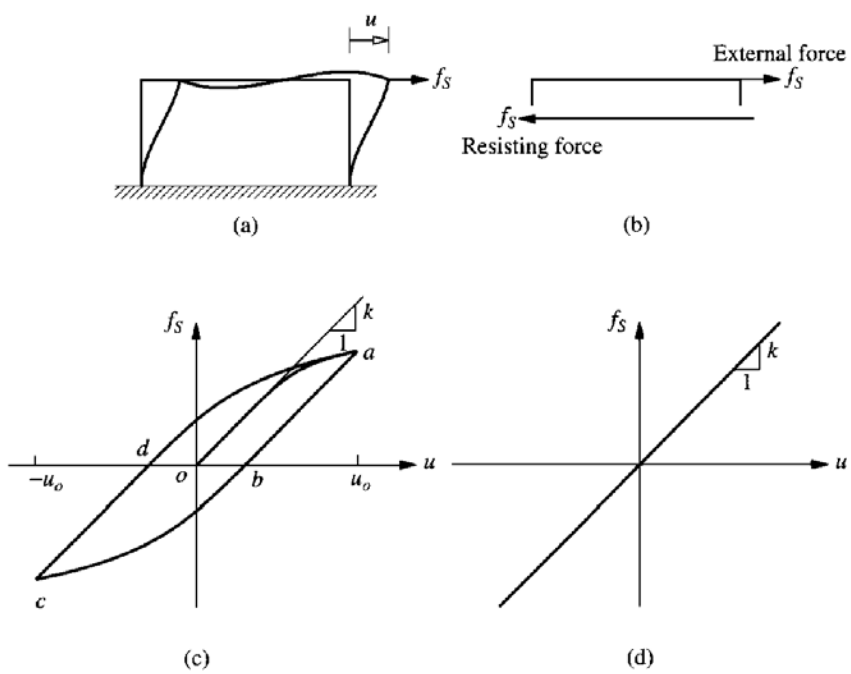
- تمرکز خواص رفتاری محیط مورد نظر در نقاطی در مرز با محیط (سازه) دیگر

نیروهای مؤثر در رفتار دینامیکی

الف - نیروی سختی (فنرها، رکابی، الاستیک...) f_S

ب - نیروی میرایی (استهلاک) f_D

ج - نیروی اینرسی (لختی) f_I



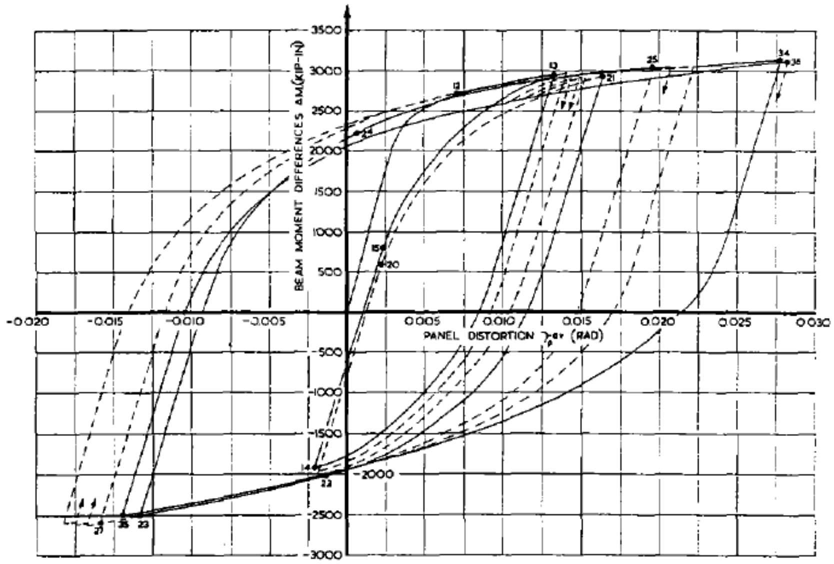
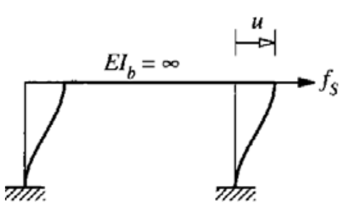
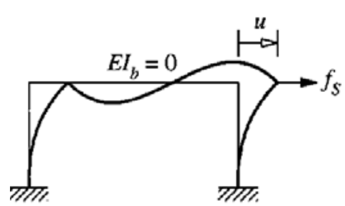


Figure 1.3.4 Force–deformation relation for a structural steel component. (From H. Krawinkler, V. V. Bertero, and E. P. Popov, "Inelastic Behavior of Steel Beam-to-Column Subassemblages," *Report No. EERC 71-7*, University of California, Berkeley, Calif., 1971.)

$$fs = ku$$





$$k = \sum_{columns} \frac{12EI_c}{h^3} = 24\frac{EI_c}{h^3}$$

$$k = \sum_{columns} \frac{3EI_c}{h^3} = 6\frac{EI_c}{h^3}$$

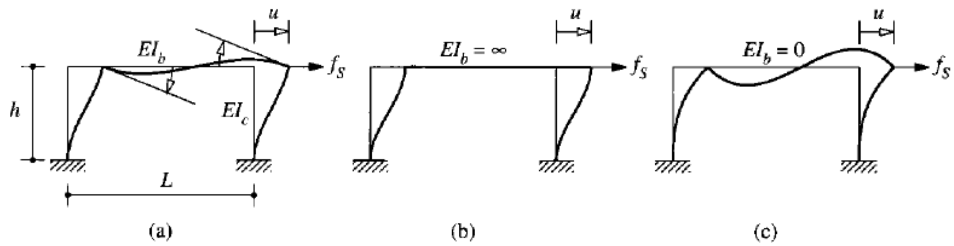
$$fs = fs(u, \dot{u})$$

یاد آوری :

$$\frac{d^2}{dt^2} = \bullet\bullet \qquad \frac{d}{dt} = \bullet$$

مثال :

$$\ddot{u} = \frac{d^2u}{dt^2} \qquad \dot{u} = \frac{du}{dt}$$



The lateral stiffness of the frame can be computed similarly for any values of I_b and I_c using the frame stiffness coefficients developed in Appendix 1. If shear deformations in elements are neglected, the result can be written in the form

$$k = \frac{24EI_c}{h^3}\frac{12\rho + 1}{12\rho + 4} \tag{1.3.5}$$

where $\rho = I_b/4I_c$ is the *beam-to-column stiffness ratio* [see Eq. (18.1.1)]. For $\rho = 0, \infty$, and $\frac{1}{4}$, Eq. (1.3.5) reduces to the results of Eqs. (1.3.3), (1.3.2), and (1.3.4), respectively. The lateral stiffness is plotted as a function of ρ in Fig. 1.3.3; it increases by a factor of 4 as ρ increases from zero to infinity.

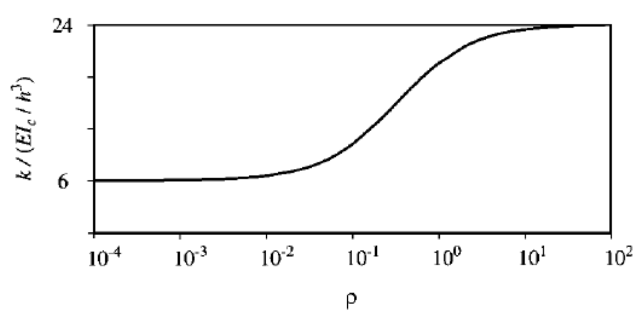
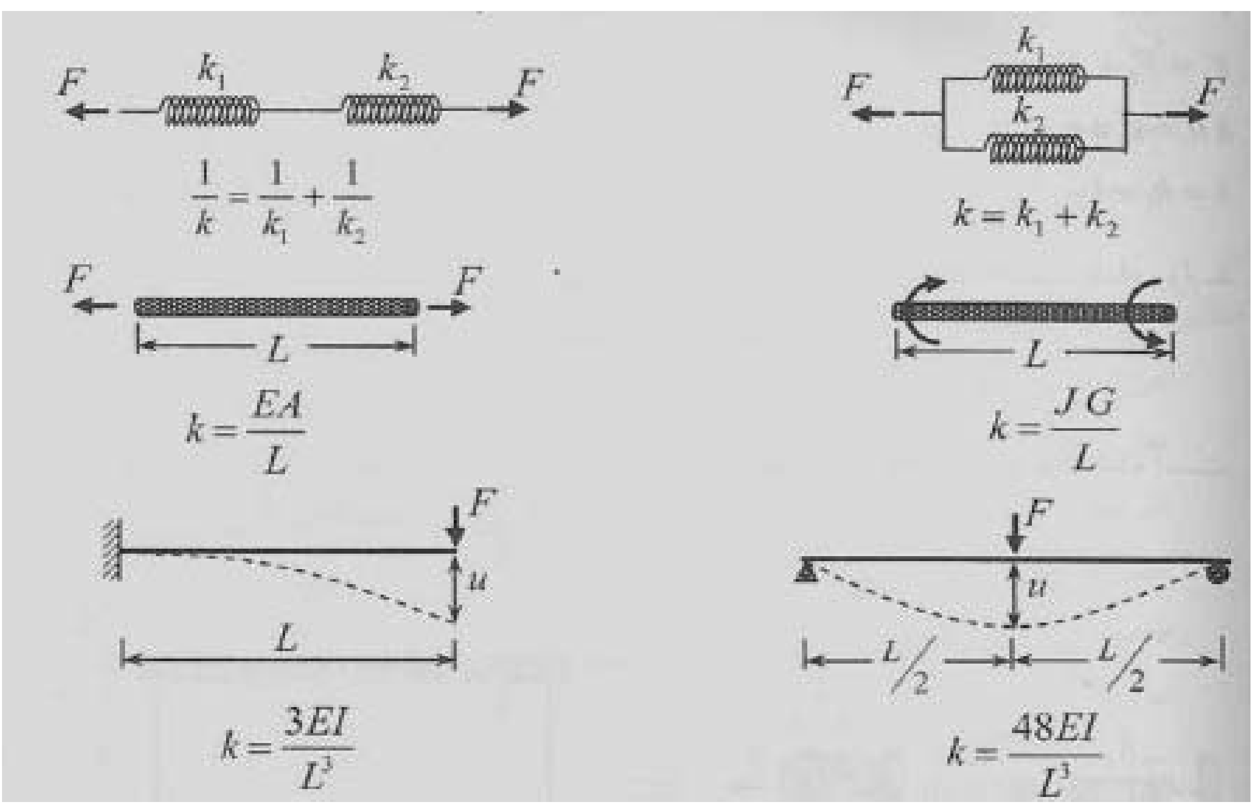


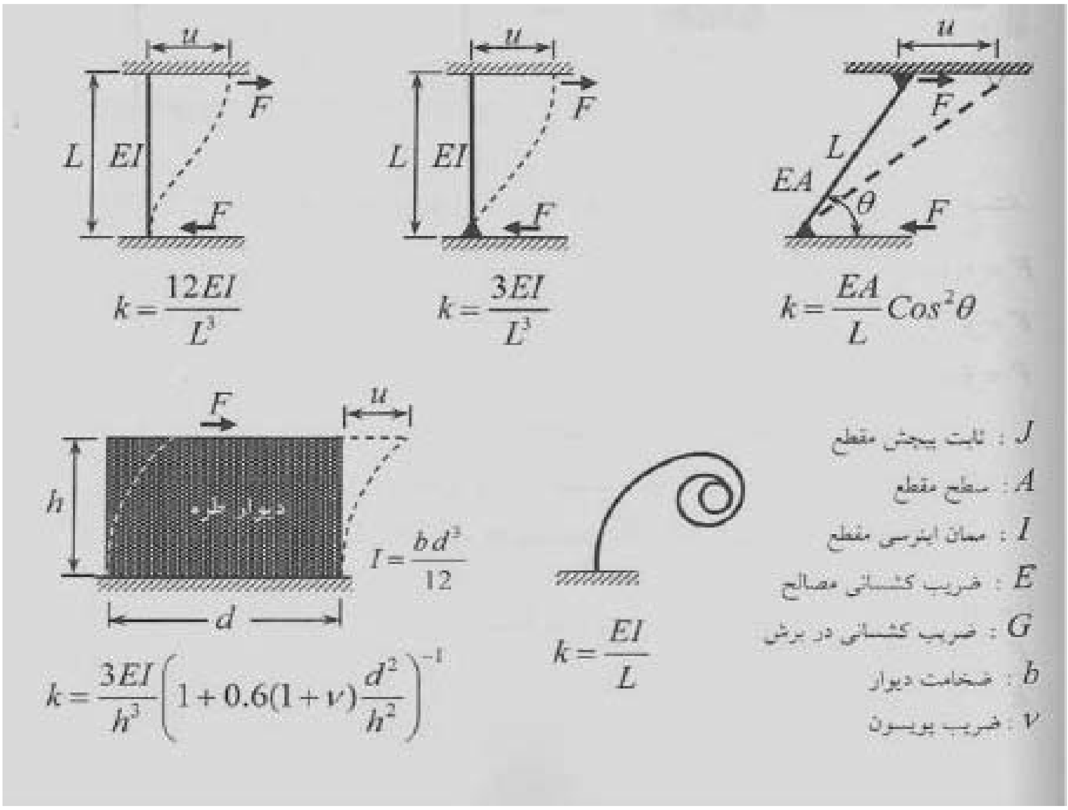
Figure 1.3.3 Variation of lateral stiffness, k, with beam-to-column stiffness ratio, ρ.

* در تعیین سختی ، جهت و نقطه مورد نظر باید معلوم باشد (درجه آزادی)

* در حالت چند درجه آزادی ← ماتریس سختی با عناصر K_{ij}

K_{ij} نیرو در درجه آزادی i وقتی تغییر مکان یا چرخش واحد در درجه آزادی j اعمال می شود و سایر درجات آزادی گرفته می شوند.

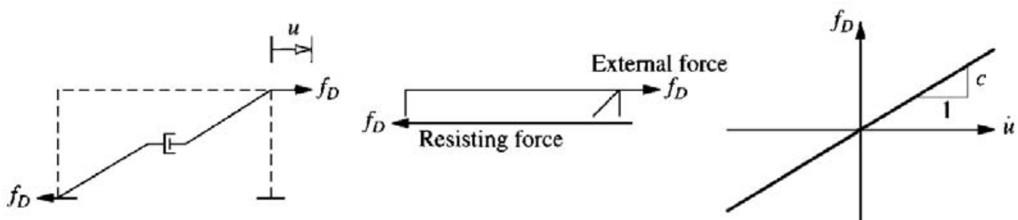




نیروهای میرائی :

پدیده ای که باعث می شود ارتعاش آزاد یک سیستم به تدریج مستقل شود میرائی نام دارد . در سازه ها میرائی بــه علــل مختلف رخ می دهد مکانیسم های متداول که باعث اتلاف انرژی در سا زه های متداول می گردند عبارتند از بــازو بســته شدن ترکهای میکروسکپی در بتن ، لقی اتصالات در فولاد ، اصطکاک اعضای سازه ای و غیر سازه ای وگرما و ...

$$f_D = c\dot{u}$$



(a) (b) (c)

Unlike the stiffness of a structure, the damping coefficient cannot be calculated from the dimensions of the structure and the sizes of the structural elements. This should not be surprising because, as we noted earlier, it is not feasible to identify all the mechanisms that dissipate vibrational energy of actual structures. Thus vibration experiments on actual structures provide the data for evaluating the damping coefficient. These may be free vibration experiments that lead to data such as those shown in Fig. 1.1.4; the measured rate at which motion decays in free vibration will provide a basis for evaluating the damping coefficient, as we shall see in Chapter 2. The damping property may also be determined from forced vibration experiments, a topic that we study in Chapter 3.

The equivalent viscous damper is intended to model the energy dissipation at deformation amplitudes within the linear elastic limit of the overall structure. Over this range of deformations, the damping coefficient c determined from experiments may vary with the deformation amplitude. This nonlinearity of the damping property is usually not considered explicitly in dynamic analyses. It may be handled indirectly by selecting a value for the damping coefficient that is appropriate for the expected deformation amplitude, usually taken as the deformation associated with the linearly elastic limit of the structure.

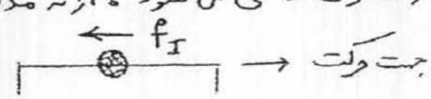
ج - نیروی اینرسی (لَختی) f_I از قانون دوم نیوتن بدست می‌آید

$$\sum \vec{F} = m\ddot{\vec{u}} \rightarrow \sum \vec{F} - m\ddot{\vec{u}} = 0$$

علامت منفی: مقاوم در برابر حرکت

$$f_I = m\ddot{u}$$

چون از قانون ناشی می‌شود، ارائه مدل فیزیکی بی معنی است



نوشتن معادلات رفتاری (معادله حرکت)

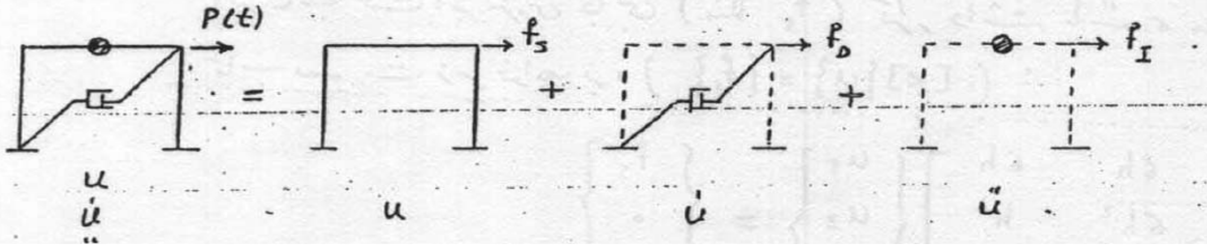
الف – کاربرد مستقیم اصل دوم نیوتن :

ب – تعادل دینامیکی (اصل دالامبر (D'Alembert's Principle)

مشابه حالت اول با بیان متفاوت : نیروی $m\ddot{u}$ از ابتدا نیروی مقاوم است ،

ج – ترکیب مولفه های نیروهای سختی ، میرایی و جرم

در حالت رفتار خطی و در اصول مشابه حالتهای الف و ب

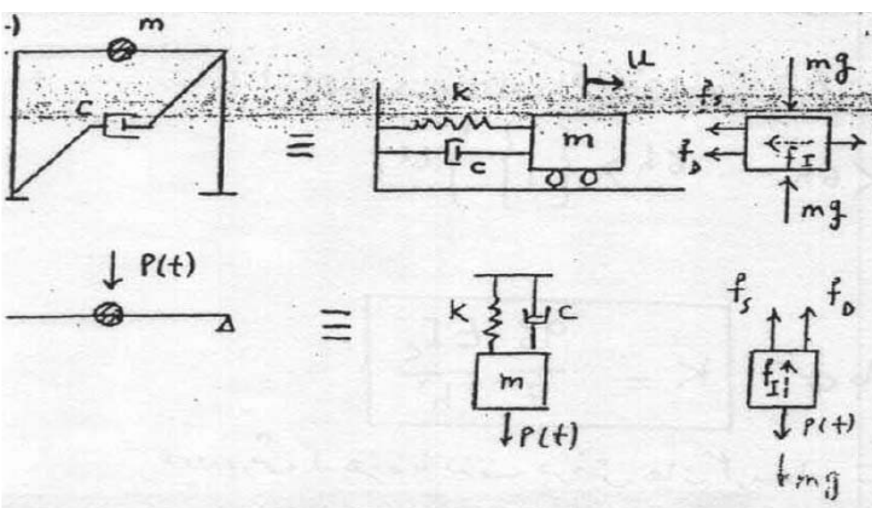


د – روش کار مجازی : تصور تغییر شکل کاذب و صفر بودن کار انجام شده

ه – اصل انرژی : اصل بقای انرژی (جنبشی و پتانسیل –)

مدل سازی رفتاری

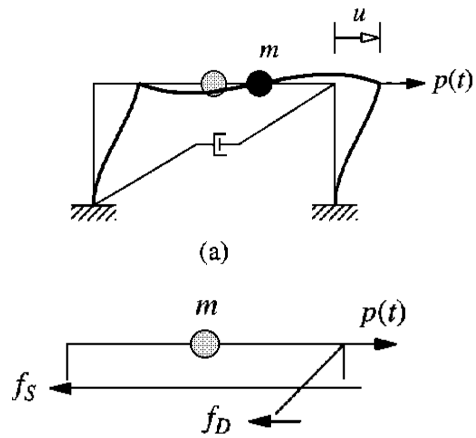
سیستم جرم – فنر – کمک فنر



۹

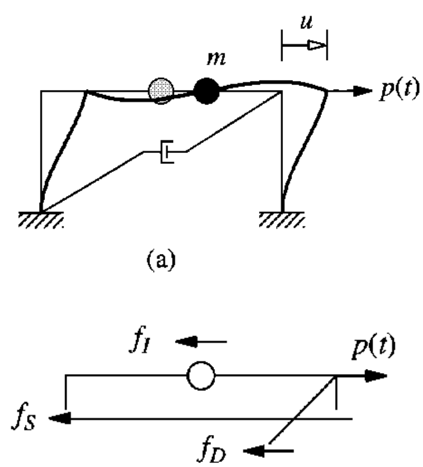
۱) با استفاده از قانون دوم نیوتن

$$fs = ku$$
$$fs = fs(u,\dot{u})$$
$$f_D = c\dot{u}$$
$$p - fs - f_D = m\ddot{u}$$
$$m\ddot{u} + f_D + fs = p(t)$$
$$m\ddot{u} + c\dot{u} + ku = p(t)$$
$$m\ddot{u} + c\dot{u} + fs(u,\dot{u}) = p(t)$$



۲) استفاده از اصل دالامبر

$$fs = ku$$
$$fs = fs(u,\dot{u})$$
$$f_D = c\dot{u}$$
$$f_I = m\ddot{u}$$
$$p(t) - fs - f_D - f_I = 0$$
$$m\ddot{u} + c\dot{u} + ku = p(t)$$
$$m\ddot{u} + c\dot{u} + fs(u,\dot{u}) = p(t)$$



۳) استفاده از روش جمع آثار قوا

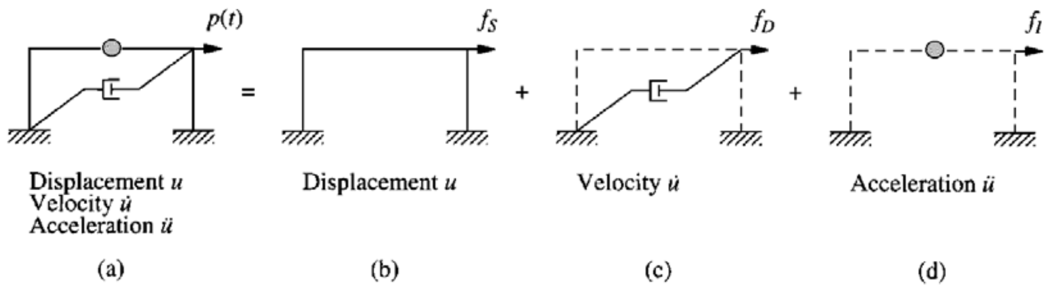
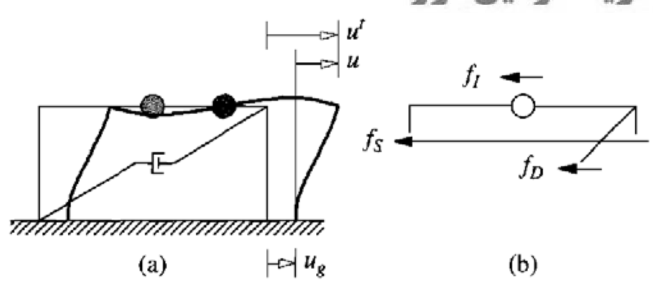


Figure 1.5.2 (a) System; (b) stiffness component; (c) damping component; (d) mass component.

معادله حرکت در حالت تحریک زمین لرزه :



$$u^t(t) = u(t) + u_g(t)$$

$$f_I + f_D + f_S = 0$$

$$f_I = m\ddot{u}^t$$

$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t)$$

$$m\ddot{u} + c\dot{u} + f_S(u,\dot{u}) = -m\ddot{u}_g(t)$$

$$p_{eff}(t) = -m\ddot{u}_g(t)$$

Although the rotational components of ground motion are not measured during earthquakes, they can be estimated from the measured translational components and it is of interest to apply the preceding concepts to this excitation.

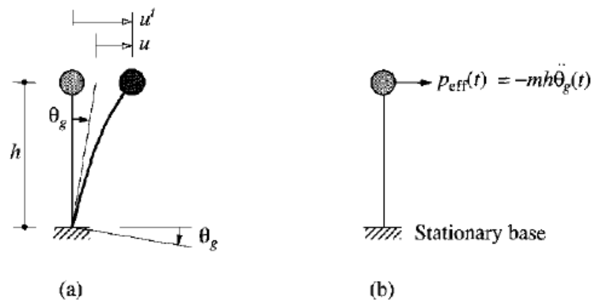


Figure 1.7.3 Effective earthquake force: rotational ground motion.

$$u^t(t) = u(t) + h\theta_g(t)$$

$$m\ddot{u} + c\dot{u} + ku = -mh\ddot{\theta}_g(t)$$

$$p_{\text{eff}}(t) = -mh\ddot{\theta}_g(t)$$

1.8.2 Element Forces

Once the deformation response history $u(t)$ has been evaluated by dynamic analysis of the structure, the element forces and stresses needed for structural design can be determined by static analysis of the structure at each instant in time (i.e., no additional dynamic analysis is necessary). This static analysis of a one-story frame can be visualized in two ways:

1. At each instant, the lateral displacement u is known to which joint rotations are related and hence they can be determined; see Eq. (b) of Example 1.1. From the known displacement and rotation of each end of a structural element (beam and column) the element forces (bending moments and shears) can be determined through the element stiffness properties (Appendix 1); and stresses can be obtained from element forces.

2. The second approach is to introduce the *equivalent static force*, a central concept in earthquake response of structures, as we shall see in Chapter 6. At any instant of time t this force f_S is the external force that will produce the deformation u at the same t in the stiffness component of the structure [i.e., the system without mass or damping (Fig. 1.5.2b)]. Thus

$$f_S(t) = ku(t) \tag{1.8.1}$$

where k is the lateral stiffness of the structure. Element forces or stresses can be determined at each time instant by static analysis of the structure subjected to the force f_S determined from Eq. (1.8.1). It is unnecessary to introduce the equivalent static force concept for the mass–spring–damper system because the spring force, also given by Eq. (1.8.1), can readily be visualized.

1.9 COMBINING STATIC AND DYNAMIC RESPONSES

In practical application we need to determine the total forces in a structure, including those existing before dynamic excitation of the structure and those resulting from the dynamic excitation. For a linear system the total forces can be determined by combining the results of two separate analyses: (1) static analysis of the structure due to dead and live loads, temperature changes, and so on; and (2) analysis of dynamic response due to the time-varying excitation. This direct superposition of the results of two analyses is valid only for linear systems.

The analysis of nonlinear systems cannot, however, be separated into two parts. The dynamic analysis of such a system must recognize the forces and deformations already existing in the structure before the onset of dynamic excitation. This is necessary to establish the initial stiffness property of the structure required to start the dynamic analysis.

روش های حل معادلات حرکت

الف - روش کلاسیک و متعارف (مستقیم) Classical Solution

حل معادله دیفرانسیل - حل عمومی و جواب خصوصی و اعمال شرایط اولیه

ب - روش انتگرال دوهامل Duhamel's Integral رفتار خطی

ج - روش‌های تبدیل Transform Methods رفتار خطی

تبدیل لاپلاس و فوریه ← تحلیل در میدان فرکانس

* مناسب برای تحلیل های اندرکنش محیط های غیر یکنواخت

* روش عددی قوی و سریع : FFT

د - روش های عددی Numerical Methods رفتار خطی و غیرخطی

الگوریتم های مختلف و پایداری روش

Example 1.6

Using Duhamel's integral, we determine the response of an SDF system, assumed to be initially at rest, to a step force, $p(t) = p_o$, $t \geq 0$. For this applied force, Eq. (1.10.2) specializes to

$$u(t) = \frac{p_o}{m\omega_n} \int_0^t \sin[\omega_n(t-\tau)]\,d\tau = \frac{p_o}{m\omega_n}\left[\frac{\cos\omega_n(t-\tau)}{\omega_n}\right]_{\tau=0}^{\tau=t} = \frac{p_o}{k}(1-\cos\omega_n t)$$

This result is the same as that obtained in Section 1.10.1 by the classical solution of the differential equation.

The Fourier transform $\hat{p}(i\omega)$ of a known excitation function $p(t)$ is defined by

$$\hat{p}(i\omega) = \mathcal{F}[p(t)] = \int_{-\infty}^{\infty} e^{-i\omega t} p(t)\,dt \qquad (1.10.3)$$

In solving the equation of motion by Fourier transformation, the first step is to transform the differential equation in variable t into an algebraic equation in the imaginary-valued variable $i\omega$. Then the algebraic equation is readily solved for $\hat{u}(i\omega)$, the transform of $u(t)$. Finally, the solution $u(t)$ of the differential equation is determined by an inverse transformation of $\hat{u}(i\omega)$. The process of inverse transformation is symbolized by

$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(i\omega)\hat{p}(i\omega)e^{i\omega t}\,d\omega \qquad (1.10.4)$$

where the complex frequency-response function $H(i\omega)$ describes the response of the system to harmonic excitation. For SDF systems the integral of Eq. (1.10.4) is evaluated by contour integration using the residue theorem of complex analysis. Closed-form results can be obtained if $p(t)$ is a simple function, and application of the Fourier transform method was restricted to such $p(t)$ until high-speed computers became available.

The Fourier transform method is now feasible for the dynamic analysis of linear systems to complicated excitations $p(t)$ or $\ddot{u}_g(t)$ that are described numerically. In such situations, the integrals of both Eqs. (1.10.3) and (1.10.4) are evaluated numerically by the discrete fast Fourier transform (DFFT) computational algorithm developed in the mid-1960s.

1.1–1.3 Starting from the basic definition of stiffness, determine the effective stiffness of the combined spring and write the equation of motion for the spring–mass systems shown in Figs. P1.1 to P1.3.

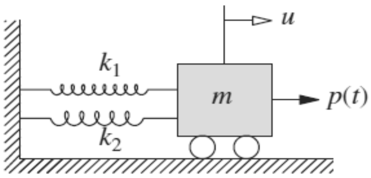


Figure P1.1

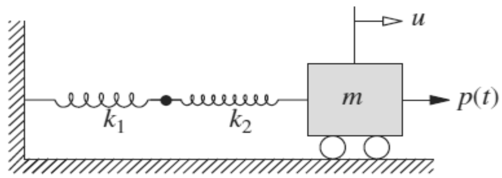


Figure P1.2

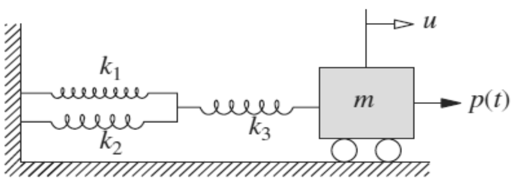


Figure P1.3

1.5 Consider the free motion in the xy plane of a compound pendulum that consists of a rigid rod suspended from a point (Fig. P1.5). The length of the rod is L and its mass m is uniformly distributed. The width of the uniform rod is b and the thickness is t. The angular displacement of the centerline of the pendulum measured from the y-axis is denoted by $\theta(t)$.

(a) Derive the equation governing $\theta(t)$.

(b) Linearize the equation for small θ.

(c) Determine the natural frequency of small oscillations.

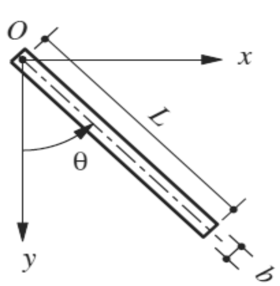


Figure P1.5

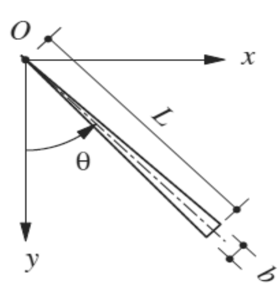


Figure P1.6

1.6 Repeat Problem 1.5 for the system shown in Fig. P1.6, which differs in only one sense: its width varies from zero at O to b at the free end.

