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Solutions Manual

To

INTRODUCTORY QUANTUM OPTICS

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Introduction

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Field Quantization

2.1 problem 2.1

Eq. (2.5) has the form

$$E_x(z,t) = \sqrt{\frac{2\omega^2}{V\varepsilon_0}} q(t)\sin(kz), \tag{2.1.1}$$

and Eq. (2.2)

$$\nabla \times \mathbf{B} = \mu_0\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \tag{2.1.2}$$

Both equations lead to

$$-\partial_z B_y = \mu_0\varepsilon_0 \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \dot{q}(t)\sin(kz), \tag{2.1.3}$$

which itself leads to Eq. (2.6)

$$B_y(z,t) = \frac{\mu_0\varepsilon_0}{k} \sqrt{\frac{2\omega^2}{V\varepsilon_0}} \dot{q}(t)\cos(kz). \tag{2.1.4}$$

2.2 problem 2.2

$$H = \frac{1}{2}\int dV \left[\varepsilon_0 E_x^2(z,t) + \frac{1}{\mu_0} B_y^2(z,t)\right]. \tag{2.2.1}$$

From the previous problem

$$E_x(z,t) = \sqrt{\frac{2\omega^2}{V\varepsilon_0}} q(t)\sin(kz), \tag{2.2.2}$$

so

$$\varepsilon_0 E_x^2(z,t) = \frac{2\omega^2}{V} q^2(t)\sin^2(kz). \tag{2.2.3}$$

Also

$$B_y(z,t) = \frac{\mu_0\varepsilon_0}{k}\sqrt{\frac{2\omega^2}{V\varepsilon_0}}\dot{q}(t)\cos(kz), \tag{2.2.4}$$

and

$$\frac{1}{\mu_0} B_y^2(z,t) = \frac{2}{V} p^2(t)\cos^2(kz), \tag{2.2.5}$$

where we have used that $c^2 = (\mu_0\varepsilon_0)^{-1}$, $p(t) = \dot{q}(t)$, and $ck = \omega$. Eq. 2.2.1 becomes then

$$H = \frac{1}{V}\int dV \left[\omega^2 q^2(t)\sin^2(kz) + p^2(t)\cos^2(kz)\right]. \tag{2.2.6}$$

Using these simple trigonometric identities $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^2 x = \frac{1-\cos 2x}{2}$, we can simplify equation 2.2.6 further to:

$$H = \frac{1}{2V}\int dV \left[\omega^2 q^2(t)(1+\cos 2kz) + p^2(t)(1-\cos 2kz)\right]. \tag{2.2.7}$$

Because of the periodic boundaries both cosine terms drop out, also $\frac{1}{V}\int dV = 1$ and we end up by

$$H = \frac{1}{2}\left(p^2 + w^2 q^2\right). \tag{2.2.8}$$

It is easy to see that this Hamiltonian has the form of a simple harmonic oscillator.

2.3 problem 2.3

Let f be a function defined as:

$$f(\lambda) = e^{i\lambda\hat{A}}\hat{B}e^{-i\lambda\hat{A}}. \tag{2.3.1}$$

If we expand f as

$$f(\lambda) = c_0 + c_1(i\lambda) + c_2\frac{(i\lambda)^2}{2!} + ..., \tag{2.3.2}$$

where

$$\begin{aligned} c_0 &= f(0) \\ c_1 &= f'(0) \\ c_2 &= f''(0)\cdots \end{aligned}$$

Also

$$\begin{aligned} c_0 &= f(0) = \hat{B} \\ c_1 &= f'(0) = \left[\hat{A}e^{i\lambda\hat{A}}\hat{B}e^{-i\lambda\hat{A}} - e^{i\lambda\hat{A}}\hat{B}\hat{A}e^{-i\lambda\hat{A}}\right]\Big|_{\lambda=0} = \left[\hat{A},\hat{B}\right] \\ c_2 &= \left[\hat{B},\left[\hat{A},\hat{B}\right]\right]. \end{aligned}$$

The same way we can determine the other coefficients.

2.4 problem 2.4

Let

$$f(x) = e^{\hat{A}x}e^{\hat{B}x} \tag{2.4.1}$$

$$\begin{aligned} \frac{df(x)}{dx} &= \hat{A}e^{\hat{A}x}e^{\hat{B}x} + e^{\hat{A}x}\hat{B}e^{\hat{B}x} \\ &= \left(\hat{A} + e^{\hat{A}x}\hat{B}e^{-\hat{A}x}\right)f(x) \end{aligned}$$

It is easy to prove that

$$\left[\hat{B},\hat{A}^n\right] = n\hat{A}^{n-1}\left[\hat{B},\hat{A}\right] \tag{2.4.2}$$

$$\begin{aligned}
\left[\hat{B}, e^{-\hat{A}x}\right] &= \sum \left[\hat{B}, \frac{\left(-\hat{A}x\right)^n}{n!}\right] \\
&= \sum (-1)^n \frac{x^n}{n!} \left[\hat{B}, \hat{A}^n\right] \\
&= \sum (-1)^n \frac{x^n}{(n-1)!} \hat{A}^{n-1} \left[\hat{B}, \hat{A}\right] \\
&= -e^{-\hat{A}x} \left[\hat{B}, \hat{A}\right] x
\end{aligned}$$

So

$$\hat{B}e^{-\hat{A}x} - e^{-\hat{A}x}\hat{B} = -e^{-\hat{A}x} \left[\hat{B}, \hat{A}\right] x$$

$$e^{-\hat{A}x}\hat{B}e^{\hat{A}x} = \hat{B} - e^{-\hat{A}x} \left[\hat{B}, \hat{A}\right] x \tag{2.4.3}$$

$$e^{\hat{A}x}\hat{B}e^{-\hat{A}x} = \hat{B} + e^{\hat{A}x} \left[\hat{A}, \hat{B}\right] x \tag{2.4