1.7 Develop the equation governing the longitudinal motion of the system of Fig. P1.7. The rod is made of an elastic material with elastic modulus E; its cross-sectional area is A and its length is L. Ignore the mass of the rod and measure u from the static equilibrium position.

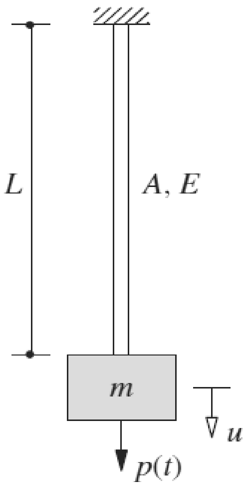


Figure P1.7

1.8 A rigid disk of mass m is mounted at the end of a flexible shaft (Fig. P1.8). Neglecting the weight of the shaft and neglecting damping, derive the equation of free torsional vibration of the disk. The shear modulus (of rigidity) of the shaft is G.

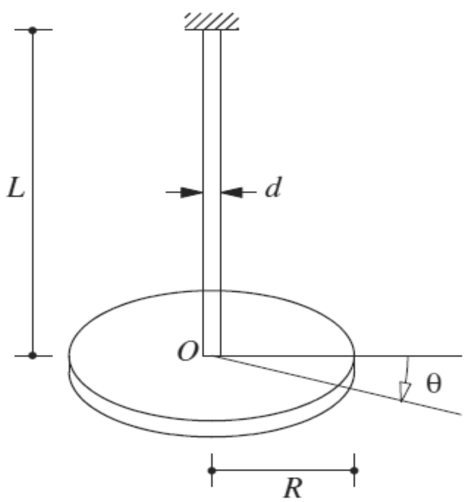


Figure P1.8

1.9–1.11 Write the equation governing the free vibration of the systems shown in Figs. P1.9 to P1.11. Assuming the beam to be massless, each system has a single DOF defined as the vertical deflection under the weight w. The flexural rigidity of the beam is EI and the length is L.

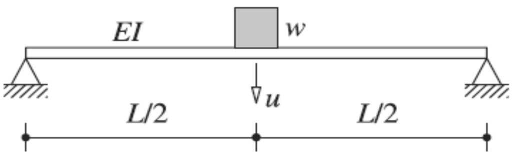


Figure P1.9

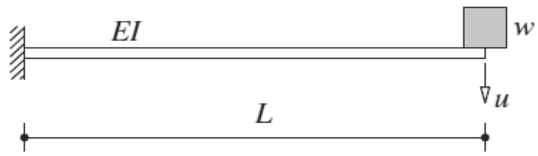


Figure P1.10

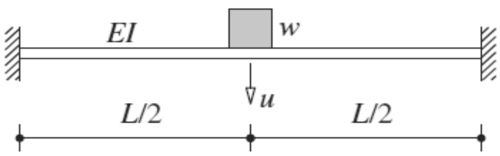


Figure P1.11

1.12 Determine the natural frequency of a weight w suspended from a spring at the midpoint of a simply supported beam (Fig. P1.12). The length of the beam is L, and its flexural rigidity is EI. The spring stiffness is k. Assume the beam to be massless.

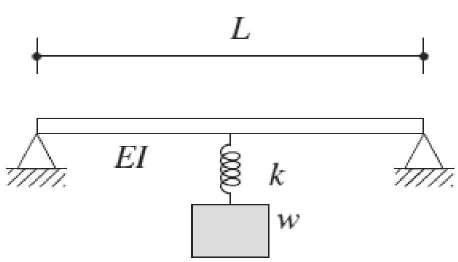


Figure P1.12

1.19 An automobile is crudely idealized as a lumped mass m supported on a spring–damper system as shown in Fig. P1.19. The automobile travels at constant speed v over a road whose roughness is known as a function of position along the road. Derive the equation of motion.

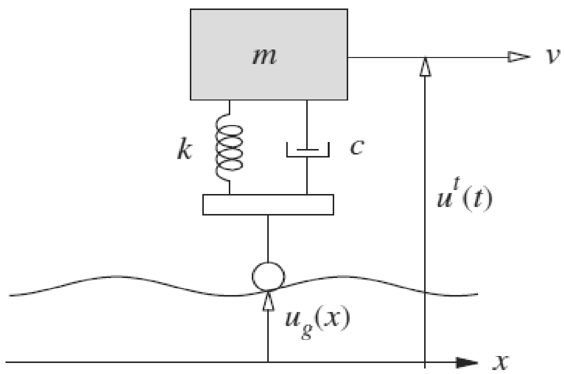
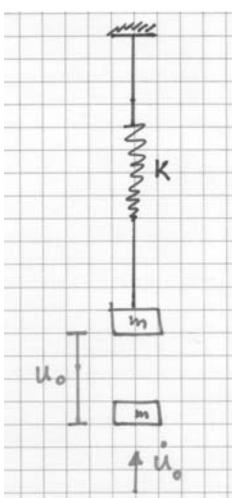


Figure P1.19

2

Free Vibration

ارتعاش آزاد نامیرا :



$$m\ddot{u} + ku = 0$$

$$u_0 = u(0)$$

$$\dot{u}_0 = \dot{u}(0)$$

$$\ddot{u} + \omega_n^{\ 2} u = 0$$

$$\omega_n^{\ 2} = \frac{k}{m}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$ فرکانس زاویه ای طبیعی

$$T_n = \frac{2\pi}{\omega_n}$$ زمان تناوب طبیعی

$$f_n = \frac{1}{T_n}$$ فرکانس طبیعی

$$u = e^{st}$$

$$(ms^2 + k)e^{st} = 0$$

$$(ms^2 + k) = 0$$

$$S_{1,2} = \pm i\omega_n$$

$$u(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$u(t) = A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

$$e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

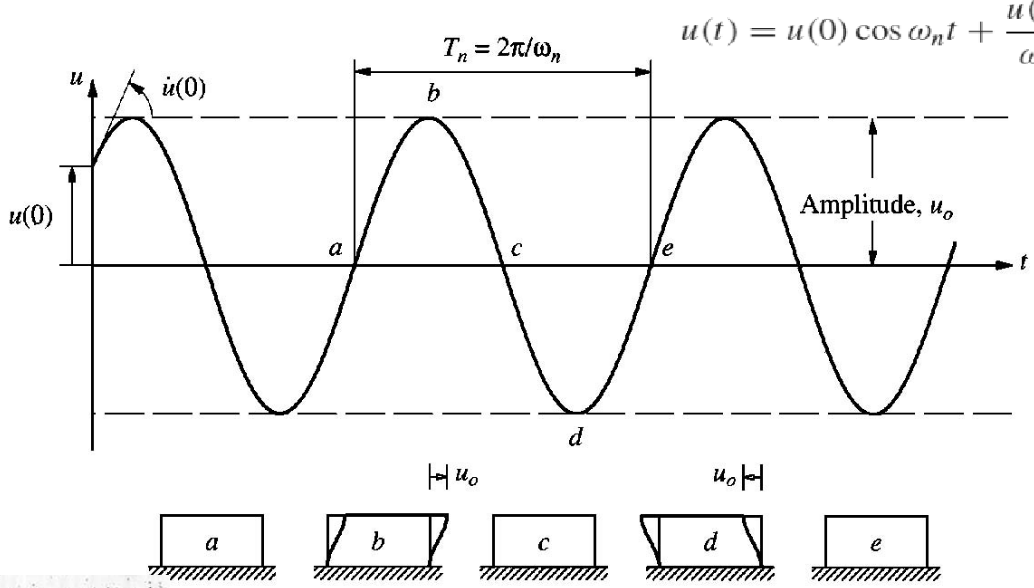
$$u(t) = A\cos\omega_n t + B\sin\omega_n t$$

$$\dot{u}(t) = -\omega_n A\sin\omega_n t + \omega_n B\cos\omega_n t$$

$$u(0) = A \dot{u}(0) = \omega_n B$$

$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$$

$$u(t) = u(0)\cos\omega_n t + \frac{\dot{u}(0)}{\omega_n}\sin\omega_n t$$

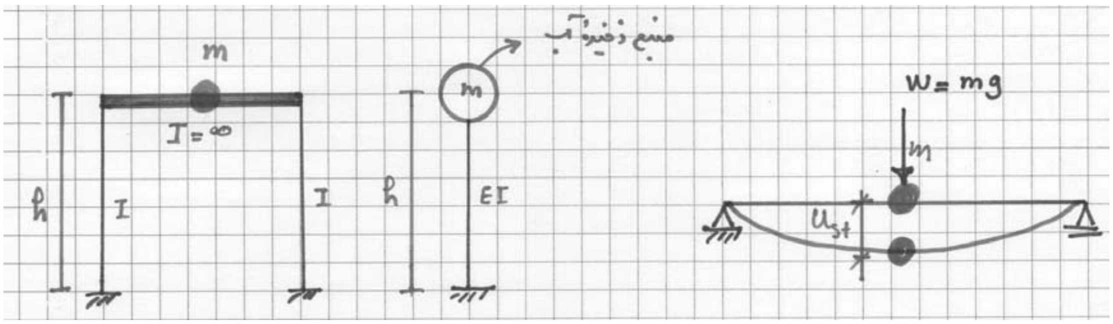


$$u(t) = \rho \sin(\omega t + \alpha)$$

$$\rho = u_{max} = \sqrt{u_0^2 + \left(\frac{\dot{u}_0}{\omega}\right)^2}$$

$$\alpha = \text{Arc tg} \frac{\omega u_0}{\dot{u}_0}$$

$$T_n = \frac{2\pi}{\omega_n} \qquad f_n = \frac{1}{T_n} \qquad f_n = \frac{\omega_n}{2\pi}$$



$$k = 2\times\frac{12EI}{h^3} \qquad k = \frac{3EI}{h^3}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$T_n = \frac{2\pi}{\omega_n}$$

$$F = k.\Delta$$

$$k = \frac{F}{\Delta} = \frac{W}{u_{st}}$$

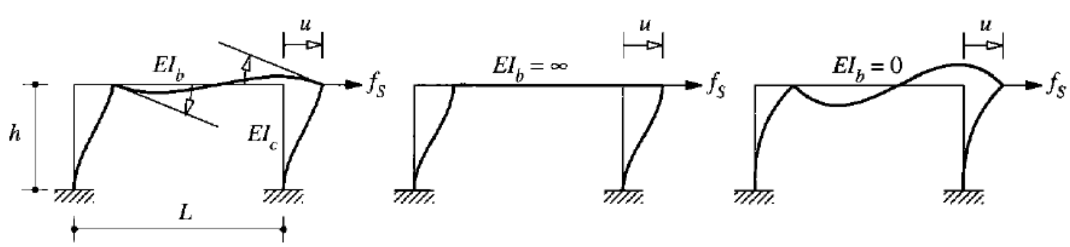
$$m = \frac{W}{g}$$

The natural frequency of the one-story frame of Fig. 1.3.2a with lumped mass m and columns clamped at the base is

$$\omega_n = \sqrt{\frac{k}{m}} \qquad k = \frac{24EI_c}{h^3}\,\frac{12\rho+1}{12\rho+4} \tag{2.1.10}$$

where the lateral stiffness comes from Eq. (1.3.5) and $\rho = I_b/4I_c$. For the extreme cases of a rigid beam, $\rho = \infty$, and a beam with no stiffness, $\rho = 0$, the lateral stiffnesses are given by Eqs. (1.3.2) and (1.3.3) and the natural frequencies are

$$(\omega_n)_{\rho=\infty} = \sqrt{\frac{24EI_c}{mh^3}} \qquad (\omega_n)_{\rho=0} = \sqrt{\frac{6EI_c}{mh^3}} \tag{2.1.11}$$



(a) (b) (c)

Figure 1.3.2

The natural frequency is doubled as the beam-to-column stiffness ratio, ρ, increases from 0 to ∞; its variation with ρ is shown in Fig. 2.1.3.

The natural frequency is similarly affected by the boundary conditions at the base of the columns. If the columns are hinged at the base rather than clamped and the beam is rigid, $\omega_n = \sqrt{6EI_c/mh^3}$, which is one-half of the natural frequency of the frame with clamped-base columns.

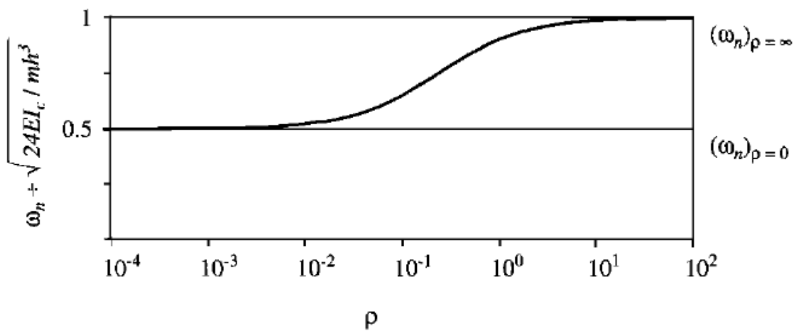


Figure 2.1.3 Variation of natural frequency, ω_n, with beam-to-column stiffness ratio, ρ.

ارتعاش آزاد میرا :

$f_I + f_D + f_s = 0$

$m\ddot{u} + c\dot{u} + ku = 0 \rightarrow m \neq 0 \rightarrow \ddot{u} + \frac{c}{m}\dot{u} + \frac{k}{m}u = 0$ ①

جواب فرضی : $u(t) = \bar{c}e^{st}$, $\dot{u} = \bar{c}se^{st}$, $\ddot{u} = \bar{c}s^2e^{st} \rightarrow$ ① در

$\bar{c}e^{st}(s^2 + \frac{c}{m}s + \omega^2) = 0 \rightarrow s^2 + \frac{c}{m}s + \omega^2 = 0 \Rightarrow$

$S = -\frac{c}{2m} \pm \sqrt{(\frac{c}{2m})^2 - \omega^2} \rightarrow$ سه حالت برای زیر رادیکال متصور است

اگر $\frac{c}{2m} = \omega \rightarrow$ مقدار c را بحرانی نامند $\Rightarrow C_{cr} = 2m\omega$

اگر از درصد استهلاک استفاده $\xi = \frac{C_{موجود}}{C_{cr}} \rightarrow S = -\xi\omega \pm \omega\sqrt{\xi^2 - 1}$

$\frac{c}{2m} = \frac{c\omega}{2m\omega} = \frac{c\omega}{C_{cr}} = \xi\omega$

ارتعاش آزاد میرا :

$$m\ddot{u} + c\dot{u} + ku = 0$$

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 = 0$$

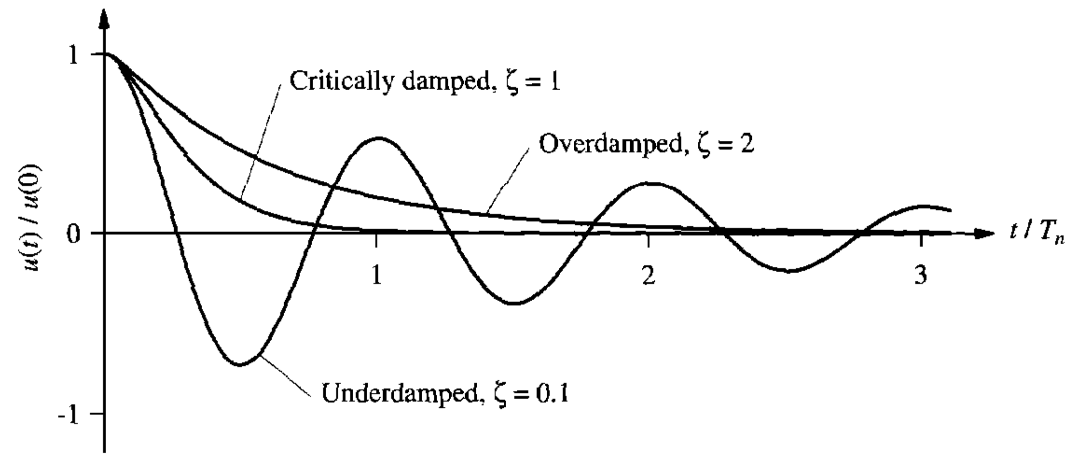
نسبت میرایی $\zeta = \frac{c}{2m\omega_n} = \frac{c}{c_{cr}}$

ضریب میرائی بحرانی $c_{cr} = 2m\omega_n = 2\sqrt{km} = \frac{2k}{\omega_n}$

بحرانی $C = Ccr$

فوق بحرانی $C > Ccr$

زیر بحرانی $C < Ccr$



معادله حرکت در حالت زیر بحرانی

$C < C_{cr} or \zeta < 1$

$$m\ddot{u} + c\dot{u} + ku = 0$$

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 = 0$$

$u_0 = u(0)$

$\dot{u}_0 = \dot{u}(0)$

$$u = e^{st}$$

$$(s^2 + 2\zeta\omega_n s + \omega_n^2)e^{st} = 0$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s_{1,2} = \omega_n(-\zeta \pm i\sqrt{1-\zeta^2})$$

$$u(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$u(t) = e^{-\zeta\omega_n t}(A_1 e^{i\omega_n t} + A_2 e^{-i\omega_n t})$$

$$\omega_D = \omega_n \sqrt{1-\zeta^2}$$

$$u(t) = e^{-\zeta\omega_n t}(A\cos\omega_D t + B\sin\omega_D t)$$

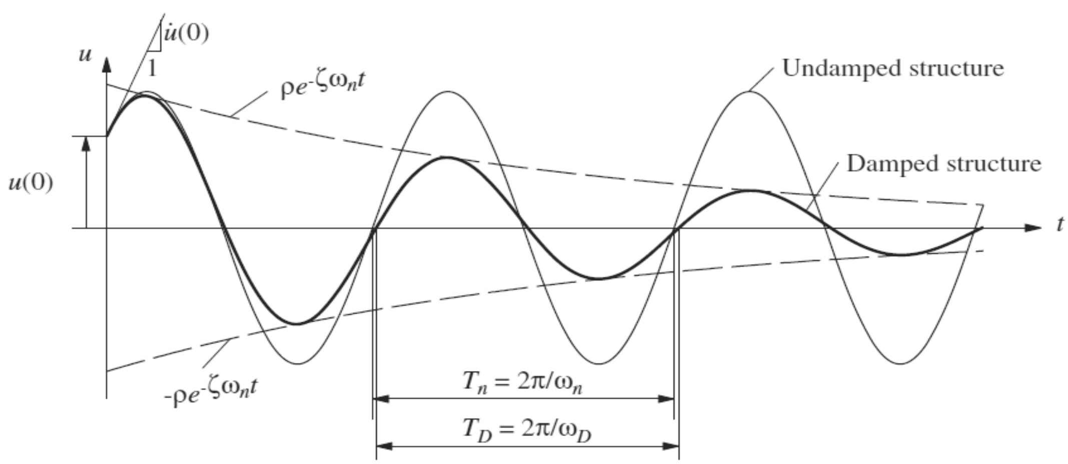
$$A = u(0)$$

$$B = \frac{\dot{u}(0)+\zeta\omega_n u(0)}{\omega_D}$$

$$u(t) = e^{-\zeta\omega_n t}[u(0)\cos\omega_D t + \left(\frac{\dot{u}(0)+\zeta\omega_n u(0)}{\omega_D}\right)\sin\omega_D t]$$

$$\omega_D = \omega_n \sqrt{1-\zeta^2}$$

$$T_D = \frac{T_n}{\sqrt{1-\zeta^2}}$$



$$\rho = \sqrt{[u(0)]^2 + \left[\frac{\dot{u}(0)+\zeta\omega_n u(0)}{\omega_D}\right]^2}$$

The more important effect of damping is on the rate at which free vibration decays. This is displayed in Fig. 2.2.4, where the free vibration due to initial displacement $u(0)$ is plotted for four systems having the same natural period T_n but differing damping ratios: $\zeta = 2$, 5, 10, and 20%.

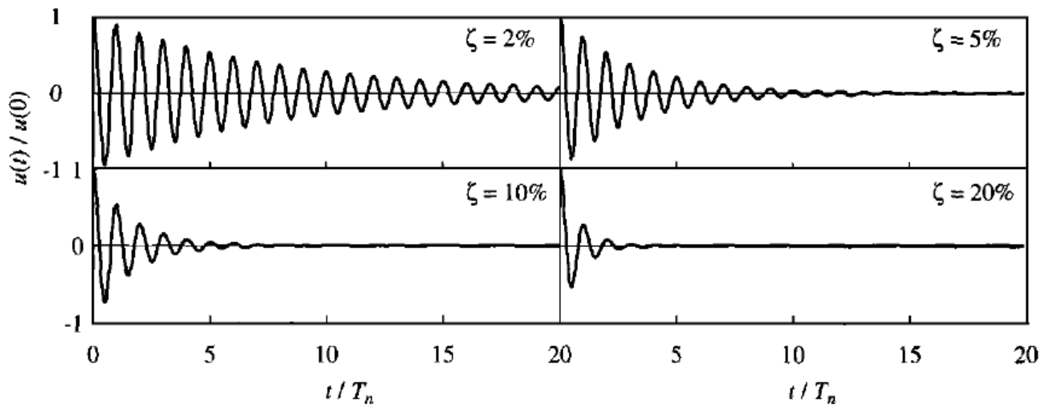
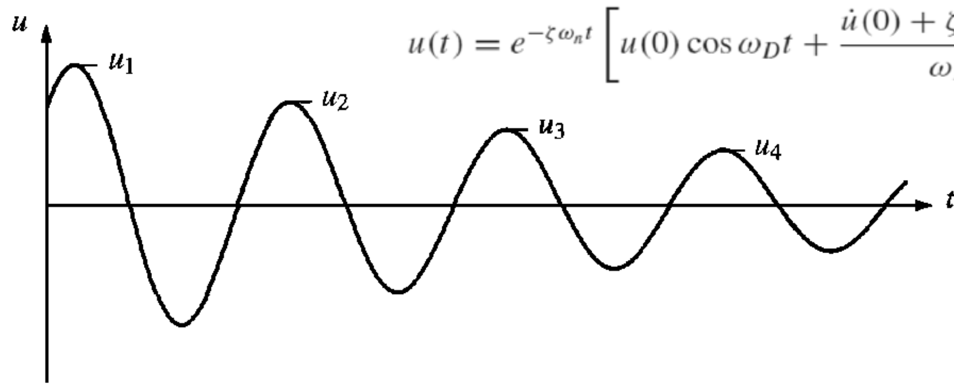


Figure 2.2.4 Free vibration of systems with four levels of damping: $\zeta = 2$, 5, 10, and 20%.



$$u(t) = e^{-\zeta\omega_n t}\left[u(0)\cos\omega_D t + \frac{\dot{u}(0) + \zeta\omega_n u(0)}{\omega_D}\sin\omega_D t\right]$$

$$\frac{u(t)}{u(t+T_D)} = \exp(\zeta\omega_n T_D) = \exp\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$\frac{u_i}{u_{i+1}} = \exp\left(\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right)$$

$$\delta = In\frac{u_i}{u_{i+1}} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$If\ \ \zeta\ \ is\ \ small, \sqrt{1-\zeta^2} \cong 1$$

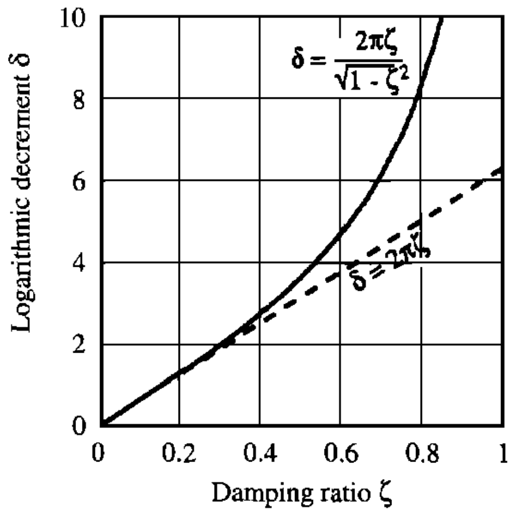
$$\delta \cong 2\pi\zeta$$

$$\frac{u_1}{u_{j+1}} = \frac{u_1}{u_2}\frac{u_2}{u_3}\frac{u_3}{u_4}\cdots\frac{u_j}{u_{j+1}} = e^{j\delta}$$

$$\delta = \frac{1}{j} Ln \frac{u_1}{u_{j+1}} \cong 2\pi\zeta$$

$$\zeta = \frac{1}{2\pi j} Ln \frac{u_i}{u_{i+j}}$$

$$\zeta = \frac{1}{2\pi\zeta} Ln \frac{\ddot{u}_i}{\ddot{u}_{i+j}}$$



$$\delta = \frac{1}{j} Ln \frac{u_i}{u_{i+j}} \cong 2\pi\zeta$$

$$\frac{u_{i+j}}{u_i} = 0.5 \qquad\qquad j_{50\%} \cong \frac{0.11}{\zeta}$$

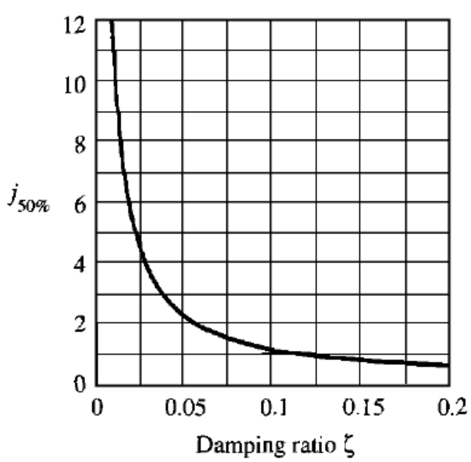


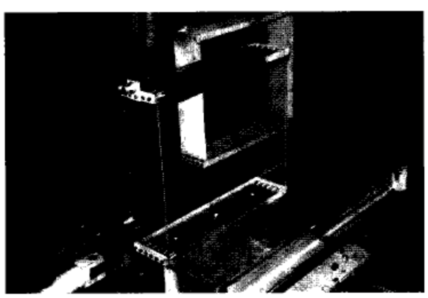
Figure 2.2.7 Number of cycles required to reduce the free vibration amplitude by 50%.

2.2.4 Free Vibration Tests

Because it is not possible to determine analytically the damping ratio ζ for practical structures, this elusive property should be determined experimentally. Free vibration experiments provide one means of determining the damping. Such experiments on two one-story models led to the free vibration records presented in Fig. 1.1.4; a part of such a record is shown in Fig. 2.2.8. For lightly damped systems the damping ratio can be determined from

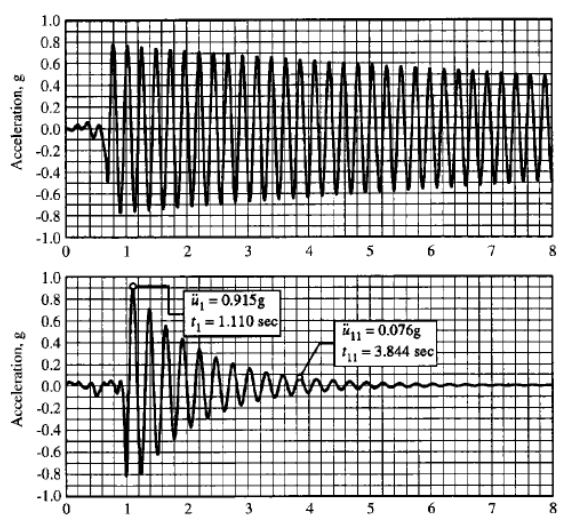
$$\zeta = \frac{1}{2\pi j} \ln \frac{u_i}{u_{i+j}} \qquad \text{or} \qquad \zeta = \frac{1}{2\pi j} \ln \frac{\ddot{u}_i}{\ddot{u}_{i+j}} \tag{2.2.14}$$

The first of these equations is equivalent to Eq. (2.2.12), which was derived from the equation for $u(t)$. The second is a similar equation in terms of accelerations, which are easier to measure than displacements. It can be shown to be valid for lightly damped systems.



(a)

Figure 1.1.4 (a) Photograph of aluminum and plexiglass model frames mounted on a small shaking table used for classroom demonstration at the University of California at Berkeley (courtesy of T. Merport); (b) free vibration record of aluminum model; (c) free vibration record of plexiglass model.



(b)

(c)

The natural period T_D of the system can also be determined from the free vibration record by measuring the time required to complete one cycle of vibration. Comparing

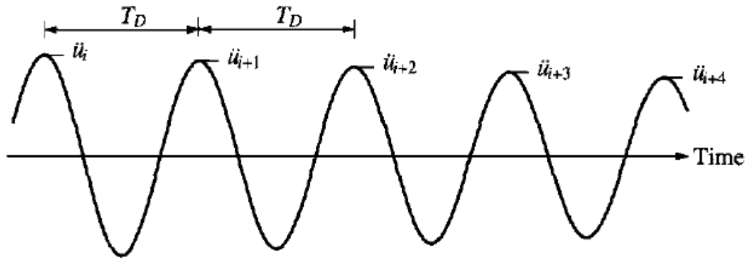


Figure 2.2.8 Acceleration record of a freely vibrating system.

this with the natural period obtained from the calculated stiffness and mass of an idealized system tells us how accurately these properties were calculated and how well the idealization represents the actual structure.

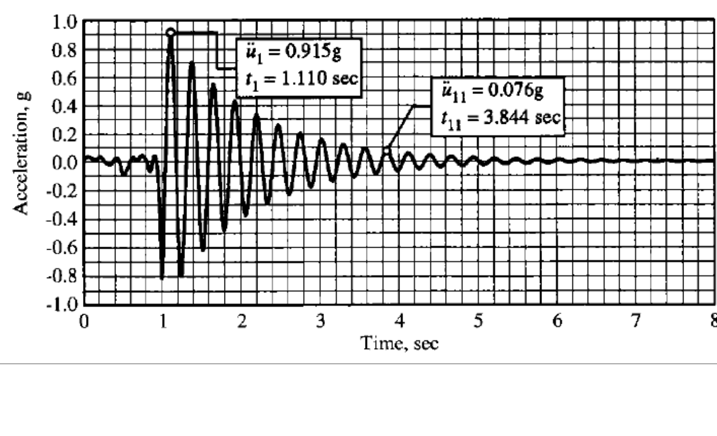
Example 2.4

Determine the natural vibration period and damping ratio of the plexiglass frame model (Fig. 1.1.4a) from the acceleration record of its free vibration shown in Fig. 1.1.4c.

Solution The peak values of acceleration and the time instants they occur can be read from the free vibration record or obtained from the corresponding data stored in a computer during the experiment. The latter provides the following data:

Peak	Time, t_i (sec)	Peak, $\ddot{u}_i$ (g)
1	1.110	0.915
11	3.844	0.076

$$T_n = \frac{3.844 - 1.110}{10} = 0.273 \text{ sec} \qquad \zeta = \frac{1}{2\pi(10)} \ln \frac{0.915\text{g}}{0.076\text{g}} = 0.0396 \text{ or } 3.96\%$$



(c)

Example 2.5

A free vibration test is conducted on an empty elevated water tank such as the one in Fig. 1.1.2. A cable attached to the tank applies a lateral (horizontal) force of 16.4 kips and pulls the tank horizontally by 2 in. The cable is suddenly cut and the resulting free vibration is recorded. At the end of four complete cycles, the time is 2.0 sec and the amplitude is 1 in. From these data compute the following: **(a)** damping ratio; **(b)** natural period of undamped vibration; **(c)** effective stiffness; **(d)** effective weight; **(e)** damping coefficient; and **(f)** number of cycles required for the displacement amplitude to decrease to 0.2 in.

Solution **(a)** Assuming small damping:

$$j_{50\%} \simeq \frac{0.11}{\zeta} \qquad \zeta = \frac{0.11}{4} = 0.0275 = 2.75\%$$

Assumption of small damping implicit in Eq. (2.2.13) is valid.

(b) $T_D = \dfrac{2.0}{4} = 0.5$ sec, $\quad T_n \simeq T_D = 0.5$ sec.

(c) $k = \dfrac{16.4}{2} = 8.2$ kips/in.

(d) $\omega_n = \dfrac{2\pi}{T_n} = \dfrac{2\pi}{0.5} = 12.57,$

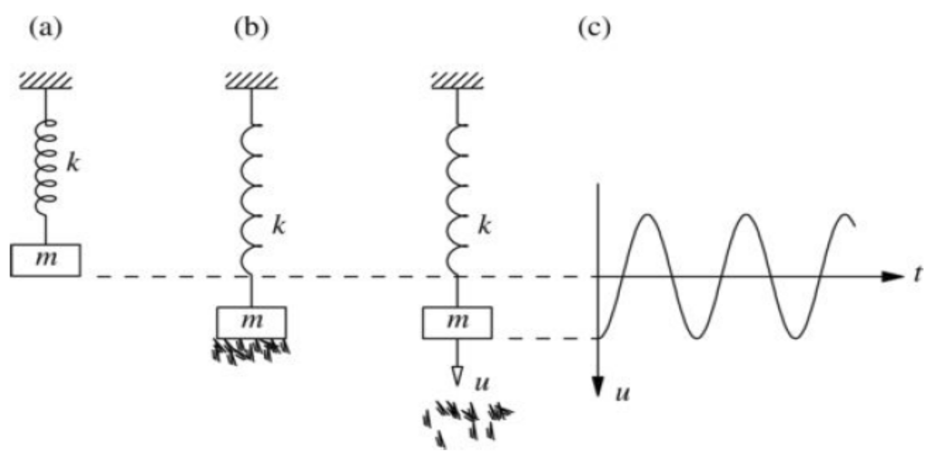
$$m = \frac{k}{\omega_n^2} = \frac{8.2}{(12.57)^2} = 0.0519 \text{ kip-sec}^2/\text{in.};$$

$$w = (0.0519)386 = 20.03 \text{ kips.}$$

(e) $c = \zeta(2\sqrt{km}) = 0.0275\left[2\sqrt{8.2(0.0519)}\right] = 0.0359$ kip-sec/in.

(f) $\zeta \simeq \dfrac{1}{2\pi j}\ln\dfrac{u_1}{u_{1+j}}, \quad j \simeq \dfrac{1}{2\pi(0.0275)}\ln\dfrac{2}{0.2} = 13.32$ cycles ~ 13 cycles.

2.2 An electromagnet weighing 400 lb and suspended by a spring having a stiffness of 100 lb/in. (Fig. P.2.2a) lifts 200 lb of iron scrap (Fig. P2.2b). Determine the equation describing the motion when the electric current is turned off and the scrap is dropped (Fig. P2.2c).



2.4 The weight of the wooden block shown in Fig. P2.4 is 10 lb and the spring stiffness is 100 lb/in. A bullet weighing 0.5 lb is fired at a speed of 60 ft/sec into the block and becomes embedded in the block. Determine the resulting motion $u(t)$ of the block.

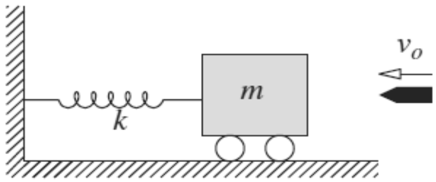


Figure P2.4

2.5 A mass m_1 hangs from a spring k and is in static equilibrium. A second mass m_2 drops through a height h and sticks to m_1 without rebound (Fig. P2.5). Determine the subsequent motion $u(t)$ measured from the static equilibrium position of m_1 and k.

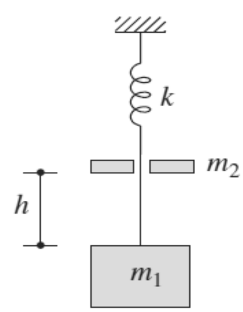


Figure P2.5

2.8 Show that the motion of a critically damped system due to initial displacement $u(0)$ and initial velocity $\dot{u}(0)$ is

$$u(t) = \{u(0) + [\dot{u}(0) + \omega_n u(0)]\, t\}\, e^{-\omega_n t}$$

2.9 Show that the motion of an overcritically damped system due to initial displacement $u(0)$ and initial velocity $\dot{u}(0)$ is

$$u(t) = e^{-\zeta \omega_n t}\left(A_1 e^{-\omega'_D t} + A_2 e^{\omega'_D t}\right)$$

where $\omega'_D = \omega_n \sqrt{\zeta^2 - 1}$ and

$$A_1 = \frac{-\dot{u}(0) + \left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n u(0)}{2\omega'_D}$$

$$A_2 = \frac{\dot{u}(0) + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n u(0)}{2\omega'_D}$$

2.14 The vertical suspension system of an automobile is idealized as a viscously damped SDF system. Under the 3000-lb weight of the car the suspension system deflects 2 in. The suspension is designed to be critically damped.
(a) Calculate the damping and stiffness coefficients of the suspension.
(b) With four 160-lb passengers in the car, what is the effective damping ratio?
(c) Calculate the natural vibration frequency for case (b).

2.15 The stiffness and damping properties of a mass–spring–damper system are to be determined by a free vibration test; the mass is given as $m = 0.1$ lb-sec^2/in. In this test the mass is displaced 1 in. by a hydraulic jack and then suddenly released. At the end of 20 complete cycles, the time is 3 sec and the amplitude is 0.2 in. Determine the stiffness and damping coefficients.

3

Response to Harmonic and Periodic Excitations

PREVIEW

The response of SDF systems to harmonic excitation is a classical topic in structural dynamics, not only because such excitations are encountered in engineering systems (e.g., force due to unbalanced rotating machinery), but also because understanding the response of structures to harmonic excitation provides insight into how the system will respond to other types of forces. Furthermore, the theory of forced harmonic vibration has several useful applications in earthquake engineering.

HARMONIC VIBRATION OF UNDAMPED SYSTEMS

$$m\ddot{u} + ku = p_o \sin \omega t$$

$$u = u(0) \qquad \dot{u} = \dot{u}(0)$$

$$u_c(t) = A \cos \omega_n t + B \sin \omega_n t$$

$$u_p(t) = C \sin \omega t$$

$$\ddot{u}_p(t) = -\omega^2 C \sin \omega t$$

$$u_p(t) = \frac{p_o}{k} \frac{1}{1 - (\omega/\omega_n)^2} \sin \omega t \qquad \omega \neq \omega_n$$

$$u(t) = A \cos \omega_n t + B \sin \omega_n t + \frac{p_o}{k} \frac{1}{1 - (\omega/\omega_n)^2} \sin \omega t$$

$$u(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{p_o}{k}\frac{1}{1-(\omega/\omega_n)^2}\sin\omega t$$

$$\dot{u}(t) = -\omega_n A\sin\omega_n t + \omega_n B\cos\omega_n t + \frac{p_o}{k}\frac{\omega}{1-(\omega/\omega_n)^2}\cos\omega t$$

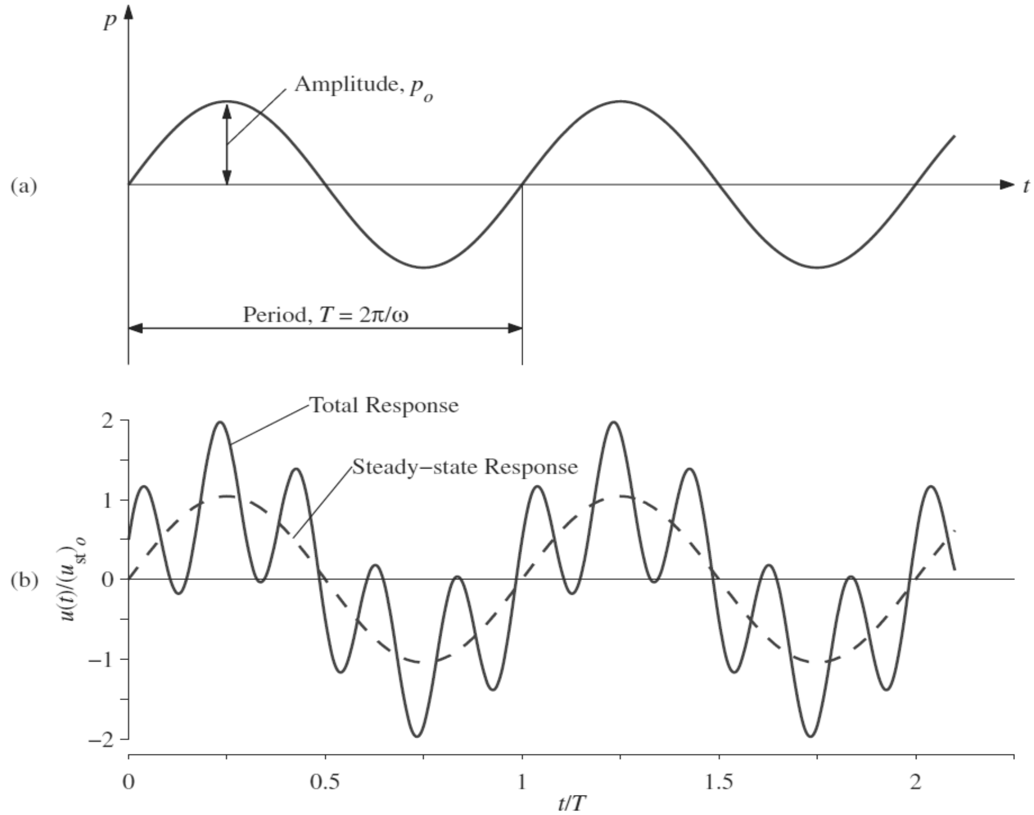
$$u(0) = A \qquad \dot{u}(0) = \omega_n B + \frac{p_o}{k}\frac{\omega}{1-(\omega/\omega_n)^2}$$

$$A = u(0) \qquad B = \frac{\dot{u}(0)}{\omega_n} - \frac{p_o}{k}\frac{\omega/\omega_n}{1-(\omega/\omega_n)^2}$$

$$u(t) = \underbrace{u(0)\cos\omega_n t + \left[\frac{\dot{u}(0)}{\omega_n} - \frac{p_o}{k}\frac{\omega/\omega_n}{1-(\omega/\omega_n)^2}\right]\sin\omega_n t}_{\text{transient}} + \underbrace{\frac{p_o}{k}\frac{1}{1-(\omega/\omega_n)^2}\sin\omega t}_{\text{steady state}}$$

if $u(0) = \dot{u}(0) = 0$,

$$u(t) = \frac{p_o}{k}\frac{1}{1-(\omega/\omega_n)^2}\left(\sin\omega t - \frac{\omega}{\omega_n}\sin\omega_n t\right)$$



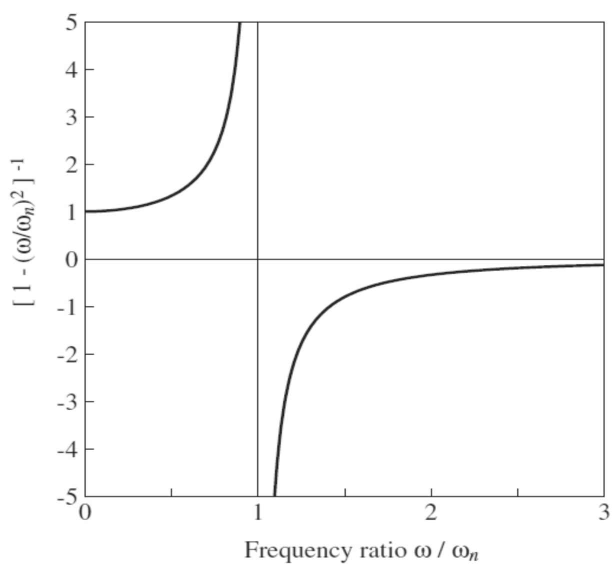
$$u(t) = (u_{st})_o \left[\frac{1}{1 - (\omega/\omega_n)^2} \right] \sin \omega t$$

$$u_{st}(t) = \frac{p_o}{k} \sin \omega t \qquad (u_{st})_o = \frac{p_o}{k}$$

$$u(t) = u_o \sin(\omega t - \phi) = (u_{st})_o R_d \sin(\omega t - \phi)$$

$$R_d = \frac{u_o}{(u_{st})_o} = \frac{1}{| \ 1 - (\omega/\omega_n)^2 \ |}$$

$$\phi = \begin{cases} 0^\circ & \omega < \omega_n \\ 180^\circ & \omega > \omega_n \end{cases}$$



If $\omega = \omega_n$,

$$u_p(t) = Ct\cos\omega_n t \qquad C = -\frac{p_o}{2k}\omega_n$$

$$u(t) = A\cos\omega_n t + B\sin\omega_n t - \frac{p_o}{2k}\omega_n t\cos\omega_n t$$

$$\dot{u}(t) = -\omega_n A\sin\omega_n t + \omega_n B\cos\omega_n t - \frac{p_o}{2k}\omega_n\cos\omega_n t + \frac{p_o}{2k}\omega_n^2 t\sin\omega_n t$$

$$A = u(0) \qquad B = \frac{\dot{u}(0)}{\omega_n} + \frac{p_o}{2k}$$

Specializing for at-rest initial conditions gives

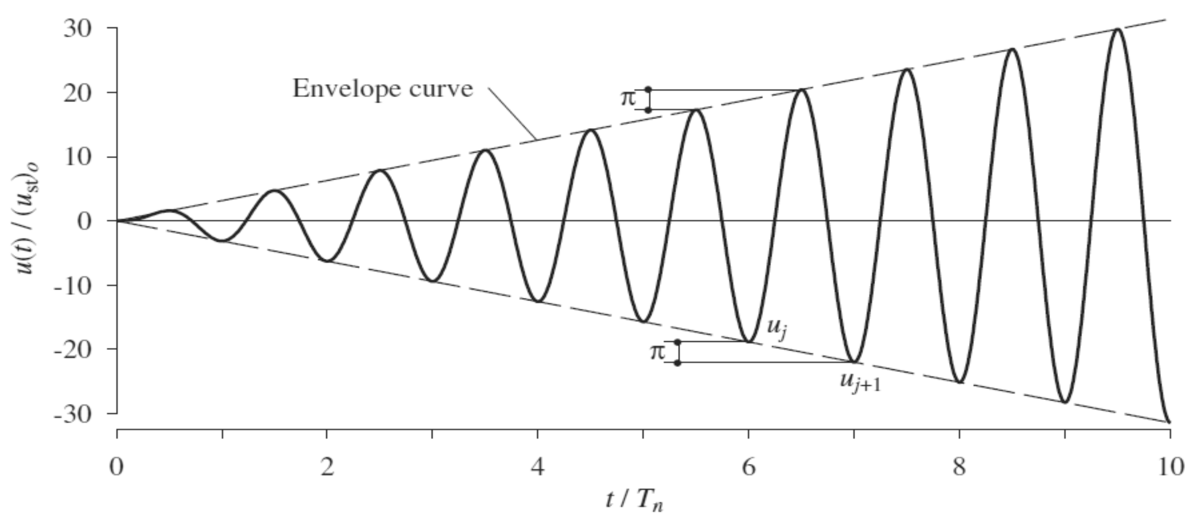
$$A = 0 \qquad B = \frac{p_o}{2k}$$

$$u(t) = -\frac{1}{2}\frac{p_o}{k}(\omega_n t\cos\omega_n t - \sin\omega_n t)$$

$$u(t) = -\frac{1}{2}\frac{p_o}{k}(\omega_n t\cos\omega_n t - \sin\omega_n t)$$

$$|u_{j+1}| - |u_j| = (u_{\text{st}})_o[\pi(j+1) - \pi j] = \frac{\pi p_o}{k}$$

$$\frac{u(t)}{(u_{\text{st}})_o} = -\frac{1}{2}\left(\frac{2\pi t}{T_n}\cos\frac{2\pi t}{T_n} - \sin\frac{2\pi t}{T_n}\right)$$



HARMONIC VIBRATION WITH VISCOUS DAMPING

$$m\ddot{u} + c\dot{u} + ku = p_o \sin \omega t$$

$$u = u(0) \qquad \dot{u} = \dot{u}(0)$$

$$u_c(t) = e^{-\zeta \omega_n t}(A \cos \omega_D t + B \sin \omega_D t)$$

$$u_p(t) = C \sin \omega t + D \cos \omega t$$

$$C = \frac{p_o}{k} \frac{1 - (\omega/\omega_n)^2}{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}$$

$$D = \frac{p_o}{k} \frac{-2\zeta \omega/\omega_n}{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}$$

$$u(t) = \underbrace{e^{-\zeta \omega_n t}(A \cos \omega_D t + B \sin \omega_D t)}_{\text{transient}} + \underbrace{C \sin \omega t + D \cos \omega t}_{\text{steady state}}$$

$$u(t) = \underbrace{e^{-\zeta \omega_n t}(A \cos \omega_D t + B \sin \omega_D t)}_{\text{transient}} + \underbrace{C \sin \omega t + D \cos \omega t}_{\text{steady state}}$$

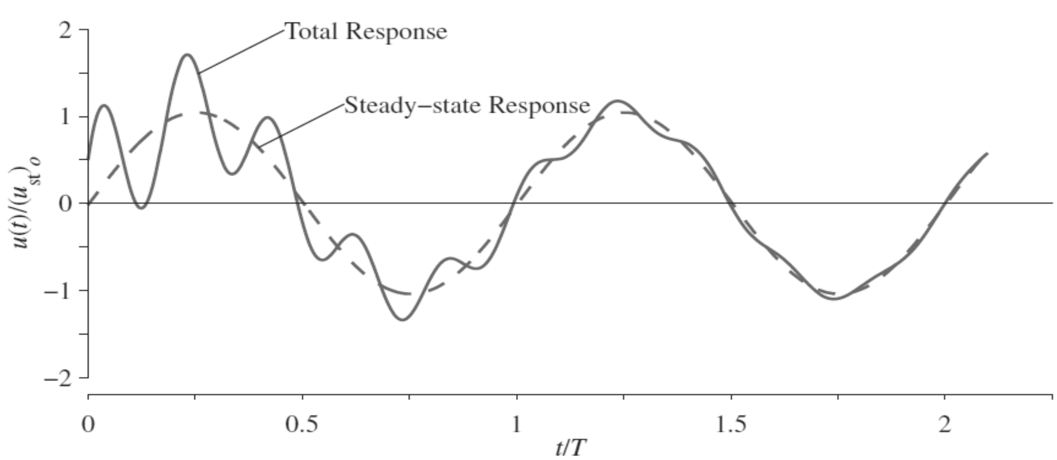


Figure 3.2.1 Response of damped system to harmonic force; $\omega/\omega_n = 0.2$, $\zeta = 0.05$, $u(0) = 0.5 p_o/k$, and $\dot{u}(0) = \omega_n p_o/k$.

$$u(t) = \underbrace{e^{-\zeta\omega_n t}(A\cos\omega_D t + B\sin\omega_D t)}_{\text{transient}} + \underbrace{C\sin\omega t + D\cos\omega t}_{\text{steady state}}$$

$$u(t) = u_o \sin(\omega t - \phi) = (u_{\text{st}})_o R_d \sin(\omega t - \phi)$$

$u_o = \sqrt{C^2 + D^2}$ and $\phi = \tan^{-1}(-D/C)$.

$$R_d = \frac{u_o}{(u_{\text{st}})_o} = \frac{1}{\sqrt{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2}}$$

$$\phi = \tan^{-1}\frac{2\zeta(\omega/\omega_n)}{1 - (\omega/\omega_n)^2}$$

Response for $\omega = \omega_n$

for $\omega = \omega_n$ and zero initial conditions,

$A = (u_{\text{st}})_o/2\zeta$ and $B = (u_{\text{st}})_o/2\sqrt{1-\zeta^2}$.

$C = 0$ and $D = -(u_{\text{st}})_o/2\zeta$

$$u(t) = (u_{\text{st}})_o \frac{1}{2\zeta}\left[e^{-\zeta\omega_n t}\left(\cos\omega_D t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D t\right) - \cos\omega_n t\right]$$

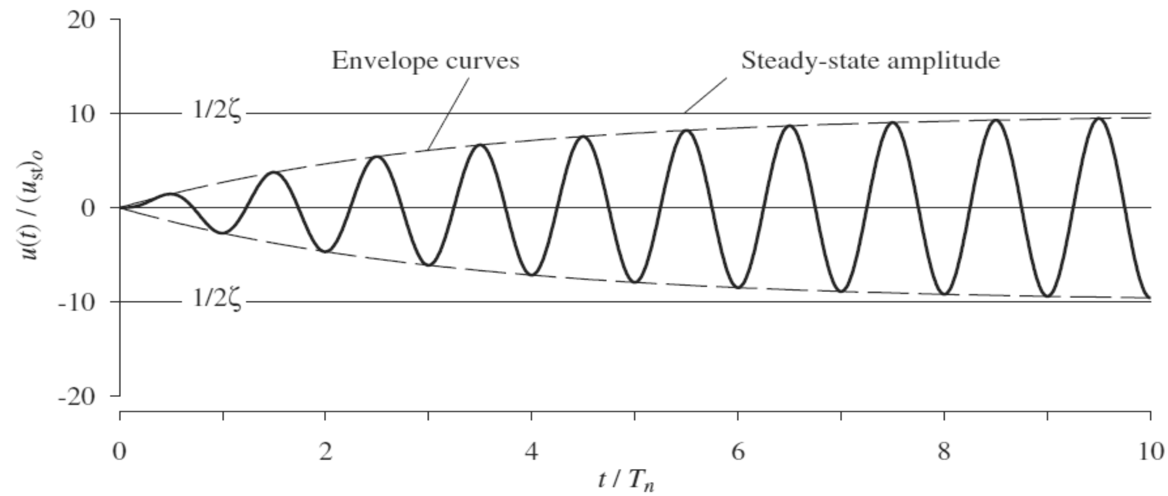


Figure 3.2.2 Response of damped system with $\zeta = 0.05$ to sinusoidal force of frequency $\omega = \omega_n$; $u(0) = \dot{u}(0) = 0$.

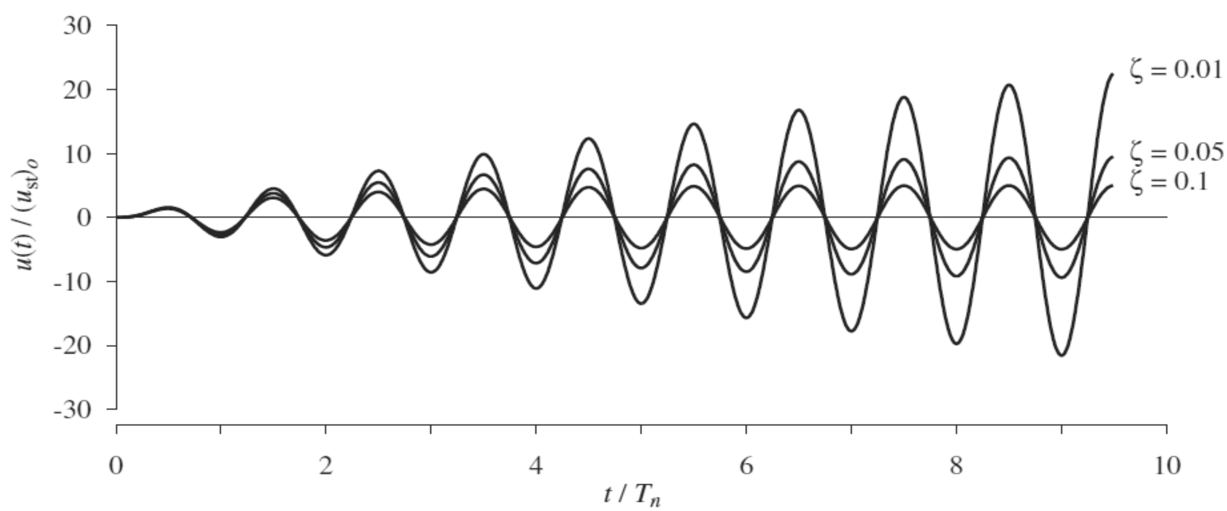


Figure 3.2.3 Response of three systems—$\zeta = 0.01$, 0.05, and 0.1—to sinusoidal force of frequency $\omega = \omega_n$; $u(0) = \dot{u}(0) = 0$.

Dynamic Response Factors

$$\frac{u(t)}{p_o/k} = R_d \sin(\omega t - \phi)$$

$$\frac{\dot{u}(t)}{p_o/\sqrt{km}} = R_v \cos(\omega t - \phi) \qquad R_v = \frac{\omega}{\omega_n} R_d$$

$$\frac{\ddot{u}(t)}{p_o/m} = -R_a \sin(\omega t - \phi) \qquad R_a = \left(\frac{\omega}{\omega_n}\right)^2 R_d$$

$$\frac{R_a}{\omega/\omega_n} = R_v = \frac{\omega}{\omega_n} R_d$$

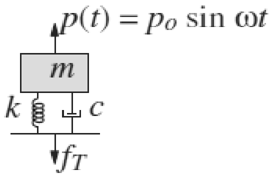
Resonant Frequencies and Resonant Responses

Displacement resonant frequency:	$\omega_n \sqrt{1 - 2\zeta^2}$
Velocity resonant frequency:	ω_n
Acceleration resonant frequency:	$\omega_n \div \sqrt{1 - 2\zeta^2}$

The three dynamic response factors at their respective resonant frequencies are

$$R_d = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad R_v = \frac{1}{2\zeta} \qquad R_a = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad (3.2.22)$$

FORCE TRANSMISSION AND VIBRATION ISOLATION



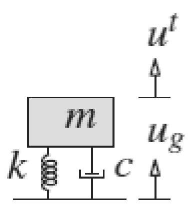
$$f_T = f_S + f_D = ku(t) + c\dot{u}(t)$$

$$f_T(t) = (u_{\mathrm{st}})_o R_d \left[k \sin(\omega t - \phi) + c\omega \cos(\omega t - \phi)\right]$$

$$(f_T)_o = (u_{\mathrm{st}})_o R_d \sqrt{k^2 + c^2\omega^2} \qquad \frac{(f_T)_o}{p_o} = R_d \sqrt{1 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}$$

$$\mathrm{TR} = \left\{ \frac{1 + [2\zeta(\omega/\omega_n)]^2}{[1 - (\omega/\omega_n)^2]^2 + [2\zeta(\omega/\omega_n)]^2} \right\}^{1/2}$$

RESPONSE TO GROUND MOTION AND VIBRATION ISOLATION



$$\ddot{u}_g(t) = \ddot{u}_{go} \sin \omega t \qquad u(t) = \frac{-m\ddot{u}_{go}}{k} R_d \sin(\omega t - \phi)$$

$$\ddot{u}^t(t) = \ddot{u}_g(t) + \ddot{u}(t)$$

$$\mathrm{TR} = \frac{\ddot{u}^t_o}{\ddot{u}_{go}} = \left\{ \frac{1 + [2\zeta(\omega/\omega_n)]^2}{\left[1 - (\omega/\omega_n)^2\right]^2 + [2\zeta(\omega/\omega_n)]^2} \right\}^{1/2}$$

$$\mathrm{TR} = \frac{u^t_o}{u_{go}} = \left\{ \frac{1 + [2\zeta(\omega/\omega_n)]^2}{\left[1 - (\omega/\omega_n)^2\right]^2 + [2\zeta(\omega/\omega_n)]^2} \right\}^{1/2}$$

Example 3.4

An automobile is traveling along a multispan elevated roadway supported every 100 ft. Long-term creep has resulted in a 6-in. deflection at the middle of each span (Fig. E3.4a). The roadway profile can be approximated as sinusoidal with an amplitude of 3 in. and a period of 100 ft. The SDF system shown is a simple idealization of an automobile, appropriate for a "first approximation" study of the ride quality of the vehicle. When fully loaded, the weight of the automobile is 4 kips. The stiffness of the automobile suspension system is 800 lb/in., and its viscous damping coefficient is such that the damping ratio of the system is 40%. Determine

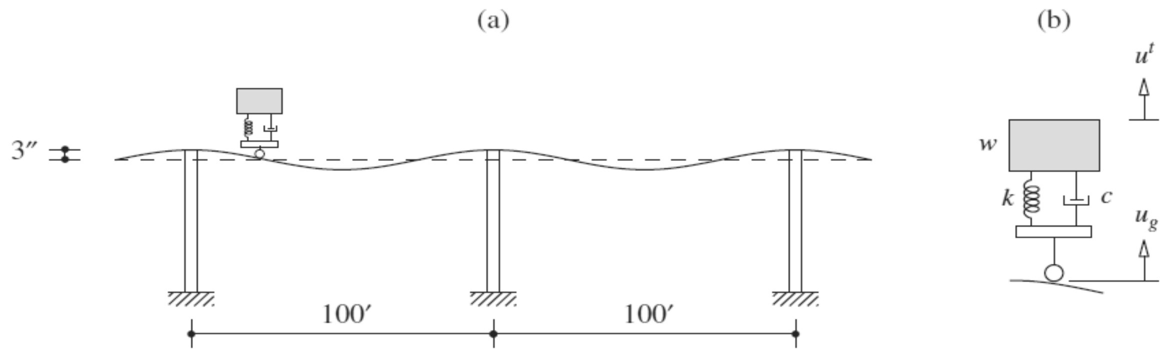


Figure E3.4

$$\omega = 2\pi/T, \quad T = L/v \quad \omega = 2\pi v/L$$

(a) *Determine* u_o^t.

$$v = 40 \text{ mph} = 58.67 \text{ ft/sec} \qquad \omega = \frac{2\pi(58.67)}{100} = 3.686 \text{ rad/sec}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{800}{4000/386}} = 8.786 \text{ rad/sec} \qquad \frac{\omega}{\omega_n} = 0.420$$

$$\frac{u_o^t}{u_{go}} = \left\{ \frac{1 + [2(0.4)(0.420)]^2}{[1 - (0.420)^2]^2 + [2(0.4)(0.420)]^2} \right\}^{1/2} = 1.186$$

$$u_o^t = 1.186 u_{go} = 1.186(3) = 3.56 \text{ in.}$$

(b) *Determine the speed at resonance.* If ζ were small, resonance would occur approximately at $\omega/\omega_n = 1$. However, automobile suspensions have heavy damping, to reduce vibration. In this case, $\zeta = 0.4$, and for such large damping the resonant frequency is significantly different from ω_n. By definition, resonance occurs for u_o^t when TR (or TR^2) is maximum over all ω. Substituting $\zeta = 0.4$ in Eq. (3.6.5) and introducing $\beta = \omega/\omega_n$ gives

$$\text{TR}^2 = \frac{1 + 0.64\beta^2}{(1 - 2\beta^2 + \beta^4) + 0.64\beta^2} = \frac{1 + 0.64\beta^2}{\beta^4 - 1.36\beta^2 + 1}$$

$$\frac{d(\text{TR})^2}{d\beta} = 0 \Rightarrow \beta = 0.893 \Rightarrow \omega = 0.893\omega_n = 0.893(8.786) = 7.846 \text{ rad/sec}$$

Resonance occurs at this forcing frequency, which implies a speed of

$$v = \frac{\omega L}{2\pi} = \frac{(7.846)100}{2\pi} = 124.9 \text{ ft/sec} = 85 \text{ mph}$$

RESPONSE TO PERIODIC EXCITATION

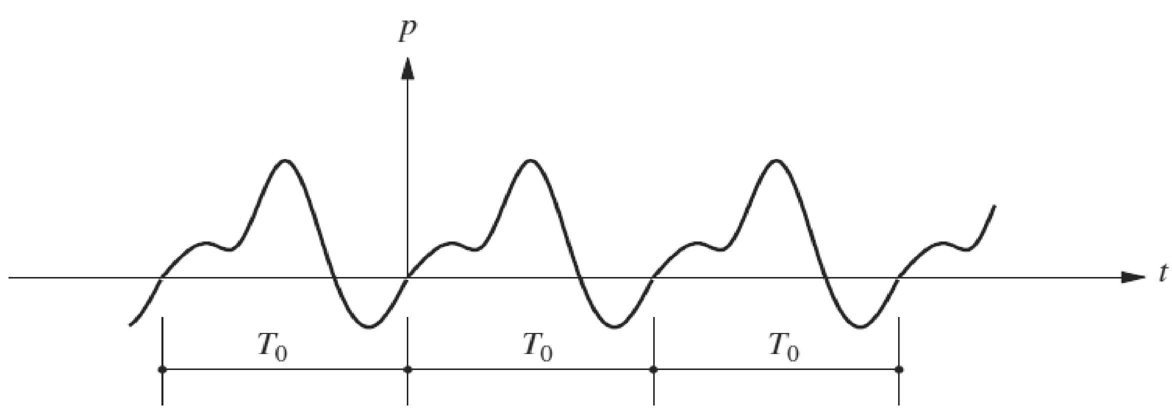


Figure 3.12.1 Periodic excitation.

$$p(t + jT_0) = p(t) \qquad j = -\infty, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \infty$$

$$p(t + jT_0) = p(t) \qquad j = -\infty, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, \infty$$

$$p(t) = a_0 + \sum_{j=1}^{\infty} a_j \cos j\omega_0 t + \sum_{j=1}^{\infty} b_j \sin j\omega_0 t \qquad \omega_0 = \frac{2\pi}{T_0}$$

$$a_0 = \frac{1}{T_0} \int_0^{T_0} p(t)\, dt$$

$$a_j = \frac{2}{T_0} \int_0^{T_0} p(t) \cos j\omega_0 t\, dt \qquad j = 1, 2, 3, \ldots$$

$$b_j = \frac{2}{T_0} \int_0^{T_0} p(t) \sin j\omega_0 t\, dt \qquad j = 1, 2, 3, \ldots$$

$$u(t) = u_0(t) + \sum_{j=1}^{\infty} u_j^c(t) + \sum_{j=1}^{\infty} u_j^s(t)$$

$$u_0(t) = \frac{a_0}{k}$$

$$u_j^c(t) = \frac{a_j}{k} \frac{2\zeta\beta_j \sin j\omega_0 t + (1 - \beta_j^2) \cos j\omega_0 t}{(1 - \beta_j^2)^2 + (2\zeta\beta_j)^2}$$

$$\beta_j = \frac{j\omega_0}{\omega_n}$$

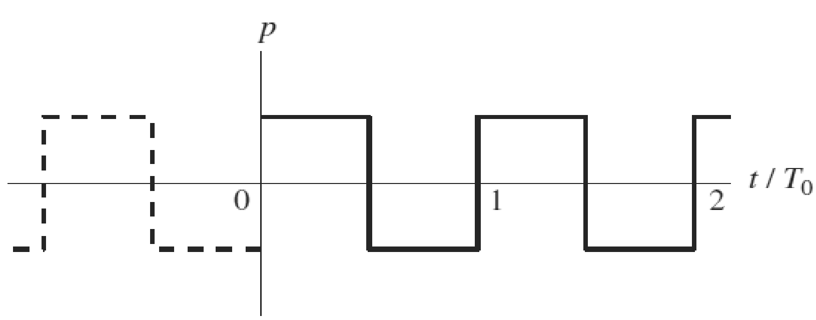
$$u_j^s(t) = \frac{b_j}{k} \frac{(1 - \beta_j^2) \sin j\omega_0 t - 2\zeta\beta_j \cos j\omega_0 t}{(1 - \beta_j^2)^2 + (2\zeta\beta_j)^2}$$

$$u(t) = \frac{a_0}{k} + \sum_{j=1}^{\infty} \frac{1}{k} \frac{1}{(1-\beta_j^2)^2 + (2\zeta\beta_j)^2} \left\{ \left[a_j(2\zeta\beta_j) + b_j(1-\beta_j^2) \right] \sin j\omega_0 t + \left[a_j(1-\beta_j^2) - b_j(2\zeta\beta_j) \right] \cos j\omega_0 t \right\} \quad (3.13.6)$$

Example 3.8

The periodic force shown in Fig. E3.8a is defined by

$$p(t) = \begin{cases} p_o & 0 \le t \le T_0/2 \\ -p_o & T_0/2 \le t \le T_0 \end{cases}$$



(a)

$$a_0 = \frac{1}{T_0} \int_0^{T_0} p(t)\, dt = 0$$

$$a_j = \frac{2}{T_0} \int_0^{T_0} p(t) \cos j\omega_0 t\, dt$$

$$= \frac{2}{T_0} \left[p_o \int_0^{T_0/2} \cos j\omega_0 t\, dt + (-p_o) \int_{T_0/2}^{T_0} \cos j\omega_0 t\, dt \right] = 0$$

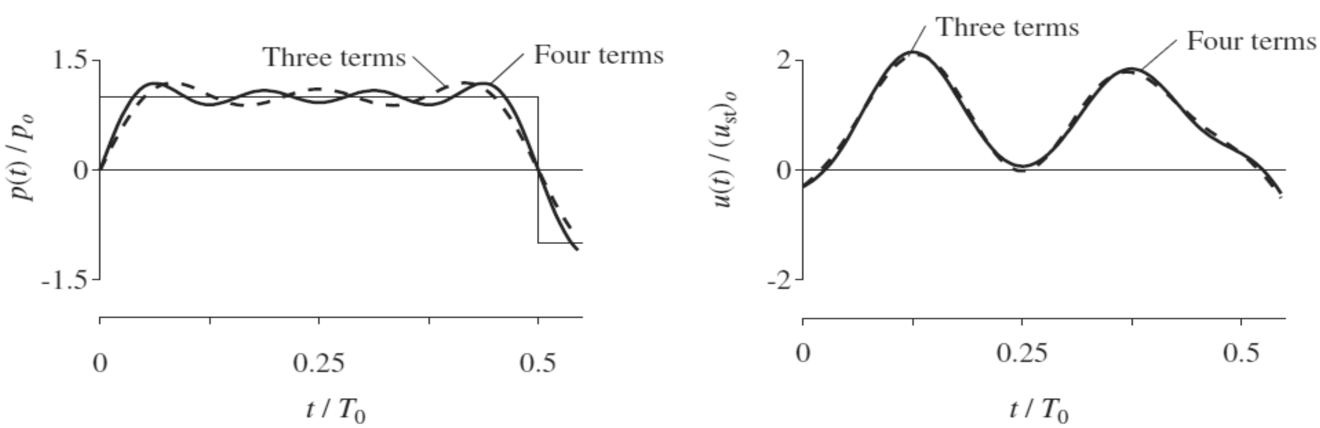
$$b_j = \frac{2}{T_0} \int_0^{T_0} p(t) \sin j\omega_0 t\, dt$$

$$= \frac{2}{T_0} \left[p_o \int_0^{T_0/2} \sin j\omega_0 t\, dt + (-p_o) \int_{T_0/2}^{T_0} \sin j\omega_0 t\, dt \right]$$

$$= \begin{cases} 0 & j \text{ even} \\ 4p_o/j\pi & j \text{ odd} \end{cases}$$

$$p(t) = \sum p_j(t) = \frac{4p_o}{\pi} \sum_{j=1,3,5}^{\infty} \frac{1}{j} \sin j\omega_0 t$$

$$u(t) = (u_{\text{st}})_o \frac{4}{\pi} \sum_{j=1,3,5}^{\infty} \frac{1}{j} \frac{(1-\beta_j^2)\sin j\omega_0 t - 2\zeta\beta_j \cos j\omega_0 t}{(1-\beta_j^2)^2 + (2\zeta\beta_j)^2}$$



3.5 An air-conditioning unit weighing 1200 lb is bolted at the middle of two parallel simply supported steel beams (Fig. P3.5). The clear span of the beams is 8 ft. The second moment of cross-sectional area of each beam is 10 in^4. The motor in the unit runs at 300 rpm and produces an unbalanced vertical force of 60 lb at this speed. Neglect the weight of the beams and assume 1% viscous damping in the system; for steel $E = 30{,}000$ ksi. Determine the amplitudes of steady-state deflection and steady-state acceleration (in g's) of the beams at their midpoints which result from the unbalanced force.

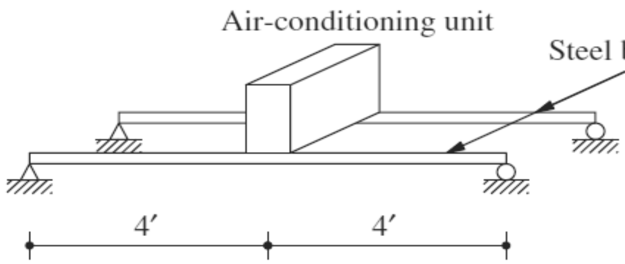


Figure P3.5

3.6 **(a)** Show that the steady-state response of an SDF system to a cosine force, $p(t) = p_o \cos \omega t$, is given by

$$u(t) = \frac{p_o}{k} \frac{\left[1 - (\omega/\omega_n)^2\right] \cos \omega t + \left[2\zeta (\omega/\omega_n)\right] \sin \omega t}{\left[1 - (\omega/\omega_n)^2\right]^2 + \left[2\zeta (\omega/\omega_n)\right]^2}$$

(b) Show that the maximum deformation due to cosine force is the same as that due to sinusoidal force.

3.7 **(a)** Show that $\omega_r = \omega_n (1 - 2\zeta^2)^{1/2}$ is the resonant frequency for displacement amplitude of an SDF system.
(b) Determine the displacement amplitude at resonance.

3.8 **(a)** Show that $\omega_r = \omega_n (1 - 2\zeta^2)^{-1/2}$ is the resonant frequency for acceleration amplitude of an SDF system.
(b) Determine the acceleration amplitude at resonance.

3.26 An SDF system with natural period T_n and damping ratio ζ is subjected to the periodic force shown in Fig. P3.26 with an amplitude p_o and period T_0.
(a) Expand the forcing function in its Fourier series.
(b) Determine the steady-state response of an undamped system. For what values of T_0 is the solution indeterminate?
(c) For $T_0/T_n = 2$, determine and plot the response to individual terms in the Fourier series. How many terms are necessary to obtain reasonable convergence of the series solution?

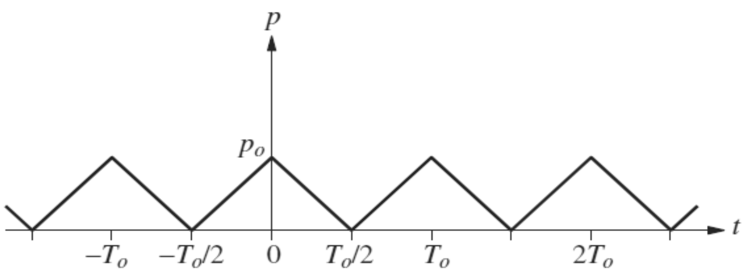


Figure P3.26

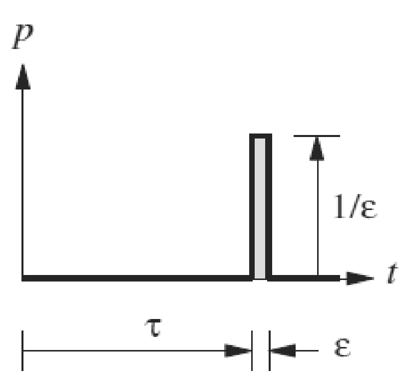
4

Response to Arbitrary, Step, and Pulse Excitations

RESPONSE TO UNIT IMPULSE

A very large force that acts for a very short time but with a time integral that is finite is called an *impulsive* force. Shown in Fig. 4.1.1 is the force $p(t) = 1/\varepsilon$, with time duration ε starting at the time instant $t = \tau$. As ε approaches zero the force becomes infinite; however, the *magnitude of the impulse*, defined by the time integral of $p(t)$, remains equal to unity. Such a force in the limiting case $\varepsilon \to 0$ is called the *unit impulse*. The *Dirac delta function* $\delta(t - \tau)$ mathematically defines a unit impulse centered at $t = \tau$.

(a)



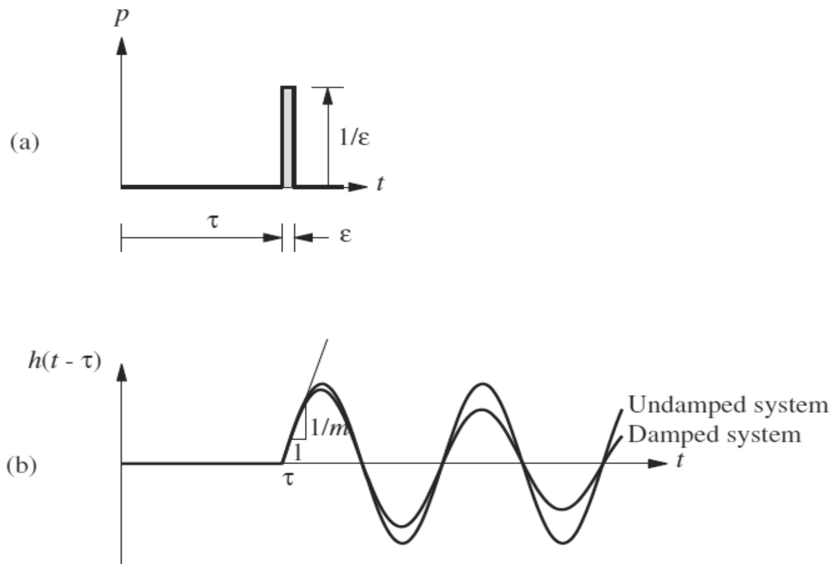


Figure 4.1.1 (a) Unit impulse; (b) response to unit impulse.

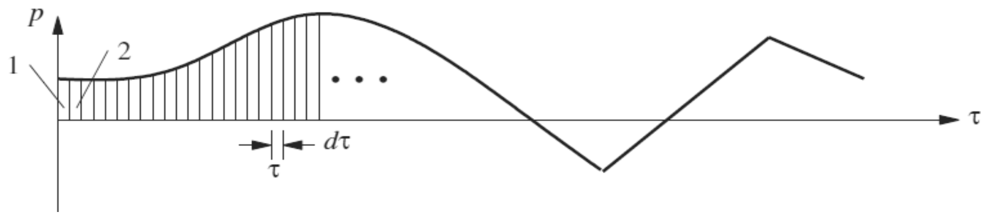
$$\frac{d}{dt}(m\dot{u}) = p \qquad \int_{t_1}^{t_2} p\,dt = m(\dot{u}_2 - \dot{u}_1) = m\,\Delta\dot{u} \qquad \dot{u}(\tau) = \frac{1}{m}$$

$$u(\tau) = 0$$

$$h(t-\tau) \equiv u(t) = \frac{1}{m\omega_D} e^{-\zeta\omega_n(t-\tau)} \sin[\omega_D(t-\tau)] \qquad t \geq \tau$$

unit impulse-response function

RESPONSE TO ARBITRARY FORCE



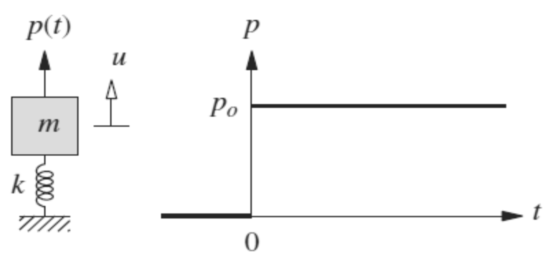
$$du(t) = [p(\tau)\,d\tau]h(t-\tau) \qquad t > \tau \qquad\qquad u(t) = \int_0^t p(\tau)h(t-\tau)\,d\tau$$

$$u(t) = \frac{1}{m\omega_D}\int_0^t p(\tau)e^{-\zeta\omega_n(t-\tau)}\sin[\omega_D(t-\tau)]\;d\tau$$

Duhamel's integral

$$u(t) = \frac{1}{m\omega_n}\int_0^t p(\tau)\sin[\omega_n(t-\tau)]\,d\tau$$

STEP FORCE



The equation of motion has been solved (Section 1.10.2) using Duhamel's integral to obtain

$$u(t) = (u_{st})_o(1 - \cos\omega_n t) = (u_{st})_o\left(1 - \cos\frac{2\pi t}{T_n}\right) \qquad (4.3.2)$$

where $(u_{st})_o = p_o/k$, the static deformation due to force p_o.

$$m\ddot{u} + c\dot{u} + ku = p_o \qquad u_p = p_o/k$$

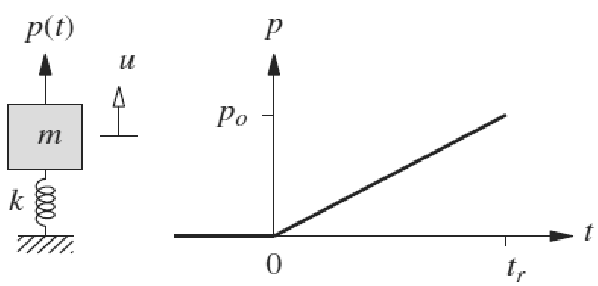
$$u(t) = e^{-\zeta\omega_n t}(A\cos\omega_D t + B\sin\omega_D t) + \frac{p_o}{k}$$

$$u(0) = \dot{u}(0) = 0$$

$$A = -\frac{p_o}{k} \qquad B = -\frac{p_o}{k}\frac{\zeta}{\sqrt{1-\zeta^2}}$$

$$u(t) = (u_{\text{st}})_o\left[1 - e^{-\zeta\omega_n t}\left(\cos\omega_D t + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_D t\right)\right]$$

RAMP OR LINEARLY INCREASING FORCE



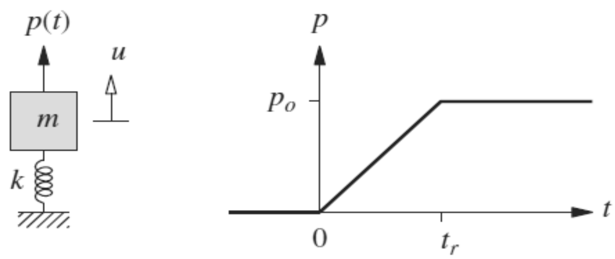
$$p(t) = p_o\frac{t}{t_r}$$

Duhamel's integral

$$u(t) = \frac{1}{m\omega_n}\int_0^t \frac{p_o}{t_r}\tau\sin\omega_n(t-\tau)\,d\tau$$

$$u(t) = (u_{\text{st}})_o\left(\frac{t}{t_r} - \frac{\sin\omega_n t}{\omega_n t_r}\right) = (u_{\text{st}})_o\left(\frac{t}{T_n}\frac{T_n}{t_r} - \frac{\sin 2\pi t/T_n}{2\pi t_r/T_n}\right)$$

STEP FORCE WITH FINITE RISE TIME



$$p(t) = \begin{cases} p_o(t/t_r) & t \le t_r \\ p_o & t \ge t_r \end{cases}$$

$$u(t) = (u_{\text{st}})_o \left(\frac{t}{t_r} - \frac{\sin \omega_n t}{\omega_n t_r} \right) \qquad t \le t_r$$

$$u(t) = u(t_r) \cos \omega_n (t - t_r) + \frac{\dot{u}(t_r)}{\omega_n} \sin \omega_n (t - t_r) + (u_{\text{st}})_o [1 - \cos \omega_n (t - t_r)]$$

RESPONSE TO PULSE EXCITATIONS

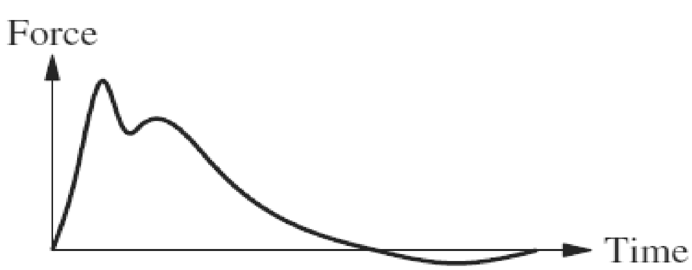
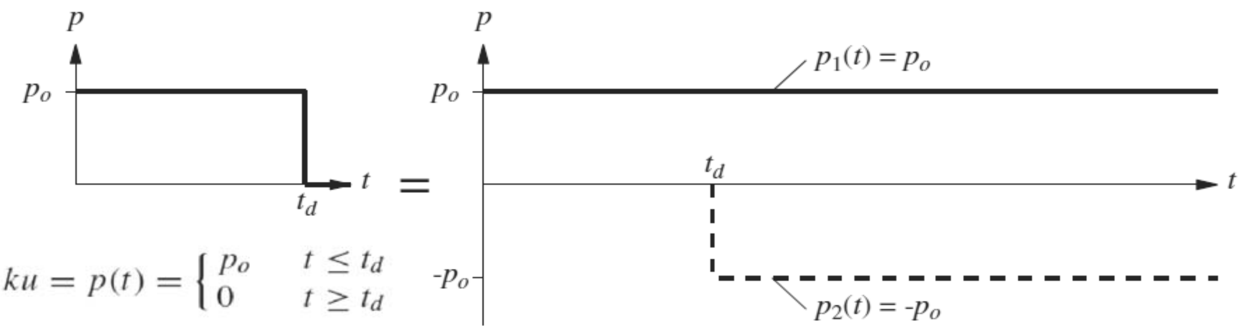


Figure 4.6.1 Single-pulse excitation.

(a)



$$m\ddot{u} + ku = p(t) = \begin{cases} p_o & t \le t_d \\ 0 & t \ge t_d \end{cases}$$

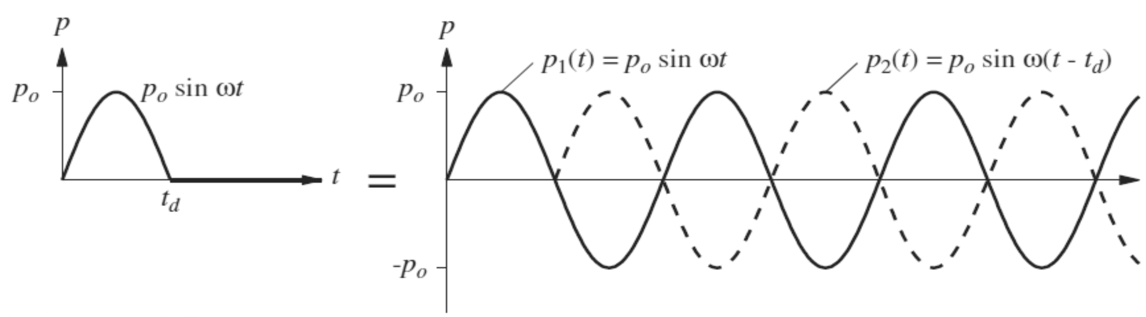
1. *Forced vibration phase.* During this phase, the system is subjected to a step force. The response of the system is given by Eq. (4.3.2), repeated for convenience:

$$\frac{u(t)}{(u_{st})_o} = 1 - \cos\omega_n t = 1 - \cos\frac{2\pi t}{T_n} \qquad t \le t_d \tag{4.7.2}$$

2. *Free vibration phase.* After the force ends at t_d, the system undergoes free vibration, defined by modifying Eq. (2.1.3) appropriately:

$$u(t) = u(t_d)\cos\omega_n(t - t_d) + \frac{\dot{u}(t_d)}{\omega_n}\sin\omega_n(t - t_d) \tag{4.7.3}$$

(b)



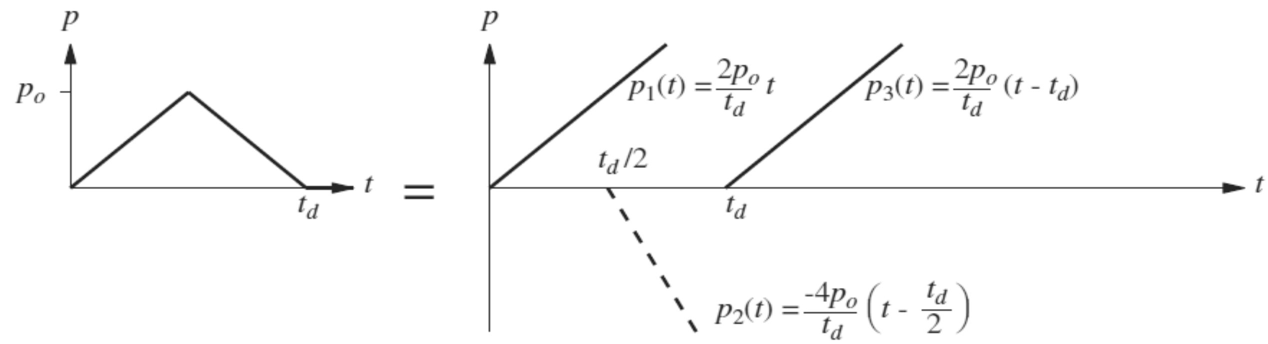
Forced Vibration Phase.

$$\frac{u(t)}{(u_{st})_o} = \frac{1}{1 - (T_n/2t_d)^2}\left[\sin\left(\pi\frac{t}{t_d}\right) - \frac{T_n}{2t_d}\sin\left(2\pi\frac{t}{T_n}\right)\right] \qquad t \le t_d$$

Free Vibration Phase.

$$\frac{u(t)}{(u_{st})_o} = \frac{(T_n/t_d)\cos(\pi t_d/T_n)}{(T_n/2t_d)^2 - 1}\sin\left[2\pi\left(\frac{t}{T_n} - \frac{1}{2}\frac{t_d}{T_n}\right)\right] \qquad t \ge t_d$$

(c)



$$\frac{u(t)}{(u_{\mathrm{st}})_o} = \begin{cases} 2\left(\dfrac{t}{t_d} - \dfrac{T_n}{2\pi t_d}\sin 2\pi\dfrac{t}{T_n}\right) & 0 \le t \le \dfrac{t_d}{2} \\ 2\left\{1 - \dfrac{t}{t_d} + \dfrac{T_n}{2\pi t_d}\left[2\sin\dfrac{2\pi}{T_n}\left(t - \dfrac{1}{2}t_d\right) - \sin 2\pi\dfrac{t}{T_n}\right]\right\} & \dfrac{t_d}{2} \le t \le t_d \\ 2\left\{\dfrac{T_n}{2\pi t_d}\left[2\sin\dfrac{2\pi}{T_n}\left(t - \dfrac{1}{2}t_d\right) - \sin\dfrac{2\pi}{T_n}(t - t_d) - \sin 2\pi\dfrac{t}{T_n}\right]\right\} & t \ge t_d \end{cases}$$

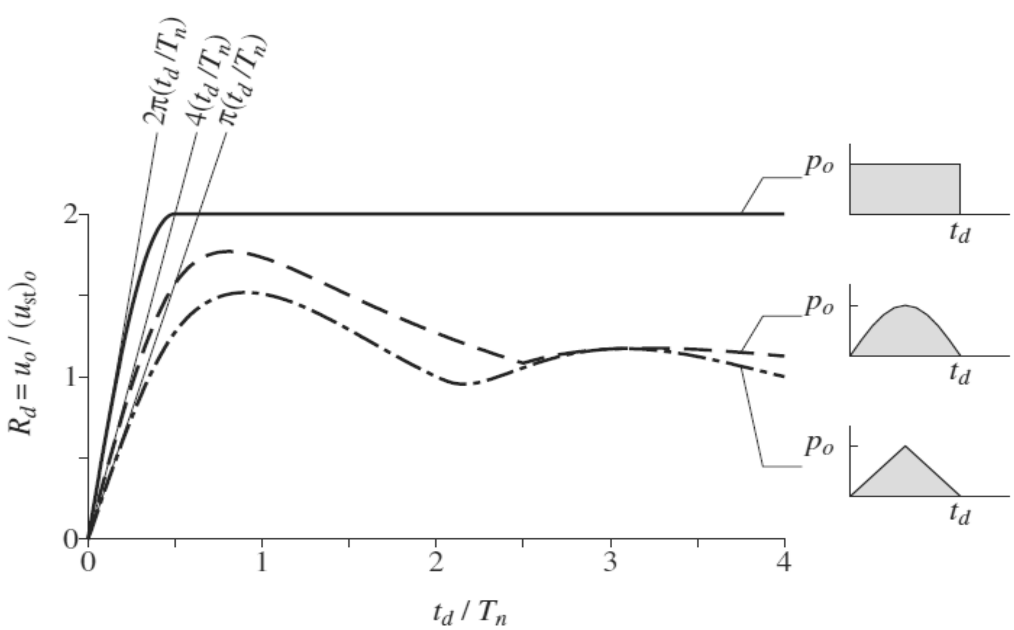


Figure 4.10.1 Shock spectra for three force pulses of equal amplitude.

4.1 Show that the maximum deformation u_0 of an SDF system due to a unit impulse force, $p(t) = \delta(t)$, is

$$u_o = \frac{1}{m\omega_n} \exp\left(-\frac{\zeta}{\sqrt{1-\zeta^2}} \tan^{-1} \frac{\sqrt{1-\zeta^2}}{\zeta}\right)$$

Plot this result as a function of ζ. Comment on the influence of damping on the maximum response.

4.2 Consider the deformation response $g(t)$ of an SDF system to a unit step function $p(t) = 1$, $t \geq 0$, and $h(t)$ due to a unit impulse $p(t) = \delta(t)$. Show that $h(t) = \dot{g}(t)$.

4.3 An SDF undamped system is subjected to a force $p(t)$ consisting of a sequence of two impulses, each of magnitude I, as shown in Fig. P4.3.
(a) Plot the displacement response of the system for $t_d/T_n = \frac{1}{8}, \frac{1}{4}$, and 1. For each case show the response to individual impulses and the combined response.

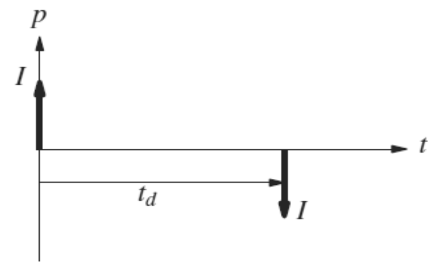


Figure P4.3

5

Numerical Evaluation of Dynamic Response

TIME-STEPPING METHODS

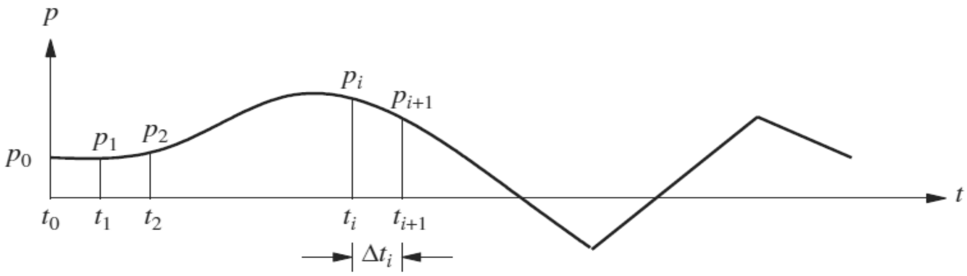
For an inelastic system the equation of motion to be solved numerically is

$$m\ddot{u} + c\dot{u} + f_S(u) = p(t) \qquad \text{or} \qquad -m\ddot{u}_g(t)$$

subject to the initial conditions

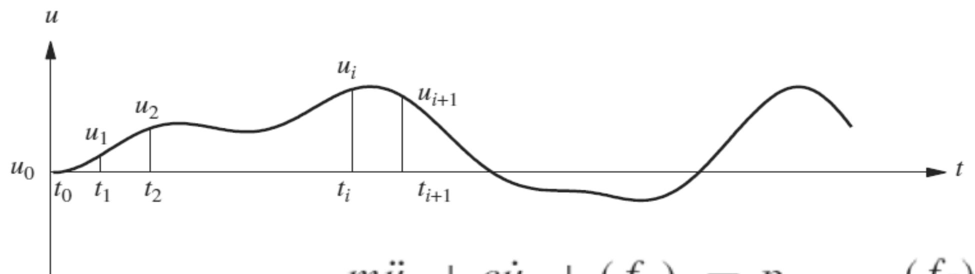
$$u_0 = u(0) \qquad \dot{u}_0 = \dot{u}(0)$$

The system is assumed to have linear viscous damping, but other forms of damping, including nonlinear damping, could be considered, as will become obvious later. However, this is rarely done for lack of information on damping, especially at large amplitudes of motion. The applied force $p(t)$ is given by a set of discrete values $p_i = p(t_i)$, $i = 0$ to N



$$\Delta t_i = t_{i+1} - t_i$$

$$m\ddot{u}_{i+1} + c\dot{u}_{i+1} + (f_S)_{i+1} = p_{i+1}$$



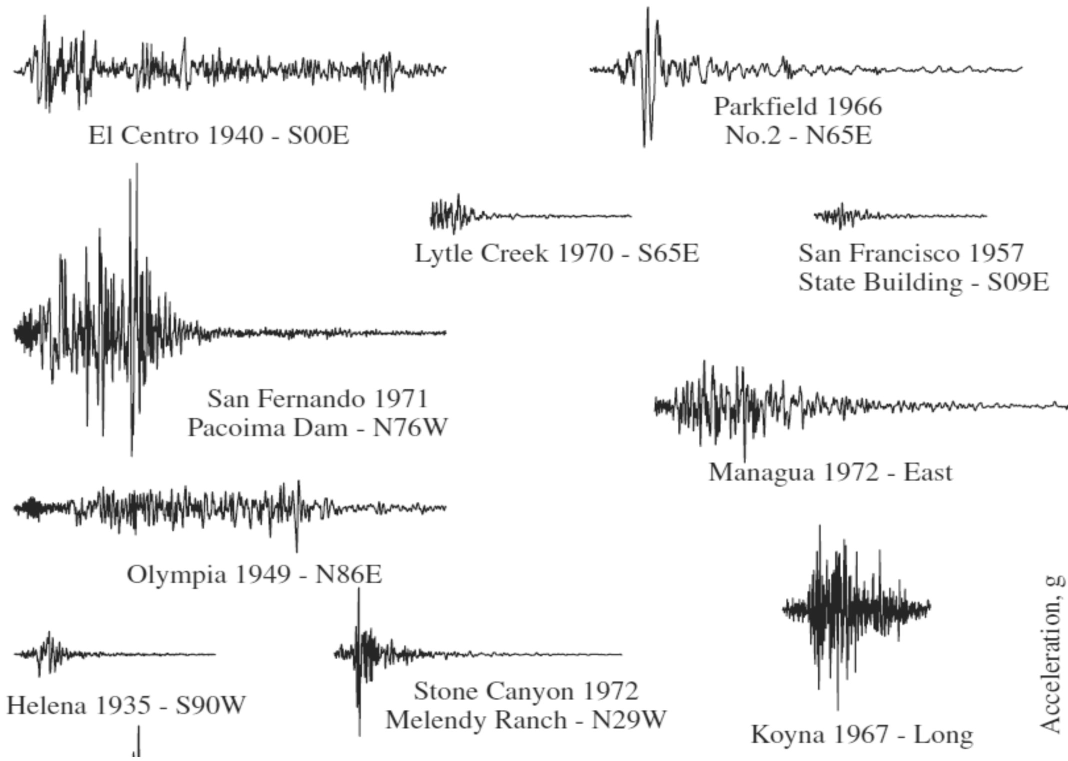
$$m\ddot{u}_i + c\dot{u}_i + (f_S)_i = p_i \qquad (f_S)_i = ku_i$$

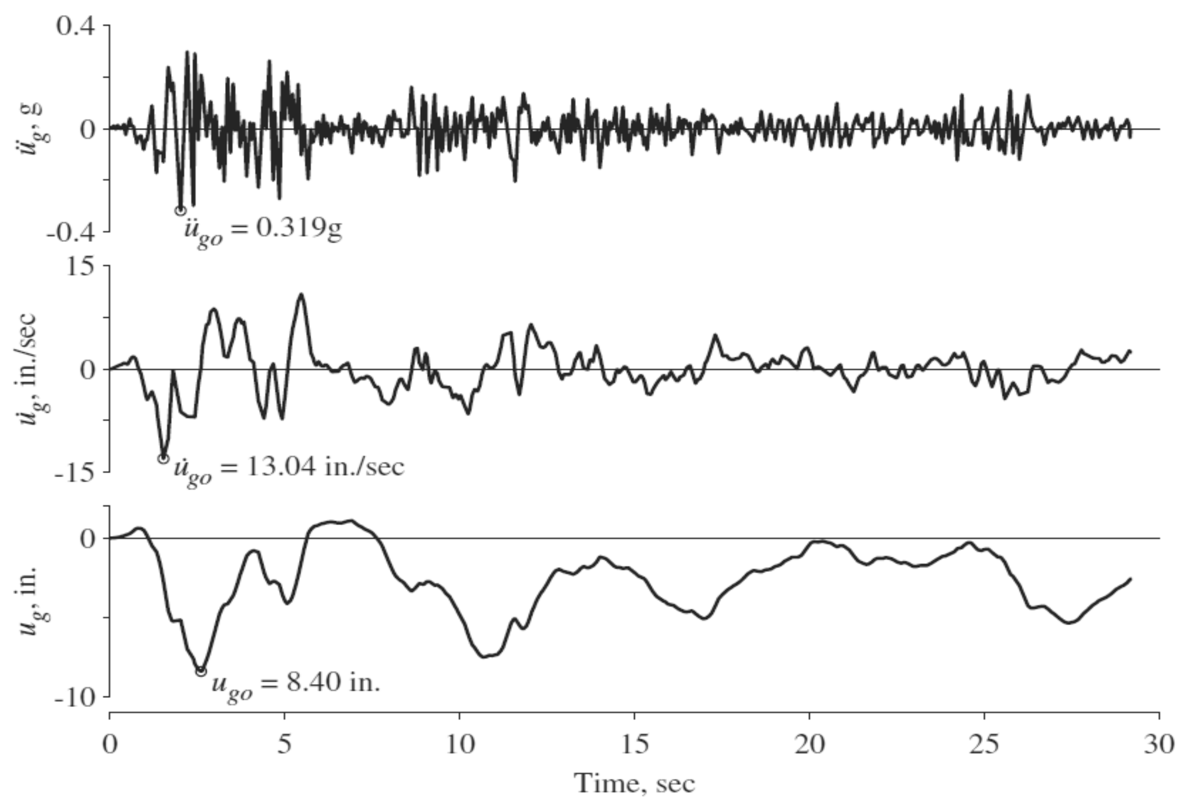
Stepping from time i to $i + 1$ is usually not an exact procedure. Many approximate procedures are possible that are implemented numerically. The three important requirements for a numerical procedure are (1) convergence—as the time step decreases, the numerical solution should approach the exact solution, (2) stability—the numerical solution should be stable in the presence of numerical round-off errors, and (3) accuracy—the numerical procedure should provide results that are close enough to the exact solution. These important issues are discussed briefly in this book; comprehensive treatments are available in books emphasizing numerical solution of differential equations.

Three types of time-stepping procedures are presented in this chapter: (1) methods based on interpolation of the excitation function, (2) methods based on finite difference expressions of velocity and acceleration, and (3) methods based on assumed variation of acceleration. Only one method is presented in each of the first two categories and two from the third group.

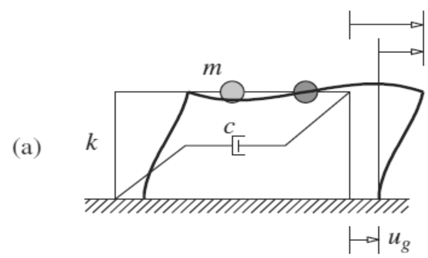
6

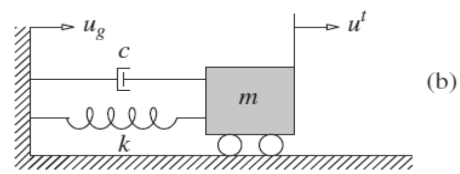
Earthquake Response of Linear Systems


El Centro 1940 - S00E
Parkfield 1966
No.2 - N65E
Lytle Creek 1970 - S65E
San Francisco 1957
State Building - S09E
San Fernando 1971
Pacoima Dam - N76W
Managua 1972 - East
Olympia 1949 - N86E
Helena 1935 - S90W
Stone Canyon 1972
Melendy Ranch - N29W
Koyna 1967 - Long
Acceleration, g


0.4
0
-0.4
$\ddot{u}_g$, g
$\ddot{u}_{go}$ = 0.319g
15
0
-15
$\dot{u}_g$, in./sec
$\dot{u}_{go}$ = 13.04 in./sec
0
-10
u_g, in.
u_{go} = 8.40 in.
0
5
10
15
20
25
30
Time, sec

تحلیل طیفی یا شبه دینامیکی سیستم های یک درجهٔ آزادی تحت اثر زلزله





$$m\ddot{u} + c\dot{u} + ku = -m\ddot{u}_g(t)$$

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g(t)$$

$$u \equiv u(t, T_n, \zeta)$$

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = -\ddot{u}_g(t)$$

$$u(t) = \frac{1}{m\omega_D}\int_0^t [-m\ddot{u}_g(t)]e^{-\zeta\omega_D(t-\tau)}\sin[\omega_D(t-\tau)]d\tau$$

با توجه به اینکه m مستقل از زمان میباشد و $\omega_D \simeq \omega_n$ خواهیم داشت

$$u(t) = -\frac{1}{\omega_n}\int_0^t \ddot{u}_g(t)e^{-\zeta\omega_n(t-\tau)}\sin[\omega_n(t-\tau)]d\tau$$

اگر عبارت انتگرال مربوط به رابطهٔ فوق $V(t)-$ در نظر گرفته شود خواهیم داشت:

$$u(t)=\frac{V(t)}{\omega_n}\rightarrow V(t)=\omega_n u(t)$$

$V(t)$ شبه سرعت نامیده میشود (چون با توجه به رابطهٔ فوق دارای واحد طول بر زمان

میباشد) مقدار حداکثر $S_V, V(t)$ (طیف سرعت) نام دارد.

$V_{max}(t)=S_v$

$V(t)=\omega_n u(t)$

$$[u(t)]_{max}=\frac{[V(t)]_{max}}{\omega_n}$$

مقدار حداکثر $u(t)$، S_d (طیف تغییر مکان) نامیده میشود.

$$S_d=[u(t)]_{max}=\frac{Sv}{\omega_n}$$

نیروی معادل استاتیکی، f_s

$$f_s = ku(t)$$

$$(f_s)_{\max} = ku_{\max} = kS_d$$

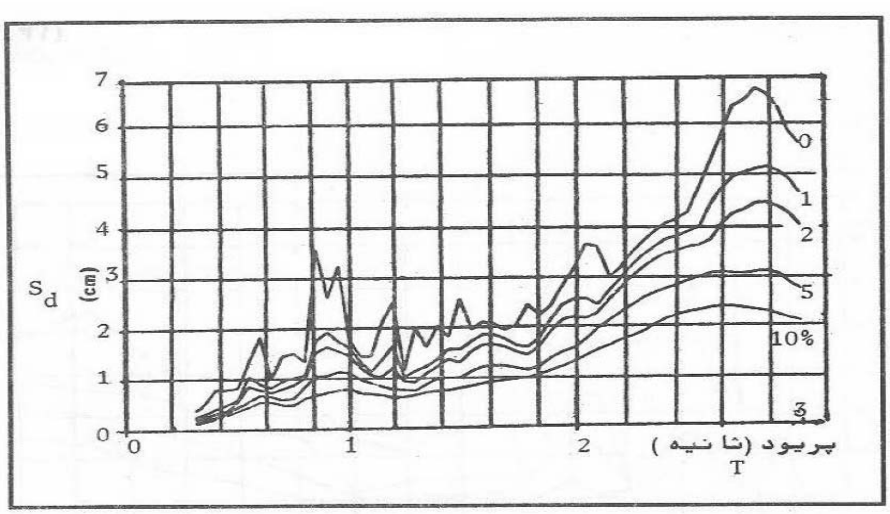
$$\omega_n^2 = k/m \rightarrow k = m\omega_n^2 \rightarrow$$

$$(f_s)_{\max} = m\omega_n^2 S_d$$

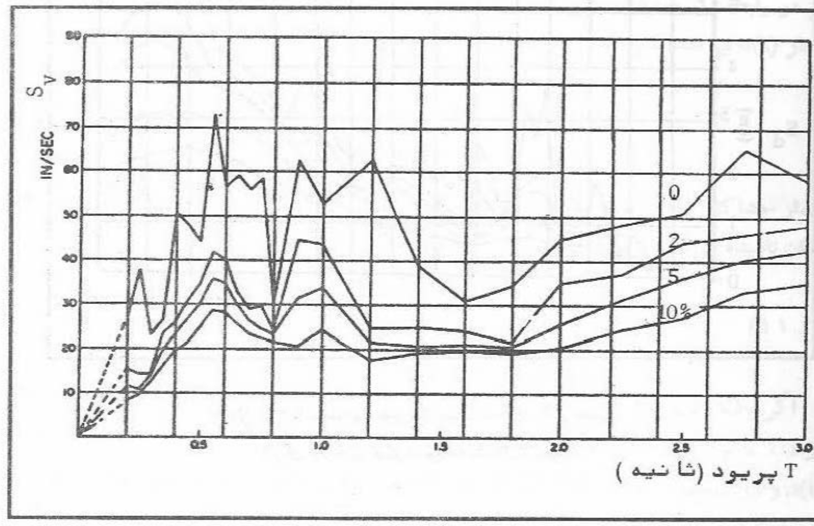
اگر نیروی وارد بر سازه را حاصل ضرب جرم در شتاب مؤثر وارد برسازه بدانیم، در رابطه فوق

$\omega_n^2 S_d$ را میتوان شتاب مؤثر فرض کرده و با S_a (طیف شتاب) نشان داد.

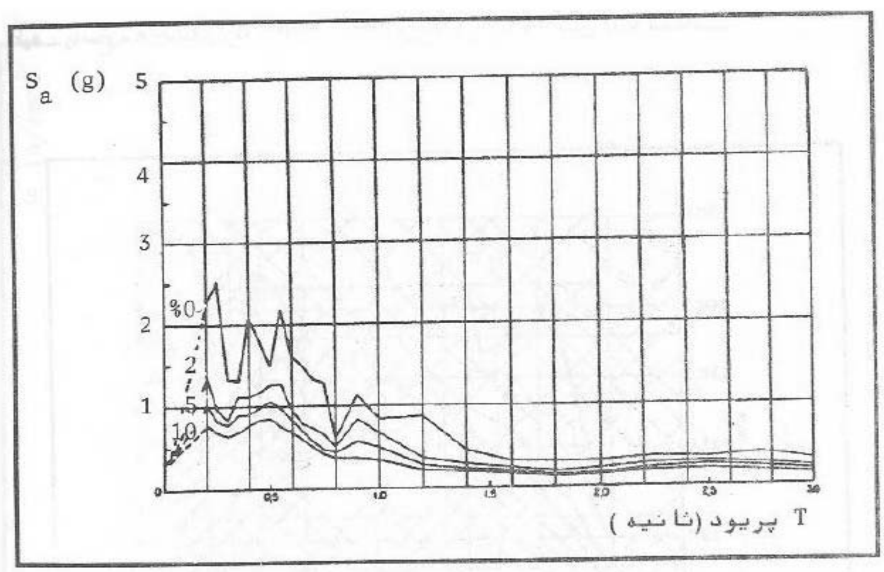
$$(f_s)_{\max} = (f_I)_{\max} = m\omega_n^2 S_d = mS_a$$



شکل ۱۱۱ ـ منحنی تغییرات طیف مکان S_d بر حسب T
(برای زلزله‌ال سنترو ـ مولفه شمال ـ جنوب)



شکل ۱۱۲ ـ منحنی تغییرات شبه سرعت S_v بر حسب T برای زلزله‌ال سنترو (N-S)



شکل ۱۱۳ ـ منحنی تغییرات طیف شتاب S_a بر حسب T برای زلزله‌ال سنترو

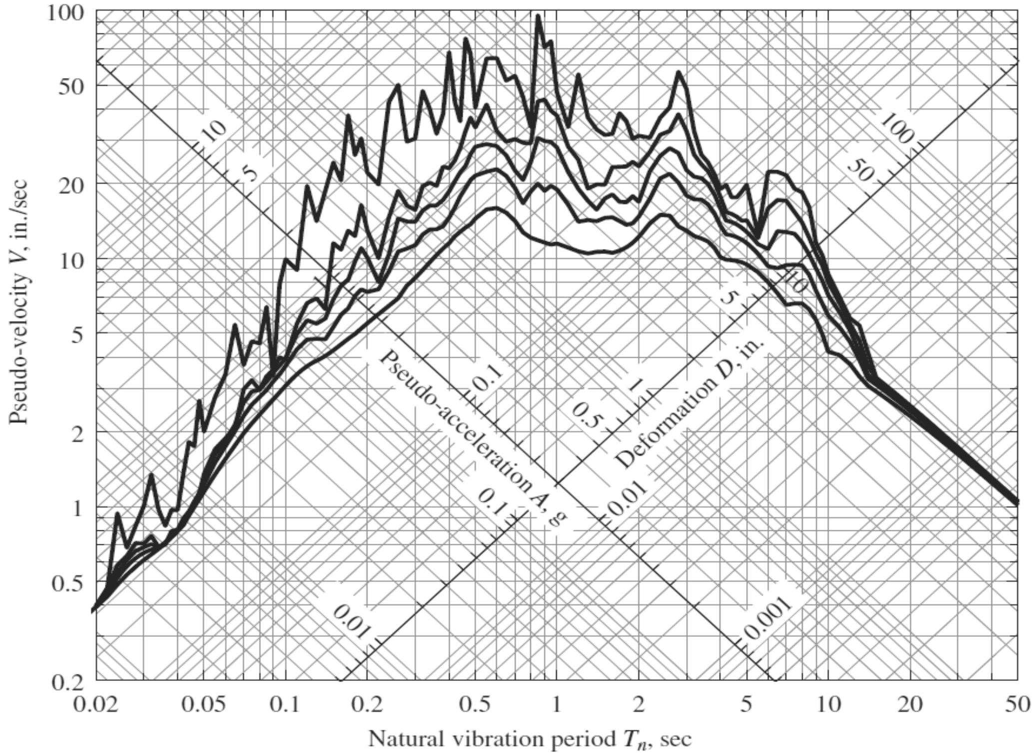


Figure 6.6.4 Combined D–V–A response spectrum for El Centro ground motion; $\zeta = 0$, 2, 5, 10, and 20%.

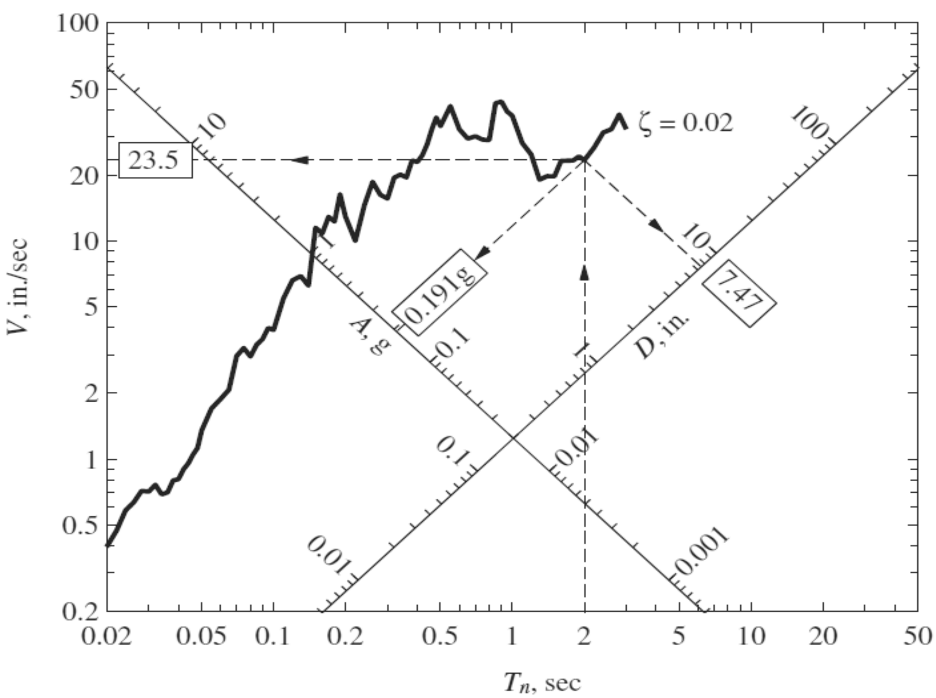


Figure 6.6.3 Combined D–V–A response spectrum for El Centro ground motion; $\zeta = 2\%$.

PEAK STRUCTURAL RESPONSE FROM THE RESPONSE SPECTRUM

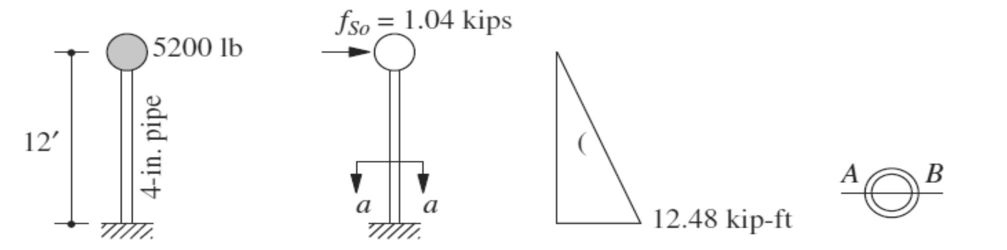
$$S_v = \omega_n S_d = S_a / \omega_n$$

$$u_o = D = \frac{T_n}{2\pi} V = \left(\frac{T_n}{2\pi} \right)^2 A$$

$$f_{So} = kD = mA$$

Example 6.2

A 12-ft-long vertical cantilever, a 4-in.-nominal-diameter standard steel pipe, supports a 5200-lb weight attached at the tip as shown in Fig. E6.2. The properties of the pipe are: outside diameter, $d_o = 4.500$ in., inside diameter $d_i = 4.026$ in., thickness $t = 0.237$ in., and second moment of cross-sectional area, $I = 7.23$ in^4, elastic modulus $E = 29{,}000$ ksi, and weight = 10.79 lb/foot length. Determine the peak deformation and bending stress in the cantilever due to the El Centro ground motion. Assume that $\zeta = 2\%$.

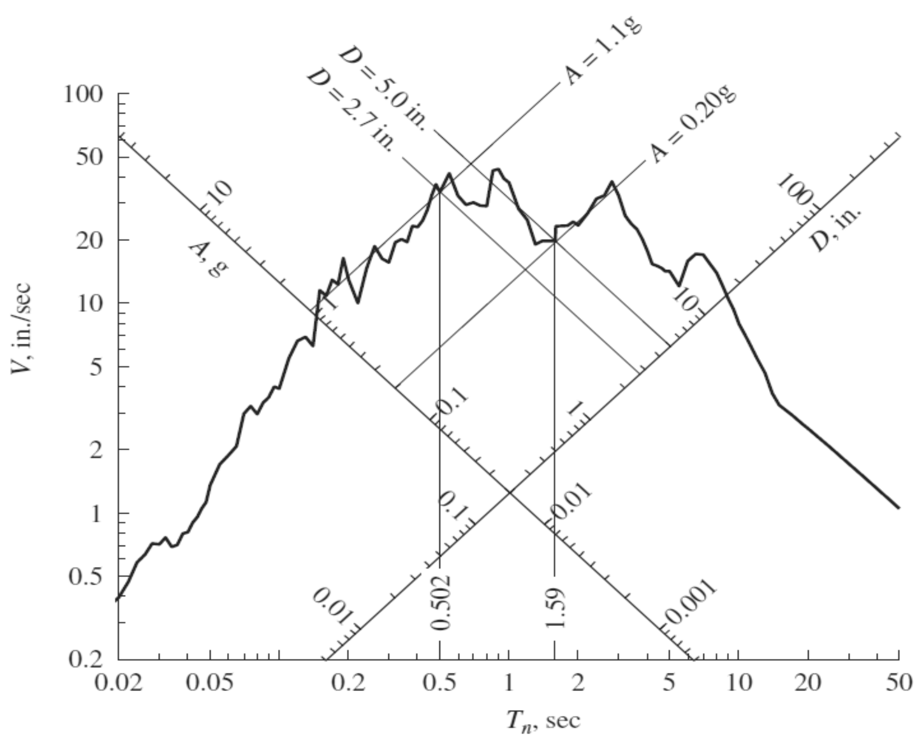


(a) (c) (d) Moment (e) Section *a-a*

$$k = \frac{3EI}{L^3} = \frac{3(29 \times 10^3)7.23}{(12 \times 12)^3} = 0.211 \text{ kip/in.} \qquad m = \frac{w}{g} = \frac{5.20}{386} = 0.01347 \text{ kip-sec}^2/\text{in.}$$

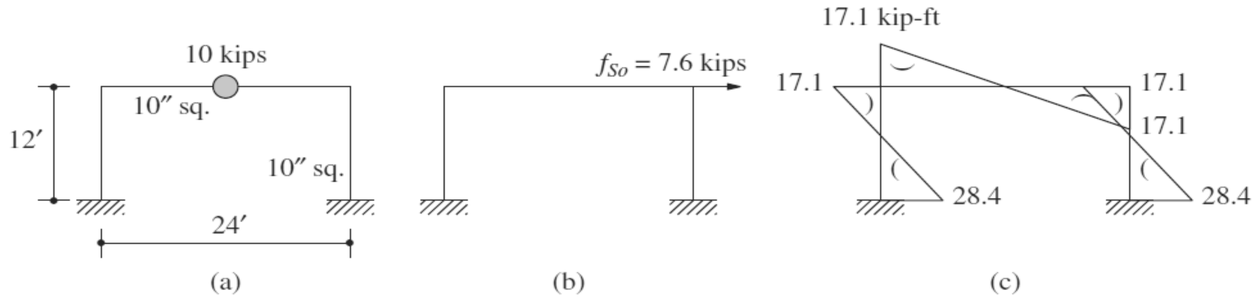
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.211}{0.01347}} = 3.958 \text{ rad/sec} \qquad T_n = 1.59 \text{ sec}$$

$$u_o = D = 5.0 \text{ in.} \qquad f_{So} = \frac{A}{g}w = 0.20 \times 5.2 = 1.04 \text{ kips}$$



Example 6.4

A small one-story reinforced-concrete building is idealized for purposes of structural analysis as a massless frame supporting a total dead load of 10 kips at the beam level (Fig. E6.4a). The frame is 24 ft wide and 12 ft high. Each column and the beam has a 10-in.-square cross section. Assume that the Young's modulus of concrete is 3×10^3 ksi and the damping ratio for the building is estimated as 5%. Determine the peak response of this frame to the El Centro ground motion. In particular, determine the peak lateral deformation at the beam level and plot the diagram of bending moments at the instant of peak response.



Solution The lateral stiffness of such a frame was calculated in Chapter 1: $k = 96EI/7h^3$, where EI is the flexural rigidity of the beam and columns and h is the height of the frame. For this particular frame,

$$k = \frac{96(3 \times 10^3)(10^4/12)}{7(12 \times 12)^3} = 11.48 \text{ kips/in.}$$

The natural vibration period is

$$T_n = \frac{2\pi}{\sqrt{k/m}} = 2\pi\sqrt{\frac{10/386}{11.48}} = 0.30 \text{ sec}$$

For $T_n = 0.3$ and $\zeta = 0.05$, we read from the response spectrum of Fig. 6.6.4: $D = 0.67$ in. and $A = 0.76$g. Peak deformation: $u_o = D = 0.67$ in. Equivalent static force: $f_{So} = (A/g)w = 0.76 \times 10 = 7.6$ kips. Static analysis of the frame for this lateral force, shown in Fig. E6.4b, gives the bending moments that are plotted in Fig. E6.4c.

6.10 A 10-ft-long vertical cantilever made of a 6-in.-nominal-diameter standard steel pipe supports a 3000-lb weight attached at the tip, as shown in Fig. P6.10. The properties of the pipe are: outside diameter = 6.625 in., inside diameter = 6.065 in., thickness = 0.280 in., second moment of cross-sectional area $I = 28.1$ in^4, Young's modulus $E = 29{,}000$ ksi, and weight = 18.97 lb/ft length. Determine the peak deformation and the bending stress in the cantilever due to the El Centro ground motion; assume that $\zeta = 5\%$.

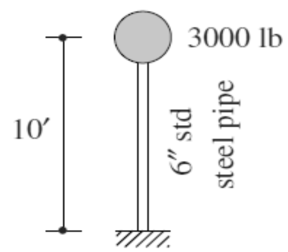
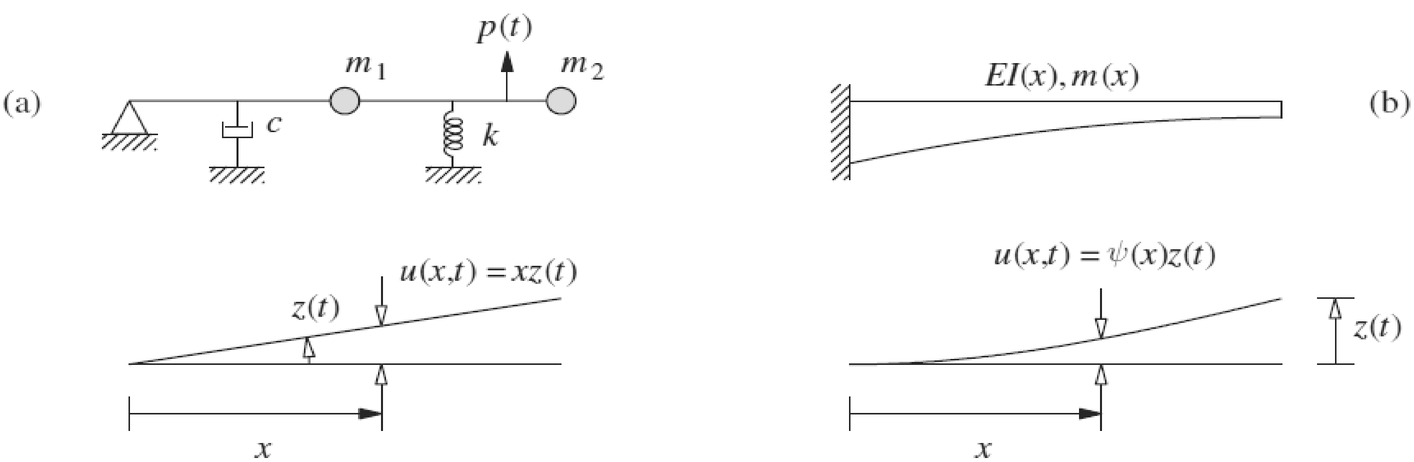


Figure P6.10

7

Generalized Single-Degree-of-Freedom Systems

GENERALIZED SDF SYSTEMS



$$u(x, t) = \psi(x) z(t)$$

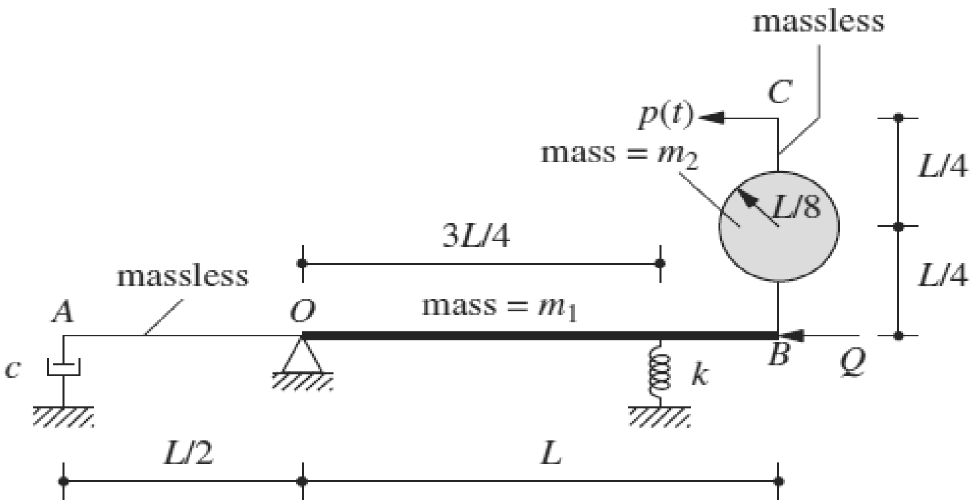
$$\psi(x) = x$$

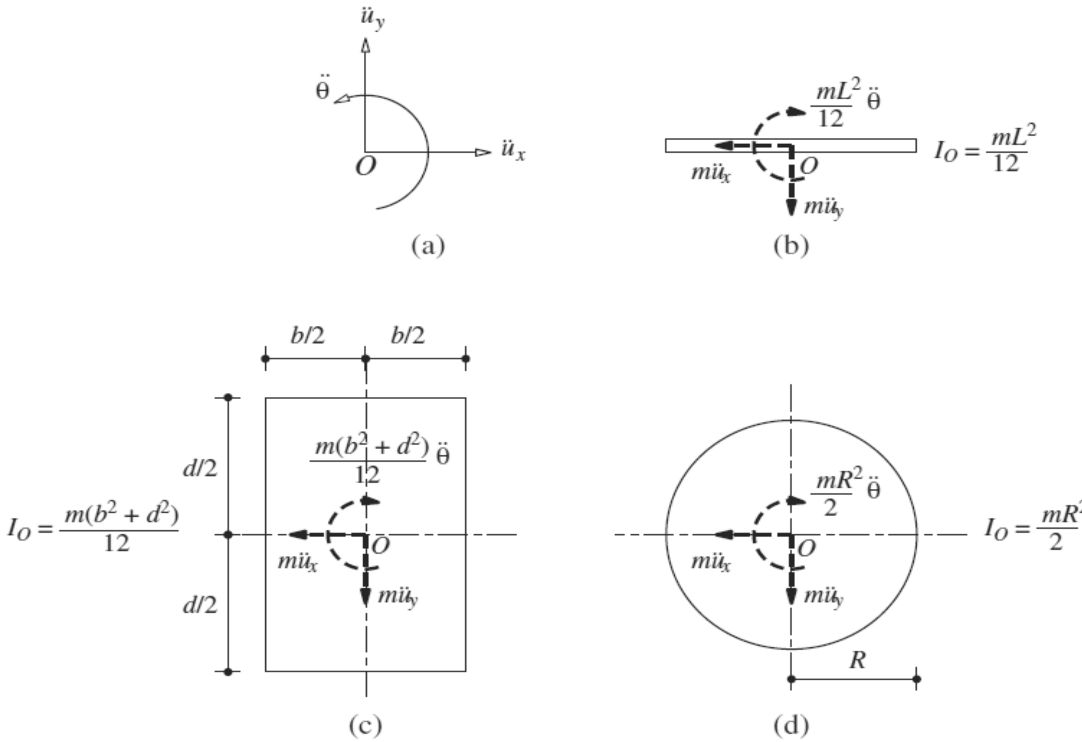
$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p}(t)$$

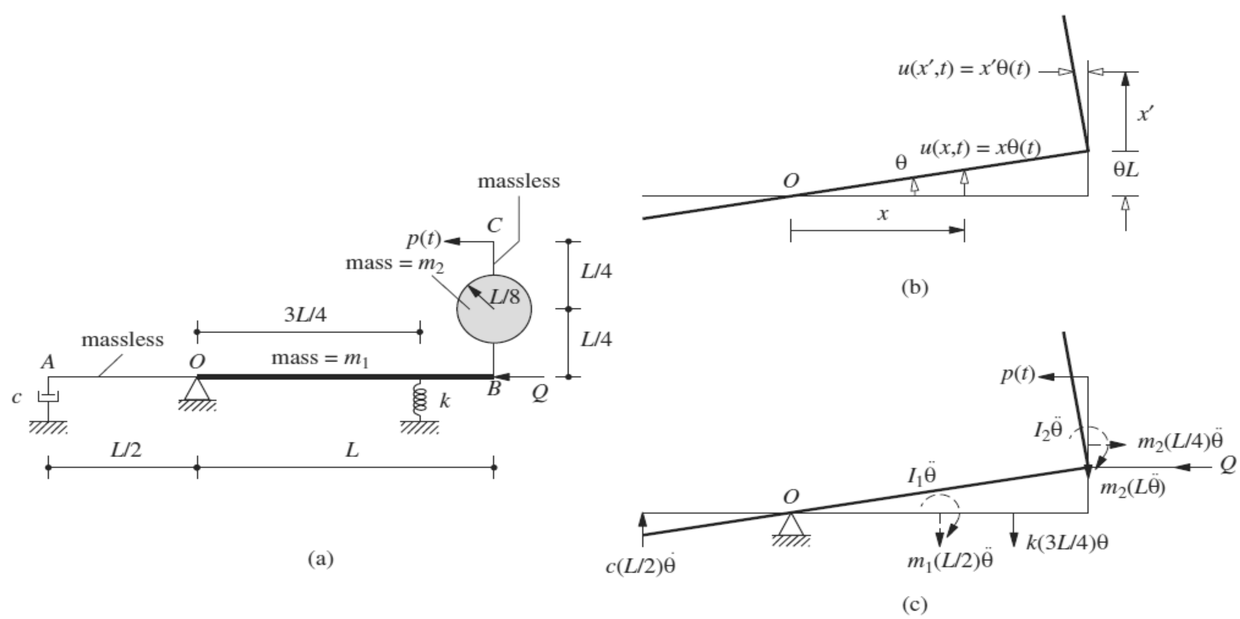
RIGID-BODY ASSEMBLAGES

Example 8.1

The system shown in Fig. E8.1a consists of a rigid bar supported by a fulcrum at O, with an attached spring and damper subjected to force $p(t)$. The mass m_1 of the part OB of the bar is distributed uniformly along its length. The portions OA and BC of the bar are massless, but a uniform circular plate of mass m_2 is attached at the midpoint of BC. Selecting the counterclockwise rotation about the fulcrum as the generalized displacement and considering small displacements, formulate the equation of motion for this generalized SDF system, determine the natural vibration frequency and damping ratio, and evaluate the dynamic response of the system without damping subjected to a suddenly applied force p_o. How would the equation of motion change with an axial force on the horizontal bar; what is the buckling load?







$$I_1\ddot{\theta}+\left(m_1\frac{L}{2}\ddot{\theta}\right)\frac{L}{2}+I_2\ddot{\theta}+(m_2L\ddot{\theta})L+\left(m_2\frac{L}{4}\ddot{\theta}\right)\frac{L}{4}+\left(c\frac{L}{2}\dot{\theta}\right)\frac{L}{2}+\left(k\frac{3L}{4}\theta\right)\frac{3L}{4}=p(t)\frac{L}{2}$$

Substituting $I_1 = m_1 L^2/12$ and $I_2 = m_2(L/8)^2/2 = m_2 L^2/128$ (see Appendix 8) gives

$$\left(\frac{m_1 L^2}{3} + \frac{137}{128} m_2 L^2\right)\ddot{\theta} + \frac{cL^2}{4}\dot{\theta} + \frac{9kL^2}{16}\theta = p(t)\frac{L}{2} \tag{a}$$

The equation of motion is

$$\tilde{m}\ddot{\theta} + \tilde{c}\dot{\theta} + \tilde{k}\theta = \tilde{p}(t) \tag{b}$$

where

$$\tilde{m} = \left(\frac{m_1}{3} + \frac{137}{128} m_2\right) L^2 \qquad \tilde{c} = \frac{cL^2}{4} \qquad \tilde{k} = \frac{9kL^2}{16} \qquad \tilde{p}(t) = p(t)\frac{L}{2} \tag{c}$$

3. *Determine the natural frequency and damping ratio.*

$$\omega_n = \sqrt{\frac{\tilde{k}}{\tilde{m}}} \qquad \zeta = \frac{\tilde{c}}{2\sqrt{\tilde{k}\tilde{m}}}$$

4. *Solve the equation of motion.*

$$\tilde{p}(t) = \frac{p(t)L}{2} = \frac{p_o L}{2} \equiv \tilde{p}_o$$

$$\theta(t) = \frac{\tilde{p}_o}{\tilde{k}}(1 - \cos\omega_n t) = \frac{8p_o}{9kL}(1 - \cos\omega_n t)$$

5. *Determine the displacements.*

$$u(x,t) = x\theta(t) \qquad u(x',t) = x'\theta(t)$$

6. *Include the axial force.* In the displaced position of the bar, the axial force Q introduces a counterclockwise moment $= QL\theta$. Thus Eq. (b) becomes

$$\tilde{m}\ddot{\theta} + \tilde{c}\dot{\theta} + (\tilde{k} - QL)\theta = \tilde{p}(t) \tag{g}$$

A compressive axial force decreases the stiffness of the system and hence its natural vibration frequency. These become zero if the axial force is

$$Q_{\text{cr}} = \frac{\tilde{k}}{L} = \frac{9kL}{16}$$

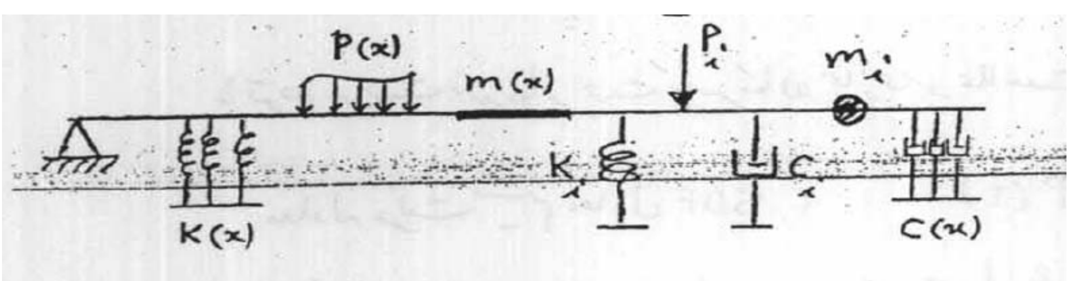
This is the critical or buckling axial load for the system.

SYSTEMS WITH DISTRIBUTED MASS AND ELASTICITY

سیستم با بینهایت درجه آزادی با سختی و جرم گسترده در حالت مرتعش (غیرصلب)

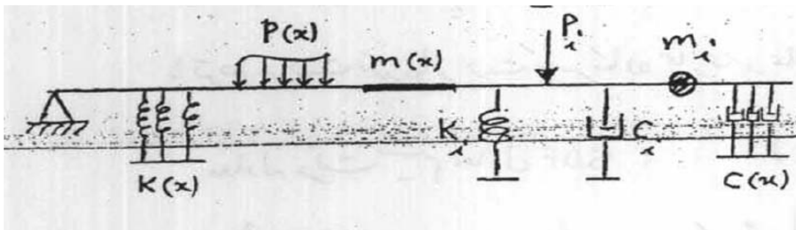
دقت مساله بستگی به تابع شکل $\Psi(x)$ دارد $\qquad U(x,t) = \Psi(x)\, Z(t)$

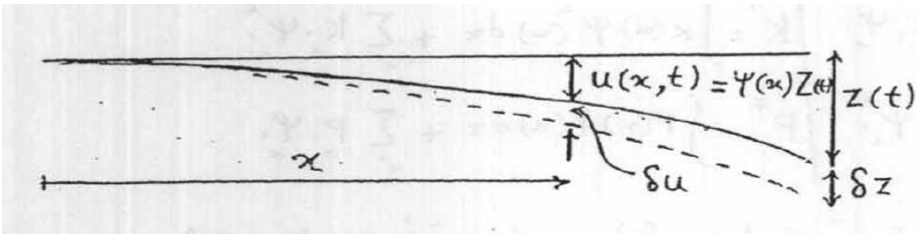
که تا چه حدی شرایط مرزی (هندسی) و شرایط فیزیکی مساله را ارضاء می نماید.



حالت کلی :

کاربرد اصل کار مجازی :





$W_{IN} = W_E$ $\qquad$ $W_S + W_D + W_I = W_E$

کار مجازی الاستیک $W_s = -\int \overbrace{k(x)\,u(x,t)\,dx}^{\text{نیروی فنر گسترده}} \cdot \delta u - \sum_i K_i\, u_i \cdot \delta u_i$

$$\begin{cases} u(x,t) = \Psi(x)\, Z(t) \rightarrow \delta u = \Psi(x)\, \delta Z \\ u_i = \Psi_i\, Z(t) \rightarrow \delta u_i = \Psi_i\, \delta Z \end{cases}$$

$$W_s = -\left\{ \int k(x)\, \Psi^2(x)\, dx + \sum_i K_i\, \Psi_i^2 \right\} Z\, \delta Z$$

کار مجازی نیروهای میرایی

$$W_D = -\int c(x)\,\dot{u}(x,t)\,dx \cdot \delta u - \sum_i c_i \dot{u}_i \cdot \delta u_i$$

$$\begin{cases} \dot{u}(x,t) = \psi(x)\,\dot{z}(t) \rightarrow & \delta u = \psi(x)\,\delta z \\ \dot{u}_i = \psi_i\,\dot{z}(t) \rightarrow & \delta u_i = \psi_i\,\delta z \end{cases}$$

$$W_D = -\left\{ \int c(x)\,\psi^2(x)\,dx + \sum_i c_i \psi_i^2 \right\} \dot{z}\,\delta z$$

به همین ترتیب؛ کار مجازی نیروی اینرسی:

$$W_I = -\left\{ \int m(x)\,\psi^2(x)\,dx + \sum_i m_i \psi_i^2 \right\} \ddot{z}\,\delta z$$

کار مجازی نیروهای خارجی:

$$W_P = \int p(x)\,dx \cdot \delta u + \sum_i p_i\,\delta u_i$$

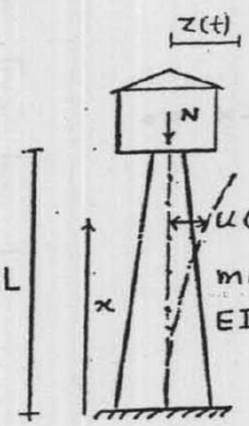
$$W_P = \left\{ \int p(x)\,\psi(x)\,dx + \sum_i p_i \psi_i \right\} \delta z$$

$W_{IN}=W_E \qquad W_S+W_D+W_I=W_E$

$$\tilde{m}\ddot{z} + \tilde{c}\dot{z} + \tilde{k}z = \tilde{p}(t)$$

$$m^* = \int m(x)\,\Psi^2(x)\,dx + \sum_i m_i \Psi_i^2 \qquad K^* = \int K(x)\,\Psi^2(x)\,dx + \sum_i K_i \Psi_i^2$$

$$C^* = \int C(x)\,\Psi^2(x)\,dx + \sum_i C_i \Psi_i^2 \qquad P^* = \int P(x)\,\Psi(x)\,dx + \sum_i P_i \Psi_i$$



$I\,e(t)$

$u(x,t)$

$m(x)$

$EI(x)$

جزئیات بیشتر در یک حالت خاص

$$u(x,t) = \Psi(x)\,Z(t)$$

از مبانی انرژی استفاده می کنیم (انرژی جنبشی و پتانسیل):

انرژی جنبشی $T = \int_0^L \frac{1}{2} m(x)\,\dot{u}^2(x,t)\,dx =$

$$\frac{1}{2}\int_0^L m(x)\,[\Psi^2(x)\,\dot{z}^2]\,dx \equiv \frac{1}{2} m^* \dot{z}^2 = T$$

جرم موثر معادل $m^* = \int_0^L m(x)\,\Psi^2(x)\,dx$

برای بیان سختی معادل مؤثر K^* از اصول مقاومت مصالح (انرژی پتانسیل تغییر شکل) :

$$V = \frac{1}{2}\int_0^L \frac{M^2}{EI}\,dx \quad , \quad u''(x,t) = \frac{d^2u(x,t)}{dx^2} = \frac{M}{EI}$$

انرژی یا کار لنگر (ممان)

$$M = EIu'' \quad , \quad u'' = \psi''(x)\,Z(t)$$

$$V = \frac{1}{2}\int_0^L \frac{(EI)^2}{EI}(u'')^2\,dx \Rightarrow V = \frac{1}{2}\int_0^L EI(x)\left(\psi''(x)\,Z(t)\right)^2 dx$$

$V = \frac{1}{2}K^*Z^2$ که $K^* = \int_0^L EI(x)\,\psi''^2(x)\,dx$

سختی هندسی معادل مؤثر : نیروی محوری (دائم) N ← افت رأس برج آب e

از مقاومت مصالح :

$$e(t) = \frac{1}{2}\int_0^L u'^2(x,t)\,dx$$

انرژی پتانسیل آن

$$V_N = -\frac{N}{2}\int_0^L u'^2(x,t)\,dx$$

انرژی پتانسیل نیروی N در اثر افت $e(t)$ کم می‌شود ← علامت منفی

اگر وزنه متغیر پایه مدنظر باشد ، $N(x)$

$$V_N = -\frac{1}{2}\int_0^L N(x)\,u'^2(x,t)\,dx = -\frac{1}{2}\int_0^L N(x)\left[\psi'(x)\,Z\right]^2 dx$$

$V_N = -\frac{1}{2}K_G^*Z^2$ که $K_G^* = \int_0^L N(x)\,\psi'^2(x)\,dx$

$$\bar{K}^* = K^* - K_G^* = \int_0^L EI(x)\,\psi''^2(x)\,dx - N_{cr}\int_0^L \psi'^2(x)\,dx = 0$$

برای کمانش
$$N_{cr} = \frac{\int_0^L EI(x)\,\psi''^2(x)\,dx}{\int_0^L \psi'^2(x)\,dx}$$

محاسبه فرکانس زاویه‌ای معادل مؤثر
$$\omega^* = \sqrt{K^*/m^*}$$

مثال - در برج آب قبلی $EI(x)$ و $m(x)$ در طول ارتفاع ثابت فرض می‌شود، مطلوبست

برآورد ω^* برای توابع مختلف شکل $\psi(x)$ (جرم گسترده m).

الف - اگر از فرمول اولر در پایداری سازه‌ها استفاده کنیم
$$\psi(x) = 1 - \cos\frac{\pi x}{2L}$$

$$m^* = \int_0^L m\,\psi^2(x)\,dx = m\int_0^L \left(1 - \cos\frac{\pi x}{2L}\right)^2 dx = 0.228\,mL$$

$$K^* = \int_0^L EI\,\psi''^2(x)\,dx = EI\int_0^L \left(\frac{\pi^2}{4L^2}\cos\frac{\pi x}{2L}\right)^2 dx = \frac{\pi^4}{32}\,\frac{EI}{L^3}$$

$$\omega^* = \sqrt{K^*/m^*} = \sqrt{\frac{\pi^4 EI}{32L^3 \times 0.228\,mL}} = \frac{3.653}{L^2}\sqrt{\frac{EI}{m}}$$

ب - اگر شکل ارتعاش برج را سهمی فرض کنیم : $\Psi(x) = \frac{x^2}{L^2}$

$$m^* = 0.2\,mL \;,\; K^* = \frac{4EI}{L^3} \;,\; \omega^* = \sqrt{\frac{4EI}{L^3 \times 0.2\,mL}} = \frac{4.472}{L^2}\sqrt{\frac{EI}{m}}$$

* جواب دقیق از حل سیستم پیوسته : $\omega^* = \frac{3.517}{L^2}\sqrt{\frac{EI}{m}}$

در حالت ب ، $\Psi(x) = \frac{x^2}{L^2} \rightarrow \frac{d^2u}{dx^2} = Z\Psi'' = Z\frac{2}{L^2}$ (عدد ثابت) ، $\frac{d^2u}{dx^2} = \frac{M}{EI} = Z\frac{2}{L^2}$

یعنی لنگر در طول ارتفاع ثابت است در صورتیکه در انتهای برج صفر و در تکیه گاه حداکثر است پس تابع مورد نظر $\Psi(x)$ یکی از شرایط مرزی را ارضاء نکرده و جواب از واقعیت دور است .

ج - در این حالت سعی می شود با ارضاء شرایط مرزی و فیزیکی بیشتر ، به یک تابع شکل $\Psi(x)$ بهتر دست یابیم . یک تابع درجه ۳ (چند جمله ای) فرض می کنیم :

چهار ضریب پس چهار شرط لازم است . $\Psi(x) = ax^3 + bx^2 + cx + d$

۱- تغییر مکان در تکیه گاه صفر است : $u = 0 \rightarrow \Psi(x=0) = 0$

۲- ضریب زاویه در تکیه گاه گیردار صفر است : $u' = 0 \rightarrow \Psi'(x=0) = 0$

۳- لنگر در انتهای برج صفر است : $u'' = 0 \rightarrow \Psi''(x=L) = 0$

۴- همیشه در نقطه شاخص (مختصات شاخص) : $u(x,t) = Z(t) \rightarrow \Psi(L) = 1$

اعمال شرایط $\Rightarrow a = -\frac{1}{2L^3}$, $b = \frac{3}{2L^2}$; $c = d = 0 \Rightarrow$

$$\Psi(x) = -\frac{x^3}{2L^3} + \frac{3x^2}{2L^2} \quad , \Psi''(x) = \frac{3}{L^2}\left(1 - \frac{x}{L}\right)$$

$$K^* = \frac{3EI}{L^3} \quad , m^* = 0.2357 mL \Rightarrow \omega^* = \frac{3.568}{L^2}\sqrt{\frac{EI}{m}}$$

برآورد سختی هندسی و نیروی N_{cr} :

حالت الف $\rightarrow \Psi(x) = 1 - \cos\frac{\pi x}{2L} \rightarrow K_G^* = \int_0^L N \Psi'^2(x)\, dx$

$$K_G^* = N\int_0^L \left(\frac{\pi}{2L}\right)^2 \left(\sin\frac{\pi x}{2L}\right)^2 dx = \frac{N\pi^2}{8L} = \frac{3EI}{L^3} \rightarrow N_{cr} = 2.467\ \frac{EI}{L^2}$$

حالت ب $\rightarrow \Psi(x) = x^2/L^2 \rightarrow K_G^* = \frac{4}{3}\frac{N}{L} \rightarrow N_{cr} = \frac{3EI}{L^2}$

حالت ج $\rightarrow \Psi(x) = -\frac{x^3}{2L^3} + \frac{3x^2}{2L^2} \rightarrow K^* = \frac{6}{5}\frac{N}{L} \rightarrow N_{cr} = 2.5\ \frac{EI}{L^2}$

حالت واقعی $\rightarrow \quad N_{cr} = 2.67\ \frac{EI}{L^2}$

8.1 Repeat parts (a), (b), and (c) of Example 8.1 with one change: Use the horizontal displacement at C as the generalized coordinate. Show that the natural frequency, damping ratio, and displacement response are independent of the choice of generalized displacement.

8.2 For the rigid-body system shown in Fig. P8.2:
(a) Formulate the equation of motion governing the rotation at O.
(b) Determine the natural frequency and damping ratio.
(c) Determine the displacement response $u(x, t)$ to $p(t) = \delta(t)$, the Dirac delta function.

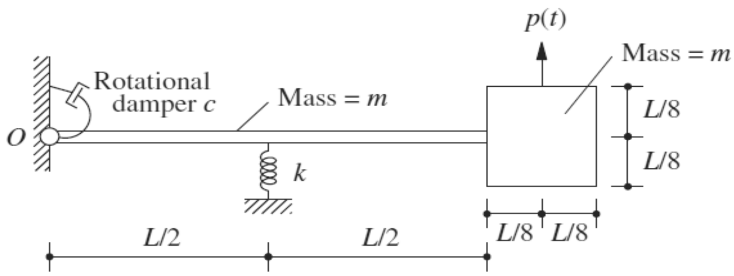


Figure P8.2

8.5 For the rigid-body system shown in Fig. P8.5:
(a) Choose a generalized coordinate.
(b) Formulate the equation of motion.
(c) Determine the natural vibration frequency and damping ratio.

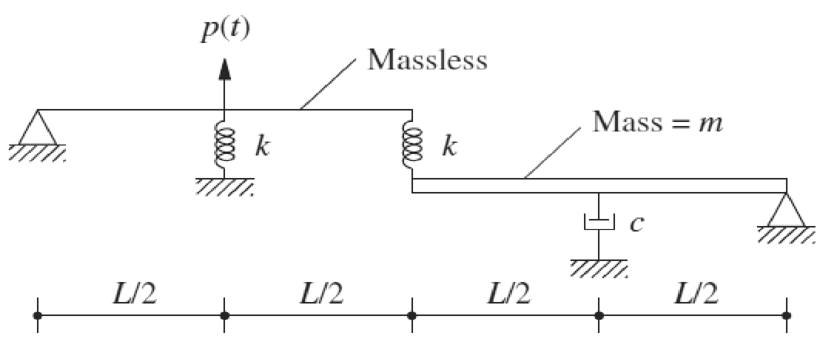


Figure P8.5

PART II MULTI-DEGREE-OF-FREEDOM SYSTEMS

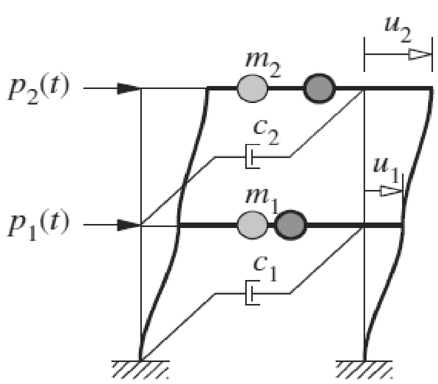
Equations of Motion, Problem Statement, and Solution Methods

Free Vibration

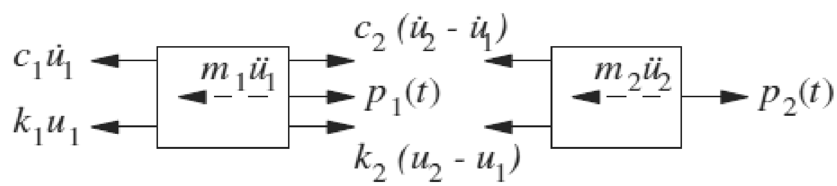
Dynamic Analysis and Response of Linear Systems

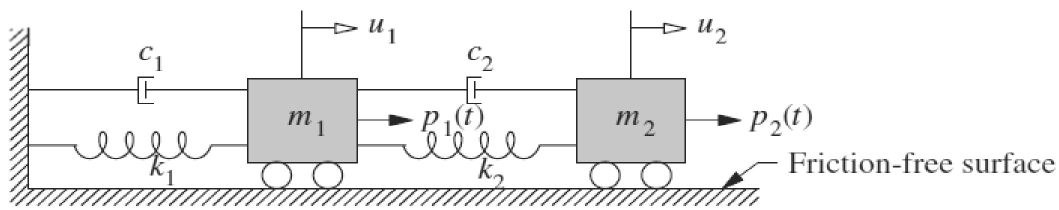
Earthquake Analysis of Linear Systems

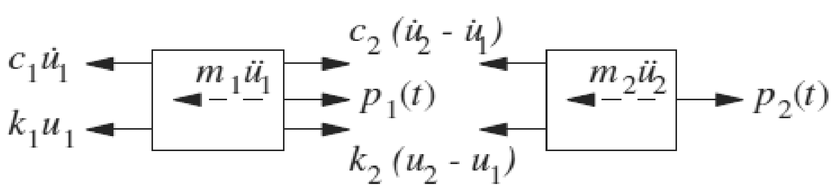
SIMPLE SYSTEM: TWO-STORY SHEAR BUILDING



(a)





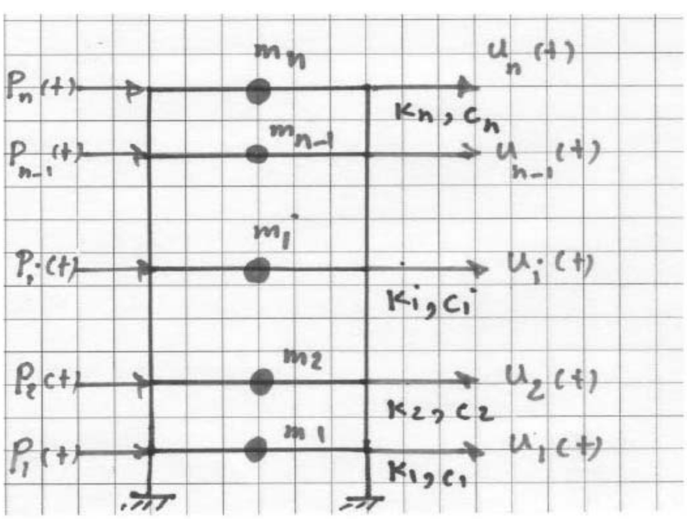


m_1 جرم : $\sum \vec{F_x} = 0 \rightarrow P_1(t) + K_2(u_2 - u_1) + c_2(\dot{u}_2 - \dot{u}_1) - m_1\ddot{u}_1 - c_1\dot{u}_1 - k_1u_1 = 0$

m_2 جرم : $\sum \vec{F_x} = 0 \rightarrow P_2(t) - m_2\ddot{u}_2 - c_2(\dot{u}_2 - \dot{u}_1) - k_2(u_2 - u_1) = 0$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix}\ddot{u}_1\\ \ddot{u}_2\end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 \\ -c_2 & c_2\end{bmatrix}\begin{Bmatrix}\dot{u}_1\\ \dot{u}_2\end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2\\ -k_2 & k_2\end{bmatrix}\begin{Bmatrix}u_1\\ u_2\end{Bmatrix} = \begin{Bmatrix}P_1(t)\\ P_2(t)\end{Bmatrix}$$

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{P(t)\}$$



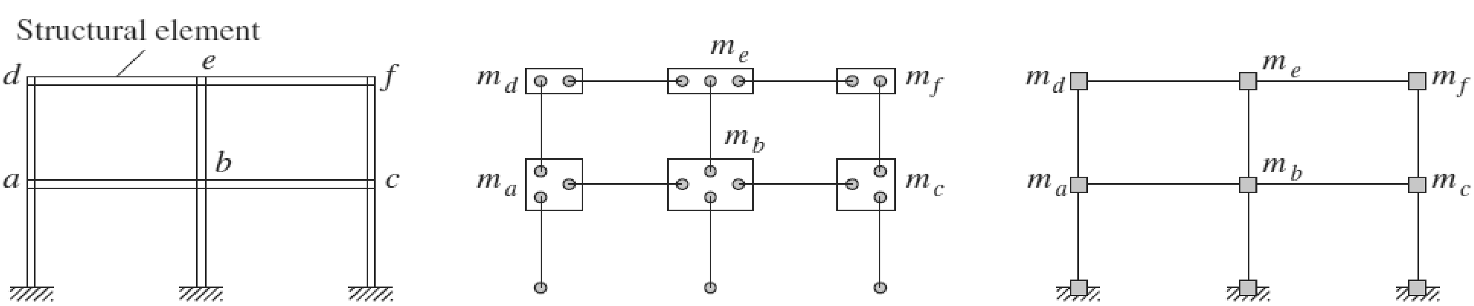
$$k_i = \sum \frac{12EI}{h^3}$$

$$[M]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{P(t)\}$$

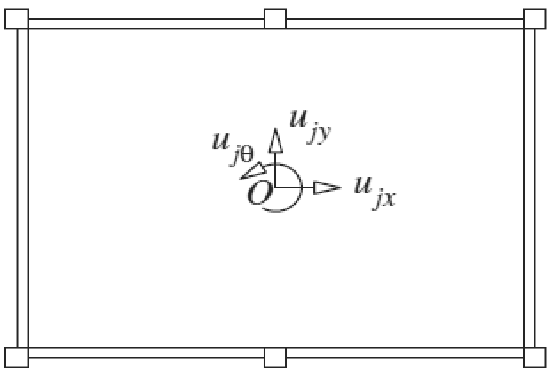
$$M = \begin{bmatrix} m_1 & & & \\ & m_2 & & \\ & & \ddots & \\ & & & m_n \end{bmatrix}_{n\times n} \quad \{u\} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \{\dot{u}\} = \begin{bmatrix} \dot{u} \\ \dot{u}_2 \\ \vdots \\ u_n \end{bmatrix} \quad \{\ddot{u}\} = \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_n \end{bmatrix}$$

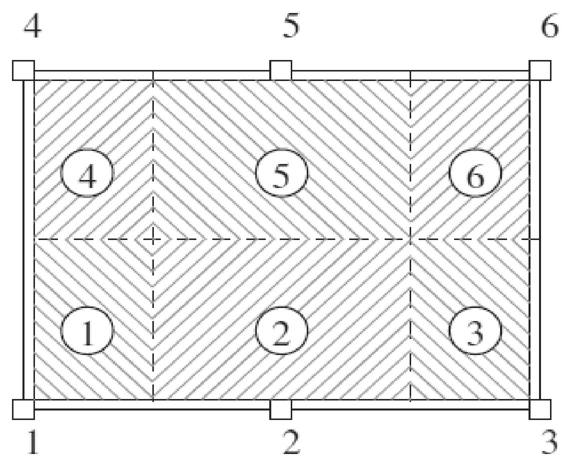
$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & \\ -k_2 & k_2 + k_3 & -k_3 & \\ & -k_3 & k_3 + k_4 & \\ & & & \ddots_{kn} \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & & & \\ -c_2 & c_2 + c_3 & -c_3 & & \\ & -c_3 & c_3 + c_4 & -c_4 & \\ & & -c_4 & & \ddots \end{bmatrix}_{Cn}$$



(a) (b) (c)





Example 9.1b

Formulate the equations of motion for the two-story shear frame in Fig. E9.1a using influence coefficients.

Solution

The two DOFs of this system are $\mathbf{u} = \langle u_1 \quad u_2 \rangle^T$.

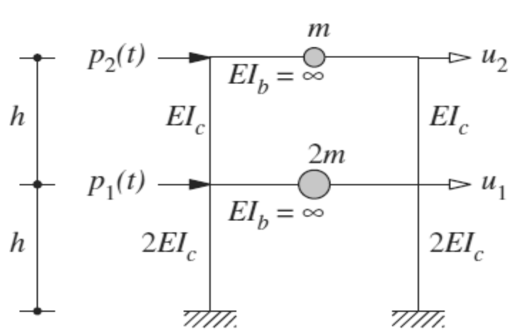


Figure E9.1a

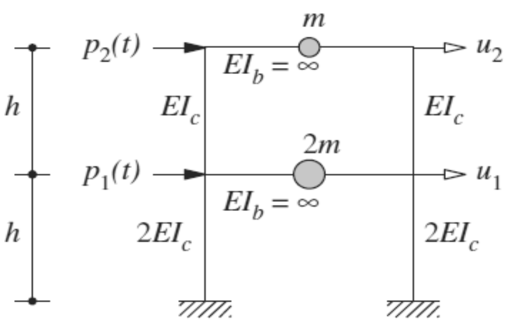
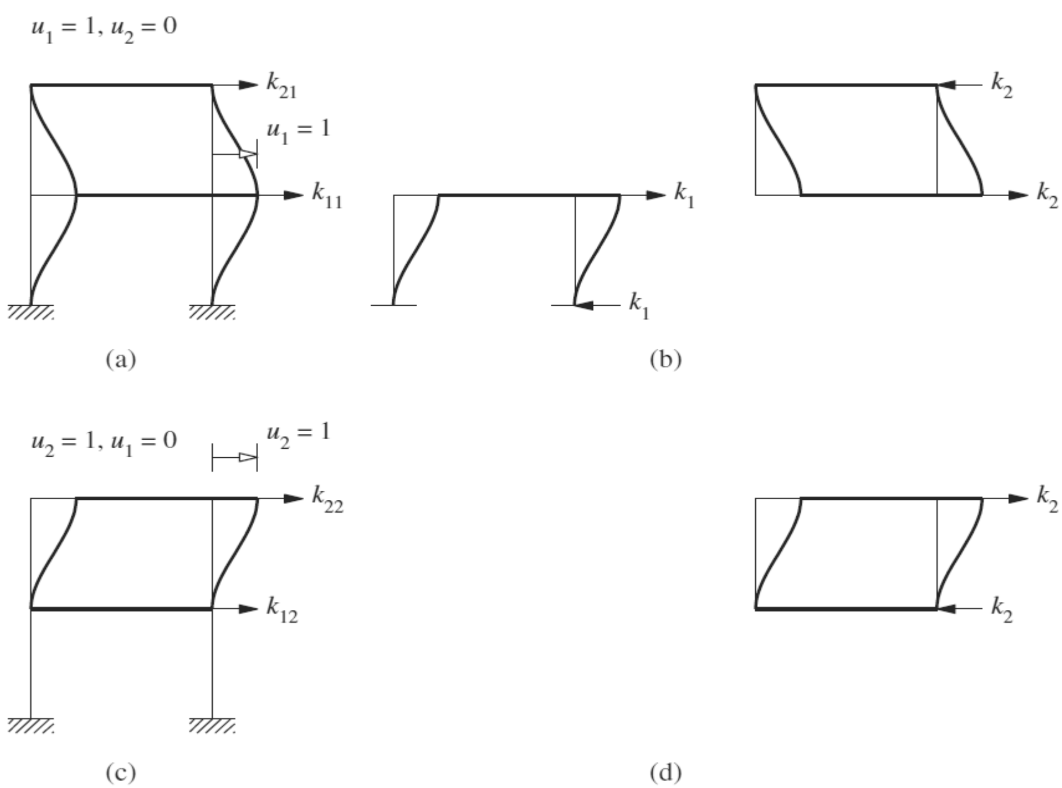


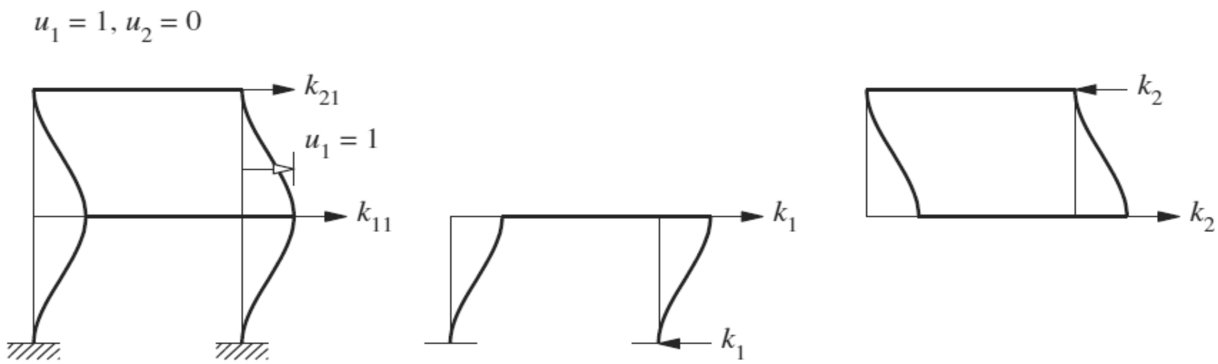
Figure E9.1a

$$\mathbf{m} = m\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$m_1 = 2m \qquad m_2 = m$$

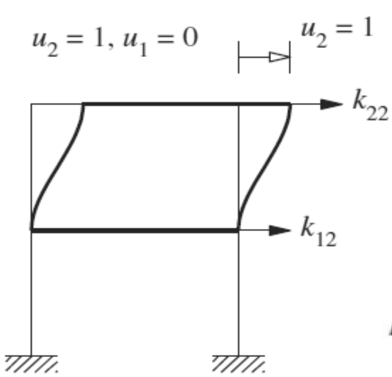
$$k_1 = 2\frac{12(2EI_c)}{h^3} = \frac{48EI_c}{h^3} \qquad k_2 = 2\frac{12(EI_c)}{h^3} = \frac{24EI_c}{h^3}$$

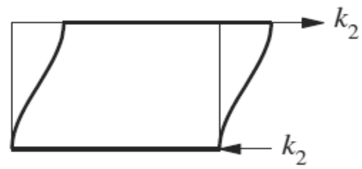




$$k_{11} = k_1 + k_2 = \frac{72EI_c}{h^3} \qquad k_{21} = -k_2 = -\frac{24EI_c}{h^3}$$

$$\mathbf{k} = \frac{24EI_c}{h^3}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$



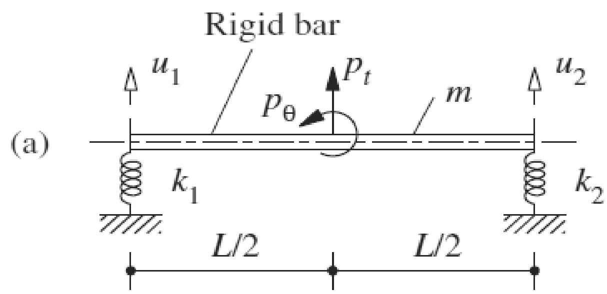


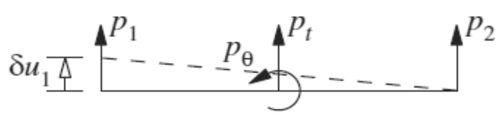
$$k_{12} = -k_2 = -\frac{24EI_c}{h^3} \qquad k_{22} = k_2 = \frac{24EI_c}{h^3}$$

$$m\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + 24\frac{EI_c}{h^3}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$

Example 9.2

A uniform rigid bar of total mass m is supported on two springs k_1 and k_2 at the two ends and subjected to dynamic forces shown in Fig. E9.2a. The bar is constrained so that it can move only vertically in the plane of the paper; with this constraint the system has two DOFs.





(b)

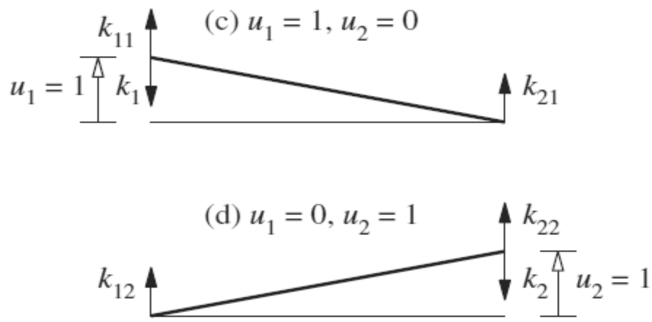
$$\delta W = p_t \frac{\delta u_1}{2} - p_\theta \frac{\delta u_1}{L}$$

$$\delta W = p_1 \delta u_1 + p_2(0)$$

$$p_1 = \frac{p_t}{2} - \frac{p_\theta}{L}$$

In a similar manner, by introducing a virtual displacement δu_2, we obtain

$$p_2 = \frac{p_t}{2} + \frac{p_\theta}{L}$$

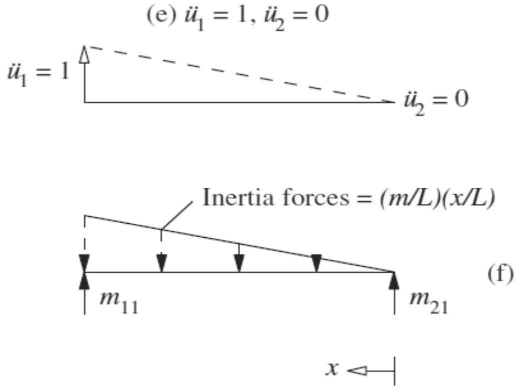


2. *Determine the stiffness matrix.* Apply a unit displacement $u_1 = 1$ with $u_2 = 0$ and identify the resulting elastic forces and the stiffness influence coefficients k_{11} and k_{21} (Fig. E9.2c). By statics, $k_{11} = k_1$ and $k_{21} = 0$. Now apply a unit displacement $u_2 = 1$ with $u_1 = 0$ and identify the resulting elastic forces and the stiffness influence coefficients (Fig. E9.2d). By statics, $k_{12} = 0$ and $k_{22} = k_2$. Thus the stiffness matrix is

$$\mathbf{k} = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \qquad \text{(e)}$$

3. *Determine the mass matrix.* Impart a unit acceleration $\ddot{u}_1 = 1$ with $\ddot{u}_2 = 0$, determine the distribution of accelerations of (Fig. E9.2e) and the associated inertia forces, and identify mass influence coefficients (Fig. E9.2f). By statics, $m_{11} = m/3$ and $m_{21} = m/6$. Similarly, imparting a unit acceleration $\ddot{u}_2 = 1$ with $\ddot{u}_1 = 0$, defining the inertia forces and mass influence coefficients, and applying statics gives $m_{12} = m/6$ and $m_{22} = m/3$. Thus the mass matrix is

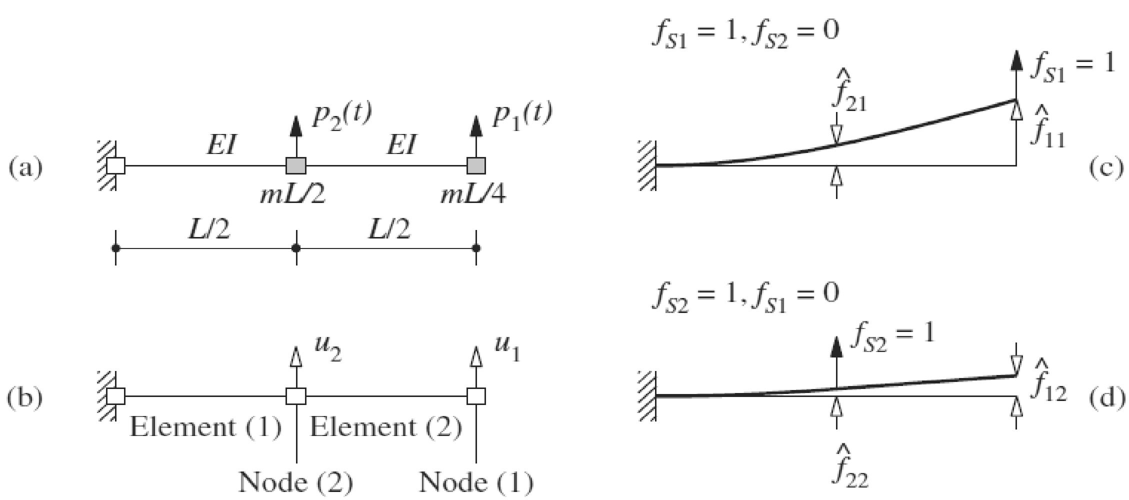
$$\mathbf{m} = \frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{(f)}$$



$$\frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} (p_t/2) - (p_\theta/L) \\ (p_t/2) + (p_\theta/L) \end{bmatrix}$$

Example 9.5

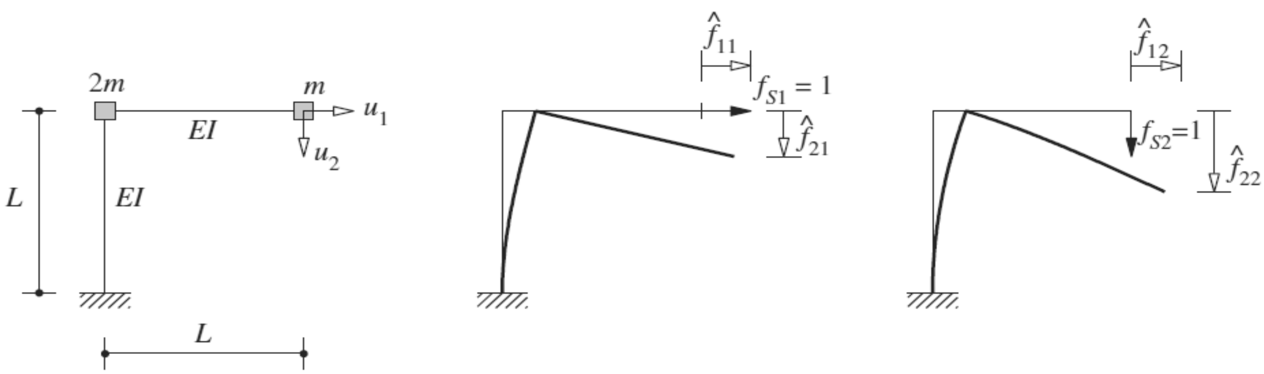
Derive the equations of motion of the beam of Example 9.4 (also shown in Fig. E9.5a) expressed in terms of the displacements u_1 and u_2 of the masses (Fig. E9.5b).



$$\hat{\mathbf{f}} = \frac{L^3}{48EI}\begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix} \qquad \mathbf{k} = \frac{48EI}{7L^3}\begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix}$$

$$\mathbf{m} = \begin{bmatrix} mL/4 & \\ & mL/2 \end{bmatrix}$$

$$\begin{bmatrix} mL/4 & \\ & mL/2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{48EI}{7L^3}\begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} p_1(t) \\ p_2(t) \end{Bmatrix}$$



$$\hat{\mathbf{f}} = \frac{L^3}{6EI}\begin{bmatrix} 2 & 3 \\ 3 & 8 \end{bmatrix}$$

This matrix is inverted to determine the stiffness matrix:

$$\mathbf{k} = \frac{6EI}{7L^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}$$

Thus the equations in free vibration of the system (without damping) are

$$\begin{bmatrix} 3m & \\ & m \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \frac{6EI}{7L^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Free Vibration

$$[m]\{\ddot{u}\} + [C]\{\dot{u}\} + [K]\{u\} = \{P(t)\}$$ معادله حرکت در حالت کلی

$$[m]\{\ddot{u}\} + [K]\{u\} = \{0\}$$ معادله ارتعاش آزاد بدون میرایی (۱)

$$\{u(t)\} = q_i(t)\,\{\phi\}_i \quad (۲)$$

$$q_i(t) = A_i \cos\omega_i t + B_i \sin\omega_i t = D_i \sin(\omega_i t + \alpha)$$

$$\{u(t)\} = \{\phi\}_i\,(A_i \cos\omega_i t + B_i \sin\omega_i t) \quad (۳)$$

$$\left[-\omega_i^2 [m]\{\phi\}_i + [K]\{\phi\}_i\right] q_i(t) = \{0\} \quad (۴)$$

$q_i(t) \neq 0$ پس:

$$\left[[K] - \omega_i^2 [m]\right]\{\phi\}_i = \{0\} \quad (۵)$$ یا $[K]\{\phi\}_i = \omega_i^2 [m]\{\phi\}_i$

$\{\phi\}_i \neq \{0\}$ $\quad \det\left[[K] - \omega_i^2 [m]\right] = 0$

$\Rightarrow \omega_i^2$ چند جمله ای مرتبه N برحسب $\Rightarrow \omega_i \quad i = 1$ تا N

$$\{\phi\}_i = \begin{Bmatrix} \phi_{1i} \\ \phi_{2i} \\ \vdots \\ \phi_{Ni} \end{Bmatrix}$$

$$\{\phi\}_i = \begin{Bmatrix} \phi_{1i} \\ \phi_{2i} \\ \vdots \\ \phi_{Ni} \end{Bmatrix}$$

بردار مود طبیعی ، شکل مود ارتعاش ، eigenvectors

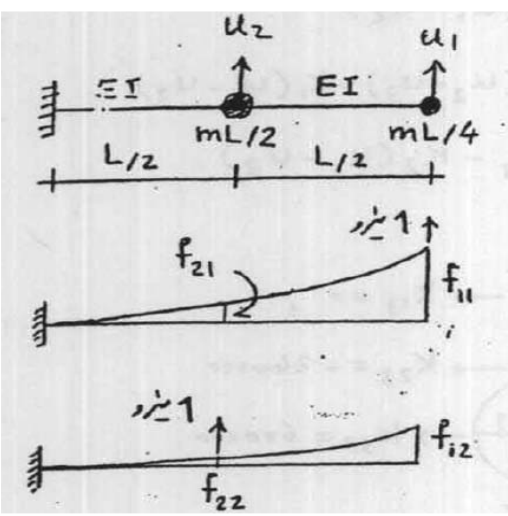
بردار مشخصه ، ...

$$\omega_1 < \omega_2 < \omega_3 \ldots <$$

ترتیب شماره گذاری فرکانس ها و پریودها

$$T_1 > T_2 > \ldots\ldots$$

مثال - مطلوبست تعیین فرکانس زاویه‌ای و مود شکل های سیستم زیر؟



$$[f] = \frac{L^3}{48EI}\begin{bmatrix} 16 & 5 \\ 5 & 2 \end{bmatrix}$$ ماتریس نرمی

$$[k] = [f]^{-1} = \frac{48EI}{7L^3}\begin{bmatrix} 2 & -5 \\ -5 & 16 \end{bmatrix}$$

$$[m] = \begin{bmatrix} mL/4 & 0 \\ 0 & mL/2 \end{bmatrix} \rightarrow [m]\{\ddot{u}\} + [k]\{u\} = \{0\}$$

$$\left[[k] - \omega^2 [m]\right] = \frac{48EI}{7L^3}\begin{bmatrix} 2-\lambda & -5 \\ -5 & 16-2\lambda \end{bmatrix} \leftarrow \lambda = \frac{7mL^4}{192\,EI}\,\omega^2$$

$$\det\begin{bmatrix} 2-\lambda & -5 \\ -5 & 16-2\lambda \end{bmatrix} = 0 \Rightarrow 2\lambda^2 - 20\lambda + 7 = 0$$

$$\lambda_1 = 0.36319 \quad , \quad \lambda_2 = 9.6368 \longrightarrow \omega_1 = 3.15623\sqrt{\frac{EI}{mL^4}}$$

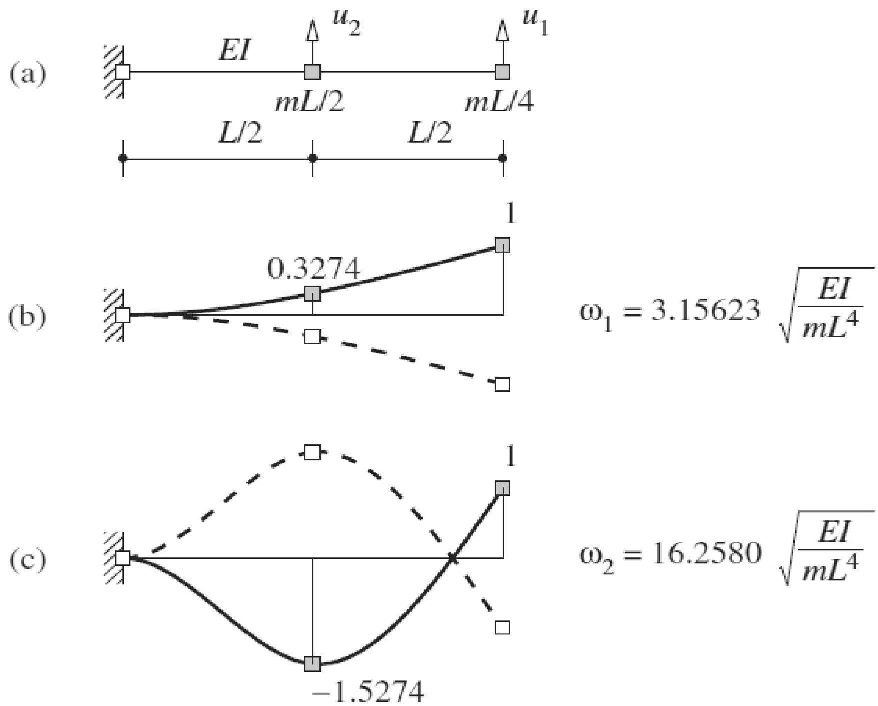
$$\omega_2 = 16.2580\sqrt{\frac{EI}{mL^4}}$$

$$\left[[K] - \omega_i^2\,[m]\right]\{\phi\}_i = \{0\}$$

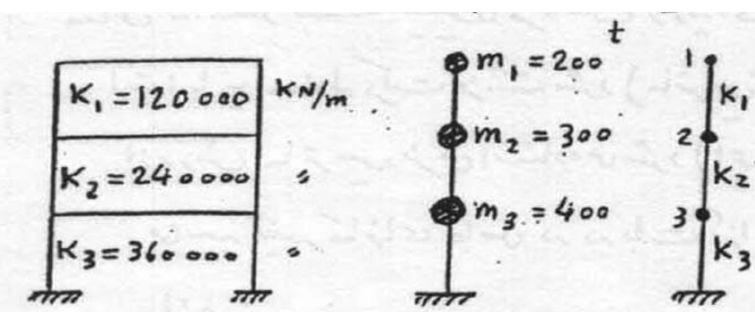
$$\left[[K] - \omega_1^2\,[m]\right]\begin{Bmatrix} 1.0 \\ \phi_2 \end{Bmatrix}_1 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \rightarrow \phi_{21} = 0.3274$$

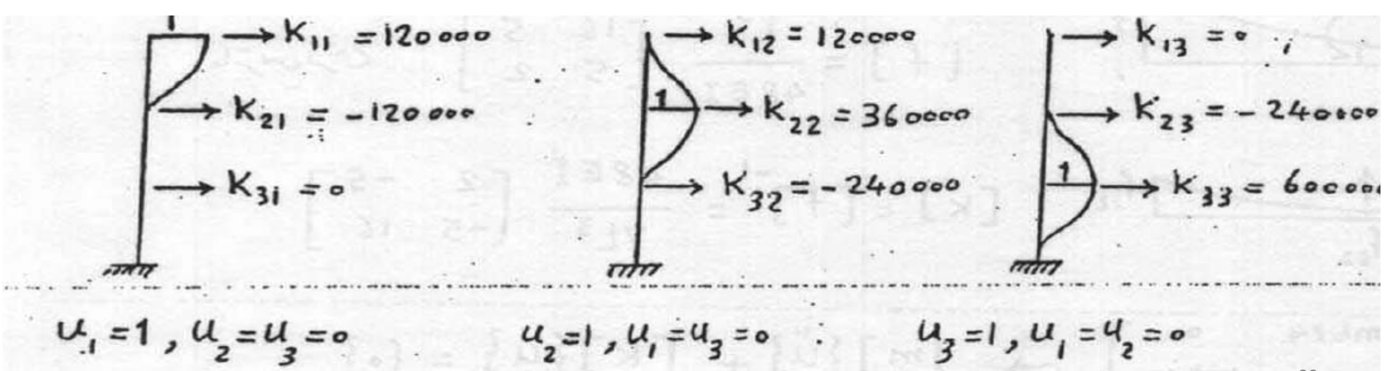
$$\left[[K] - \omega_2^2\,[m]\right]\begin{Bmatrix} 1.0 \\ \phi_2 \end{Bmatrix}_2 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \rightarrow \phi_{22} = -1.5274$$

$$\{\phi\}_1 = \begin{Bmatrix} 1.0000 \\ 0.3274 \end{Bmatrix} \quad , \quad \{\phi\}_2 = \begin{Bmatrix} 1.0000 \\ -1.5274 \end{Bmatrix}$$



مثال - در قاب سه طبقه داده شده مطلوبست تعیین ω_n و $\{\phi\}_n$ ؟





$$[k] = 120\times10^{3}\begin{bmatrix}1 & -1 & 0\\ -1 & 3 & -2\\ 0 & -2 & 5\end{bmatrix} \quad , \quad [m] = 200\begin{bmatrix}1.0 & 0 & 0\\ 0 & 1.5 & 0\\ 0 & 0 & 2.0\end{bmatrix}$$

$$\lambda_i = \omega_i^2 / 600$$

$$\left[[k]-\omega^{2}[m]\right] = 120\times10^{3}\begin{bmatrix}1-\lambda_i & -1 & 0\\ -1 & 3-1.5\lambda_i & -2\\ 0 & -2 & 5-2\lambda_i\end{bmatrix}$$

دترمینان ماتریس اخیر صفر ←

$$\lambda_i^{3} - 5.5\,\lambda_i^{2} + 7.5\,\lambda_i - 2 = 0$$

$$\lambda_1 = 0.351 \quad , \quad \lambda_2 = 1.61 \quad , \quad \lambda_3 = 3.54$$

$$\omega_1^{2} = 210 \quad , \quad \omega_2^{2} = 966 \quad , \quad \omega_3^{2} = 2124$$

$$\omega_1 = 14.5 \ \text{Rad/s} \quad , \quad \omega_2 = 31.1 \quad , \quad \omega_3 = 46.1$$

$$\left[[K] - \omega_i^2 [m]\right]\{\phi\}_i = \{0\}$$

$$\begin{bmatrix} 1-\lambda_i & -1 & 0 \\ -1 & 3-1.5\lambda_i & -2 \\ 0 & -2 & 5-2\lambda_i \end{bmatrix} \begin{Bmatrix} \phi_1 = 1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}_i = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{Bmatrix} -1 \\ 0 \end{Bmatrix} \times 1 + \begin{bmatrix} 3-1.5\lambda_i & -2 \\ -2 & 5-2\lambda_i \end{bmatrix} \begin{Bmatrix} \phi_2 \\ \phi_3 \end{Bmatrix}_i = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

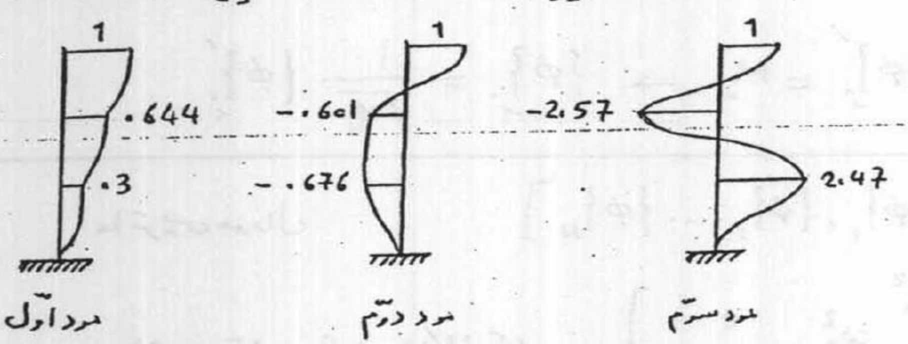
(100)

$$\begin{bmatrix} 3-1.5\lambda_i & -2 \\ -2 & 5-2\lambda_i \end{bmatrix} \begin{Bmatrix} \phi_{2i} \\ \phi_{3i} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$\lambda_1 = 0.351 \rightarrow \begin{bmatrix} 2.475 & -2 \\ -2 & 4.3 \end{bmatrix} \begin{Bmatrix} \phi_{21} \\ \phi_{31} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \Rightarrow \begin{cases} \phi_{21} = 0.644 \\ \phi_{31} = 0.300 \end{cases}$$

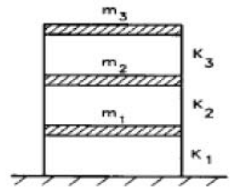
$$\lambda_2 = 1.61 \rightarrow \phi_{22} = -0.601 \quad , \quad \phi_{32} = -0.676$$

$$\lambda_3 = 3.54 \rightarrow \phi_{23} = -2.57 \quad , \quad \phi_{33} = 2.47$$



اشکال مدی و فرکانسهای سازه سه درجه زیر را بدست آورید

$m_1 = m_2 = m_3 = m_0$ و $k_1 = k_2 = k_3 = k_0$



شکل ۱. ۲۳ سازه سه درجه

$$[M] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = k_0 \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\left| k_0 \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & -1 \end{bmatrix} - m_0 \omega^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

اگر $\lambda = \omega^2 / (k_0 / m_0)$ آنگاه

$$\left| \begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} \right| = 0 \qquad \therefore \lambda^3 - 5\lambda^2 + 6\lambda - 1 = 0$$

پس:

$\lambda_1 = 0.198; \quad \lambda_2 = 1.555; \quad \lambda_3 = 3.247$

$$\omega_1 = \sqrt{\lambda_1 \frac{k_0}{m_0}} = 0.445 \sqrt{\frac{k_0}{m_0}} \; ; \; \omega_2 = 1.247 \sqrt{\frac{k_0}{m_0}} \; ; \omega_3 = 1.802 \sqrt{\frac{k_0}{m_0}}$$

$$\left[[k]-\omega_i^2[M]\right]\begin{bmatrix}\phi_{1i}\\ \phi_{2i}\\ \phi 3i\end{bmatrix}=0 \qquad \therefore \begin{bmatrix}2-\lambda_i & -1 & 0\\ -1 & 2-\lambda_i & -1\\ 0 & -1 & 1-\lambda_i\end{bmatrix}\begin{bmatrix}\phi_{1i}\\ \phi_{2i}\\ \phi_{3i}\end{bmatrix}=0$$

باید توجه داشت که دستگاه معادلات فوق بدلیل همگن بودن نمی‌تواند تمام مقادیر ϕ را بدست دهد زیرا یکی از معـادلات تکـراری اسـت (چـون قـبلاً دترمیـنان ضـرایب صـفر شده است) پس مقدار یکی از ϕها برابر ۱ فرض می‌شود، مثلاً:

$\phi_{1i}=1$ یا $\phi_{11}=\phi_{12}=\phi_{13}=1$

از خط اول معادلات فوق:

$$(2-\lambda_i)\phi_{1i}-\phi_{2i}=0 \qquad \therefore\ \phi_{2i}=2-\lambda_i$$

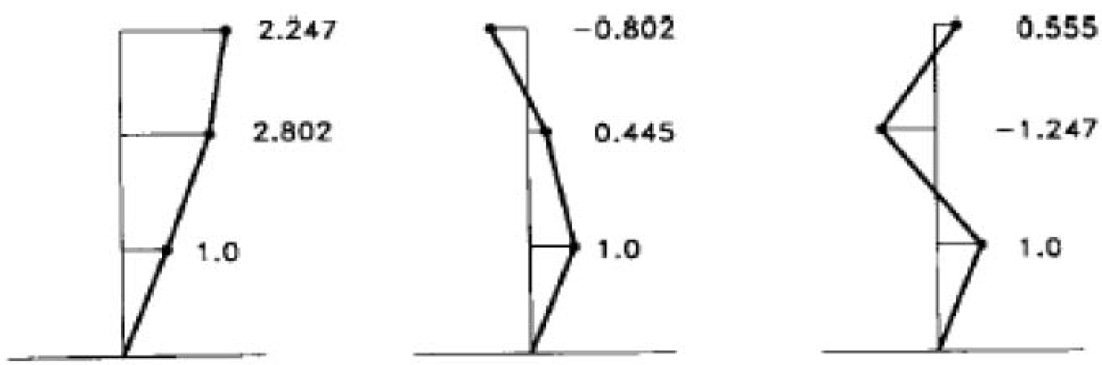
و ازخط دوم:

$$-\phi_{1i}+(2-\lambda_i)\phi_{2i}-\phi_{3i}=0 \qquad \therefore\ \phi_{3i}=(2-\lambda_i)^2-1$$

با نهادن λ_i در رابطه فوق داریم:

$$\phi_{21}=2-0.198=1.802 \qquad \phi_{22}=0.445 \qquad \phi_{23}=-1.247$$
$$\phi_{31}=2.247 \qquad \phi_{32}=-0.802 \qquad \phi_{33}=0.555$$

$$[\phi_1]=\begin{bmatrix}1.0\\1.802\\2.247\end{bmatrix} \qquad [\phi_2]=\begin{bmatrix}1.0\\0.445\\-0.802\end{bmatrix} \qquad [\phi_3]=\begin{bmatrix}1.0\\-1.247\\0.555\end{bmatrix}$$



$$\omega_1=0.445\sqrt{\frac{k_o}{m_e}} \qquad \omega_2=1.247\sqrt{\frac{k_o}{m_e}} \qquad \omega_3=1.802\sqrt{\frac{k_o}{m_e}}$$

ORTHOGONALITY OF MODES — خاصیت تعامد مودها

معادله مشخصه $\left[[K]-\omega_r^2[m]\right]\{\phi\}_r=\{0\}$ را برای مود شماره r می‌نویسیم!

طرفین رابطه را در $\{\phi\}_s^T$ ضرب می‌کنیم:

$$[K]\{\phi\}_r-\omega_r^2[m]\{\phi\}_r=\{0\}$$

$$\{\phi\}_s^T[K]\{\phi\}_r-\omega_r^2\{\phi\}_s^T[m]\{\phi\}_r=0 \quad ①$$

حال معادله را برای مود S ام می‌نویسیم!

طرفین رابطه را در $\{\phi\}_r^T$ ضرب می‌کنیم:

$$[K]\{\phi\}_s-\omega_s^2[m]\{\phi\}_s=\{0\}$$

$$\{\phi\}_r^T[K]\{\phi\}_s-\omega_s^2\{\phi\}_r^T[m]\{\phi\}_s=0 \quad ②$$

چون [K] و [m] متقارن است، رابطه ② ←

$$\{\phi\}_s^T[K]\{\phi\}_r-\omega_s^2\{\phi\}_s^T[m]\{\phi\}_r=0 \quad ③$$

رابطه ① ـ رابطه ③ $=(\omega_s^2-\omega_r^2)\{\phi\}_s^T[m]\{\phi\}_r=0 \Rightarrow$

رابطه تعامد مودها نسبت به ماتریس سختی و جرم:

$$\{\phi\}_s^T[m]\{\phi\}_r=0 \quad , \quad \{\phi\}_s^T[K]\{\phi\}_r=0$$

NORMALIZATION OF MODES — مقیاس کردن مودها

$\{\phi\}_i$ الف ـ با تقسیم مولفه های بردار مود بر بزرگترین عدد آنها، بردار مود به عدد یک مقیاس می‌شود.

$$\{\phi\}=\begin{Bmatrix}2.0\\1.0\end{Bmatrix}\rightarrow\{\phi\}=\begin{Bmatrix}1.0\\0.5\end{Bmatrix}$$

ب ـ مقیاس نمودن برحسب ماتریس جرم به نحوی که

$$\{\phi\}_i^T[m]\{\phi\}_i=1$$

$$\{\phi\}_i'^T[m]\{\phi\}_i'=M_i \rightarrow \{\phi\}_i=\frac{1}{\sqrt{M_i}}\{\phi\}_i'$$

$$[\Phi] = \left[\{\phi\}_1 , \{\phi\}_2 \ldots \{\phi\}_N \right]$$ ماتریس مودال

$$[\Omega^2] = \begin{bmatrix} \omega_1^2 & & 0 \\ & \omega_2^2 & \\ 0 & & \ddots \; \omega_N^2 \end{bmatrix}$$ ماتریس مقادیر مشخصه (فرکانسی)

MODAL ANALYSIS تحلیل دینامیکی سیستم های چند درجه آزادی به روش مودال

$$[m]\{\ddot{u}\} + [c]\{\dot{u}\} + [K]\{u\} = \{P(t)\}$$

از تحلیل ارتعاش آزاد مقادیر ω_i و $\{\phi\}_i$ معلوم است ($i = 1$ تا N).

تغییر متغیر از مجهول فیزیکی $\{u\}$ به مجهول مودال :

$$\{u\} = [\Phi]\{q\}$$

$$\{u(t)\} = \sum_{i=1}^{N} \{\phi\}_i \, q_i(t)$$

(۱۰۲)

$$[m][\Phi]\{\ddot{q}\} + [c][\Phi]\{\dot{q}\} + [K][\Phi]\{q\} = \{P(t)\}$$

طرفین رابطه را در $\{\phi\}_i^T$ ضرب می‌کنیم :

$$\underbrace{\{\phi\}_i^T[m][\Phi]\{\ddot{q}\}} + \{\phi\}_i^T[c][\Phi]\{\dot{q}\} + \{\phi\}_i^T[K][\Phi]\{q\} = \{\phi\}_i^T\{P(t)\}$$

$$\begin{Bmatrix} \phi_{1i} \\ \phi_{2i} \\ \vdots \\ \phi_{Ni} \end{Bmatrix}^T \begin{bmatrix} m_{11} & & 0 \\ & m_{22} & \\ 0 & & \ddots \; m_{NN} \end{bmatrix} \left[\begin{Bmatrix} \phi_{11} \\ \phi_{21} \\ \vdots \\ \phi_{N1} \end{Bmatrix} \begin{Bmatrix} \phi_{12} \\ \phi_{22} \\ \vdots \\ \phi_{N2} \end{Bmatrix} \cdots \begin{Bmatrix} \phi_{1i} \\ \phi_{2i} \\ \vdots \\ \phi_{Ni} \end{Bmatrix} \cdots \begin{Bmatrix} \phi_{1N} \\ \phi_{2N} \\ \vdots \\ \phi_{NN} \end{Bmatrix} \right] \begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_N \end{Bmatrix}$$

$$\underbrace{\{\phi\}_i^T[m]\{\phi\}_1 \ddot{q}_1}_{\text{صفر}} + \underbrace{\{\phi\}_i^T[m]\{\phi\}_2}_{\text{صفر}} \ddot{q}_2 + \cdots + \underbrace{\{\phi\}_i^T[m]\{\phi\}_i}_{M_i} \underbrace{\ddot{q}_i}_{\ddot{q}_i} + \cdots$$

برای در نظر گرفتن میرایی معمولاً با روش رایلی عمل می‌شود $[C] = \alpha[m] + \beta[K]$

بنابراین خاصیت تعامد مودها نسبت به ماتریس میرایی نیز برقرار می‌شود :

$$\{\phi\}_r^T[C]\{\phi\}_s = 0 \quad , \quad \{\phi\}_i^T[C]\{\phi\}_i = C_i$$

در نهایت :

$$M_i \ddot{q}_i + C_i \dot{q}_i + K_i q_i = P_i$$

$$\{\phi\}_i^T[K]\{\phi\}_i = K_i \quad , \quad \{\phi\}_i^T\{P(t)\} = P_i$$

با توجه به روابط بین جرم ، سختی و میرایی

معادله یک درجه آزادی مستقل $\ddot{q}_i + 2\xi_i \omega_i \dot{q}_i + \omega_i^2 q_i = P_i / M_i$

$i = 1$ تا N

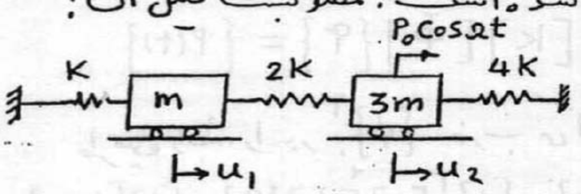
$\Rightarrow q_i$ معلوم

$$\{u\} = [\Phi]\{q\} = \sum_{i=1}^{N} \{\phi\}_i q_i$$

مودهای اولیه مهم است، می توان چند مود اول را در نظر گرفت

$$\{u\} = \sum_{i=1}^{r} \{\phi\}_i q_i \qquad r \ll N$$

مثال - یک سازه دو درجه آزادی بصورت زیر مدل شده است . مطلوبست تحلیل آن؟



$K = 1000$, $m = 0.5$

$\xi = 2\%$. واحدها هماهنگ است .

$\Omega = 1.03\,\omega_1$

$$[K] = \begin{bmatrix} 3K & -2K \\ -2K & 6K \end{bmatrix} , \quad [m] = m\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\det\left[[K] - \omega^2 [m]\right] = 0 \Rightarrow \begin{vmatrix} 3K - \omega^2 m & -2K \\ -2K & 6K - 3\omega^2 m \end{vmatrix} = 0$$

$$\Rightarrow \omega_1^2 = 1.2417\ K/m \ , \quad \omega_2^2 = 3.7584\ K/m$$

$$[\Phi] = \begin{bmatrix} 1.0000 & 1.0000 \\ 0.8792 & -0.3792 \end{bmatrix} \quad , \quad \Omega = 1.03\omega_1 = 51.32 \ \text{Rad/s}$$

$$M_1 = \{\phi\}_1^T [m] \{\phi\}_1 = 3.319\,m \quad , \quad M_2 = 1.4314\,m$$

$$K_1 = \{\phi\}_1^T [K] \{\phi\}_1 = 4.1212K \quad , \quad K_2 = 5.3798K$$

$$\{P(t)\} = \begin{Bmatrix} 0 \\ P_0 \cos\Omega t \end{Bmatrix} \quad , \quad \{\phi\}_1^T \{P(t)\} = P_1 = 0.8792\,P_0 \cos\Omega t$$

$$\{\phi\}_2^T \{P(t)\} = P_2 = -0.3792\,P_0 \cos\Omega t$$

$$\ddot{q}_i + 2\xi_i \omega_i \dot{q}_i + \omega_i^2 q_i = \frac{P_i}{M_i}$$

$$\{u\} = [\Phi]\{q\} = \sum_{i=1}^{2} \{\phi\}_i q_i$$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 2.9008 \\ 2.5500 \end{Bmatrix} \frac{P_0}{K} \cos(\Omega t - 2.5468) -$$

اثر مود اول

$$\begin{Bmatrix} 0.1084 \\ -0.0411 \end{Bmatrix} \frac{P_0}{K} \cos(\Omega t - 0.0364)$$

اثر مود دوم

تحلیل سیستمهای چند درجه آزادی به روش طیفی

$$\ddot{y}_i + 2\zeta_i\omega_i\dot{y}_i + \omega_i^2 y_i = \frac{\{\phi_i\}^T[m]\{I\}}{\{\phi_i\}^T[m]\{\phi_i\}} \times -\ddot{u}_g(t)$$

تعاریف زیر در نظر گرفته میشود:

$$\mu_i = \{\phi_i\}^T[m]\{I\} = \{I\}^T[m]\{\phi_i\}$$

μ_i ضریب شرکت پذیری مودال زلزله در مورد iام میباشد.

$$\{\phi_i\}^T[m]\{\phi_i\} = M_i$$

$$\ddot{y}_1 + 2\zeta_i\omega_i\dot{y}_i + \omega_i^2 y_i = \frac{-\ddot{u}_g(t)\mu_i}{M_i}$$

برای حل معادله یک درجه آزادی فوق با استفاده از انتگرال دیوهامل، داریم:

$$y_i(t) = \frac{-\mu_i}{M_i\omega_i}\int_0^t e^{-\zeta_i\omega_i(t-z)}\sin\omega_i(t-\tau)d\tau \times \ddot{u}_g(t)$$

$$y_i(t) = \frac{\mu_i}{M_i\omega_i}V_i(t)$$

$$\{u(t)\} = \sum_{i=1}^{n}\{\phi_i\}y_i(t)$$

$$\{u(t)\} = \sum_{i=1}^{n}\{\phi_i\}[\frac{\mu_i V_i(t)}{M_i\omega_i}]$$

در روش طیفی بجای جستجو جهت یافتن تاریخچه جواب (تغییر مکان) به مقدار حداکثر

بسنده میشود

$$\max\{u_i(t)\} = \{\phi_i\}\frac{\mu_i}{M_i\omega_i}V_{i\max}(t) = \{\phi_i\}\frac{\mu_i}{M_i\omega_i}S_{vi}$$

برای به دست آوردن جواب کل باید مقادیر رابطه فوق را برای کلیه مودهای مورد نظر بدست آورده و ترکیب نمود. لیکن مقادیر حداکثر تغییر مکان در مودهای مختلف در یک لحظه اتفاق نمیافتند و بنابراین ترکیب مستقیم آنها صحیح نخواهد بود:

$$\max\{u(t)\} \neq \sum_{i=1}^{n} \max\{u_i(t)\}$$

لذا همانند مسائل مشابه، جهت کسب بهترین نتیجه، از روش جذر مجموع مربعات نتایج استفاده میگردد، در این صورت خواهیم داشت:

$$\max\{u(t)\} \cong \left\{ \sum_{i=1}^{n} \left[\left[\frac{\{\phi_i\}\mu_i}{M_i \omega_i} \right] S_{Vi} \right]^2 \right\}^{\frac{1}{2}}$$

نیروی زلزله وارد به پای سازه ناشی از مود iام، برابر مجموع حاصل ضرب جرم هر طبقه در شتاب موثر وارد به آن طبقه در اثر زلزله خواهد بود:

$$F_i(t) = \sum_{k=1}^{n} m_k \ddot{u}_{ki}$$

$\ddot{u}_{ki}$ شتاب وارد به جرم k ناشی از مود iام میباشد. شتاب موثر وارد بر یک جرم بصورت زیر بیان میشود:

$$\ddot{u}(t) = \omega^2 u(t)$$

$$F_i(t) = \sum_{k=1}^{n} m_k \omega_i^2 u_{ki}$$

u_{ki} تغییر مکان جرم k ناشی از مود iام میباشد.

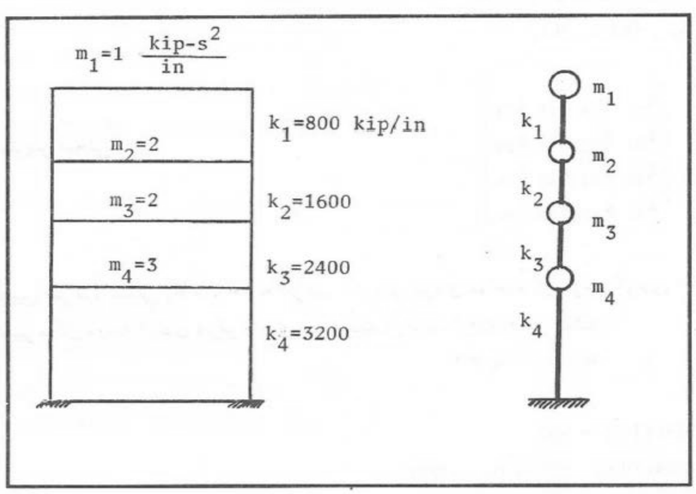
طبق روابط موجود در تحلیل دینامیکی، حداکثر نیروی ناشی از زلزله در طبقه k ام برای مورد i ام، بصورت زیر نوشته میشود:

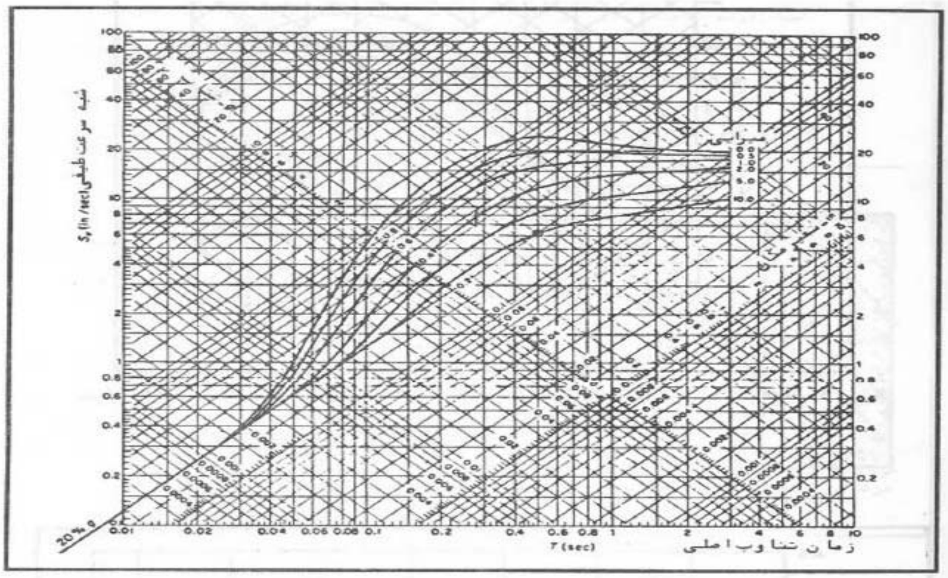
$$\max F_{ki}(t) = m_k \ddot{u}_{ki(\max)} = m_k \omega_i^2 u_{ki(\max)}$$

و البته حداکثر تغییر مکان طبقه k ام برای مود i ام یعنی $u_{ki(\max)}$ از رابطه زیر حاصل میشود.

$$\max u_{ki} = \left(\frac{\phi_{ki} \mu_i}{M_i \omega_i} \right) S_{vi}$$

مثال: برای یک ساختمان چهار طبقه با مشخصات ارائه شده روی شکل مطلوبست تعیین تغییر مکان هر طبقه و همچنین نیروی برشی پای سازه به روش طیفی در صورتی که درصد میرائی برابر پنج درصد در نظر گرفته میشود. از طیف طرح سه جانبه شکل استفاده شود.





شکل ۱۲۷ ـ طیف طراحی معتبر و رایج آمریکا بر اساس زلزله‌های بزرگ ایالات متحده

(Washington - Olympia, (California) - Taft - El Centro)

تهیه شده توسط Housner و مقیاس شده بر اساس شتاب حداکثر $a_{max} = 0.2g$

$$[m] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & -k_4 & +k_4 \end{bmatrix}$$

$$[k] = 800 \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 3 & -2 & 0 \\ 0 & -2 & 5 & -3 \\ 0 & 0 & -3 & 7 \end{bmatrix}$$

$$\det([k]-\omega^2[M])=0$$

$$\{\omega\}=\begin{bmatrix}13.249\\29.660\\41.079\\55.882\end{bmatrix}$$

$$[\phi]=\begin{bmatrix}1.00000 & 1.00000 & -0.90145 & 0.15436\\0.77910 & -0.09963 & 1.00000 & -0.44817\\0.49655 & -0.53989 & -0.15859 & 1.00000\\0.23506 & -0.43761 & -0.70797 & -0.63688\end{bmatrix}$$

ضرایب M_i را محاسبه میکنیم:

$$M_i=\{\phi_i\}^T[m]\{\phi_i\}$$

$$M_1=\begin{Bmatrix}1.00000\\0.77910\\0.49655\\0.23506\end{Bmatrix}^T\begin{bmatrix}1&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&3\end{bmatrix}\begin{Bmatrix}1.00000\\0.77910\\0.49655\\0.23506\end{Bmatrix}=2.87288$$

به همین ترتیب:

$$M_2=2.17732$$
$$M_3=4.36658$$
$$M_4=3.64239$$

مقادیر ضریب μ_i حساب میشود:

$$\mu_i = \{\phi_i\}^T [m] \{I\} = \{I\}^T [m] \{\phi_i\} = \sum_{k=1}^{n} m_k \phi_{ki}$$

$$\mu_1 = 1\times 1 + 2\times 0.77910 + 2\times 0.49655 + 3\times 0.23506 = 4.2565$$

$$\mu_2 = -1.5919$$

$$\mu_3 = -1.3425$$

$$\mu_4 = -0.6526$$

برای محاسبه حداکثر تغییر مکان نقطه انتهایی (طبقه آخر) یا طبقه شماره ۱

$$\max u_1 \cong \left\{ \sum_{i=1}^{4} \left[\left[\frac{\phi_{1i}\mu_i}{M_i \omega_i} \right] S_{vi} \right]^2 \right\}^{\frac{1}{2}}$$

ابتدا باید با استفاده از طیف مورد نظر مقادیر طیفی لازم را تعیین نمود (چهار درجه آزادی و

در حقیقت چهار پریود داریم که $T_i = \dfrac{2\pi}{\omega_i}$):

i	T_i	S_d(in)	S_v(in/s)	S_a(g)
1	0.47	0.6	8.0	0.28
2	0.21	0.14	4.0	0.30
3	0.15	0.07	2.7	0.28
4	0.11	0.03	1.6	0.24

$$\max u_1 \approx \{[\frac{1(4.2565)(8.0)}{2.873(13.294)}]^2 + [\frac{1(1.5919)(4.0)}{2.177(29.66)}]^2 +$$

$$[\frac{0.9015(1.3425)(2.7)}{4.367(41.079)}]^2 + [\frac{0.1544(0.6525)(1.6)}{3.642(55.882)}]^2\}^{1/2}$$

$$= \{0.7949 + 0.0097 + 0.0003 + 0.0000006\}^{1/2} = 0.897 in$$

$$\max u_2 \approx 0.695(in)$$

$$\max u_3 \approx 0.446(in)$$

$$\max u_4 \approx 0.214(in)$$

برای محاسبه حداکثر نیروی برشی پای ساختمان داریم:

$$\max F \cong \left\{ \sum_{i=1}^{n} \left[\left[\frac{\mu_i^2}{M_i} \right] S_{ai} \right]^2 \right\}^{\frac{1}{2}}$$

توجه شود که در سیستم آحاد انگلیسی مقدار شتاب ثقل برابر است با $g=386 in/s^2$

$$\max F \approx \{[\frac{(4.2565)^2(0.28)(386)}{2.873}]^2 + [\frac{(1.5919)^2(0.3)(386)}{2.177}]^2 +$$

$$[\frac{(1.3425)^2(0.28)(386)}{4.367}]^2 + [\frac{(0.6526)^2(0.24)(386)}{3.642}]^2\}^{1/2}$$

نیروی برشی پای سازه =696.3kip

مثال

نتایج تحلیل ارتعاش آزاد یک ساختمان ۱۷ طبقه برای سه مود اول آن به شرح ذیل ارائه شده است (شکل مودهای اول، دوم و سوم). در صورتیکه جرم متمرکز طبقات در ستون آخر جدول فوق نشان داده شده باشد، مطلوبست تعیین حداکثر مقادیر برش پایه برای سه مود اول و حداکثر نیروی وارد بر هر طبقه در هر یک از سه مود و تعیین حداکثر تغییر مکان هر طبقه در اثر هر مود. واحد جرم $\frac{t.(\sec)^2}{m}$ است. فرض میشود شتاب طرح منطقه 0.35g باشد

طبقه	$\{\phi_1\}^T$	$\{\phi_2\}^T$	$\{\phi\}^T$	m_i
17	0.040336	-0.042302	0.037938	94.9
16	0.037957	-0.031601	0.019675	109
15	0.035487	-0.020511	0.001312	109
14	0.032911	-0.009365	-0.014753	109
13	0.03024	0.001353	-0.026066	111
12	0.027462	0.010993	-0.030775	111
11	0.024621	0.019092	-0.028385	114
10	0.021777	0.025043	-0.019781	114
9	0.018935	0.028613	-0.007092	116.5
8	0.016169	0.029914	0.005978	120
7	0.013476	0.029411	0.017737	120
6	0.010687	0.027229	0.026475	120
6	0.010687	0.023582	0.030812	122.4
5	0.008383	0.023582	0.030812	122.4
4	0.006080	0.01885	0.030207	122.4
3	0.004003	0.013467	0.025066	124.7
2	0.002231	0.008053	0.016720	124.7
1	0.000837	0.003214	0.007247	137
T_1=0.777,T_2=0.220,T_3=0.099 ثانیه				

ابتدا ضرائب μ_i, M_i محاسبه میشوند.

$$M_1 = \{\phi_i\}^T [m] \{\phi_i\}$$
$$M_1 = M_2 = M_3 = 1$$
$$\mu_i = \{\phi_i\}^T [m] \{I\}$$
$$\mu_1 = 36.8428$$
$$\mu_2 = 17.7249$$
$$\mu_3 = 11.3484$$

حداکثر مقدار برش پایه در هر مود برابر است با:

$$MaxF_i = \frac{\mu_i^2}{M_i} S_{ai}$$

در این مثال نیز از طیف شکل مسأله قبلی استفاده میکنیم. باید توجه شود که طیف فوق

برای شتاب 0.20g مقیاس شده است.

$$T_1 = 0.777 \rightarrow S_{a1} = 0.2g \times \frac{0.35g}{0.20g} = 0.35g$$
$$T_2 = 0.22 \rightarrow S_{a2} = 0.3g \times \frac{0.35g}{0.20g} = 0.525g$$
$$T_3 = 0.099 \rightarrow S_{a3} = 0.22g \times \frac{0.35}{0.20g} = 0.385g$$

$$MaxF_1 = \frac{\mu_1^2}{M_1} S_{a1} = \frac{(36.8428)^2}{1} \times 0.35g = 4660ton$$

$$MaxF_2 = \frac{\mu_2^2}{M_2} S_{a2} = \frac{(17.7249)^2}{1} \times 0.525g = 1618ton$$

$$MaxF_3 = \frac{\mu_3^2}{M_3} S_{a3} = \frac{(11.3484)^2}{1} \times 0.385g = 486.4ton$$

بنابراین نیروی برشی کل ناشی از زلزله مربوط به طیف مورد نظر و با در نظر گرفتن تاثیر سه

مود اول برابر است با:

$$MaxF = (F_1^2 + F_2^2 + F_3^2)^{0.5} = (4660^2 + 1618^2 + 486.4^2)^{0.5} = 4956.8tc$$

برای تعیین حداکثر نیروی ناشی از زلزله در طبقه kام برای مود iام داریم:

$$MaxF_{ki} = M_k \omega_i^2 u_{ki(\max)}$$

که در آن $u_{ki(max)}$ حداکثر تغییر مکان طبقه kام ناشی از مود iام میباشد

$$Maxu_{ki} = \frac{\phi_{ki}\mu_i}{M_i \omega_i} S_{vi}$$

$$MaxF_{ki} = M_k \omega_i^2 \frac{\phi_{ki}\mu_i}{M_i \omega_i} S_{vi} = \frac{M_k \phi_{ki} \mu_i}{M_i} S_{ai}$$

$$MaxF_i = \frac{\mu_i^2}{M_i} S_{ai}$$

$$MaxF_{ki} = \frac{M_k \phi_{ki}}{\mu_i} (MaxF_i)$$

طبقه	مقدار ضریب $M_k\phi_{ki} / \mu_i$ برای مودهای مختلف			حداکثر نیروی ناشی از زلزله در طبقات مختلف و برای مودهای مختلف (ton)		
	مود اول	مود دوم	مود سوم	مود اول	مود دوم	مود سوم
17	0.103897	-0.226487	0.317253	484.16	-366.45	154.31
16	0.1122964	0.19433	0.1889758	523.30	-314.42	91.92
15	0.104989	-0.126133	-0.126016	489.25	-204.08	6.13
14	0.097368	-0.057590	-0.1417	453.73	-93.18	-68.92
13	0.091119	0.008473	-0.254954	424.61	13.71	-124.01
12	0.082737	0.068842	-0.301013	385.55	111.39	-146.41
11	0.076183	0.122793	-0.28514	355.01	198.68	-138.69
10	0.067383	0.161067	-0.198709	314.00	260.61	-96.66
9	0.059874	0.188064	-0.072805	279.01	304.29	-40.28
8	0.052664	0.20252	0.063212	245.41	327.68	30.75
7	0.043892	0.199116	0.187554	204.54	322.19	91.23
6	0.035395	0.18434	0.279951	164.94	298.26	136.17
5	0.02785	0.162846	0.3323276	129.78	263.48	161.64
4	0.0202	0.130169	0.325802	94.13	210.61	158.47
3	0.013549	0.094808	0.275433	63.14	153.4	133.97
2	0.007551	0.056655	0.1837248	35.19	91.68	89.36
1	0.0031124	0.024842	0.0874815	14.5	40.19	42.55

$$Maxu_{ki} = \frac{\phi_{ki}\mu_i}{M_i\omega_i} S_{vi}$$

$$Maxu_{ki} = \frac{\phi_{ki}\mu_i S_{ai}}{M_i\omega_i^2}$$

$$\frac{\phi_{ki}\mu_i S_{ai}}{M_i} = \frac{MaxF_{ki}}{M_k}$$

$$Maxu_{ki} = \frac{MaxF_{ki}}{M_k\omega_i^2}$$

طبقه	مقادیر تغییر مکان هر طبقه در سه مود اول (متر)		
	مود اول	مود دوم	مود سوم
17	0.078	-0.00473	0.0004
16	0.0734	-0.00354	0.0002
15	0.0686	-0.00229	0.0001
14	0.0636	-0.00105	-0.00016
13	0.0589	0.000145	-0.00027
12	0.0531	0.00123	-0.0003
11	0.0476	0.00214	-0.0003
10	0.0421	0.0028	-0.00021
10	0.0421	0.0028	-0.00008
9	0.0366	0.0032	-0.00008
8	0.0313	0.00335	0.00006
7	0.0261	0.00329	0.00019
6	0.0210	0.00305	0.00028
5	0.0162	0.00264	0.0003
4	0.0118	0.00211	0.00032
3	0.00774	0.00151	0.00026
2	0.00431	0.0009	0.00018
1	0.00162	0.00036	0.00008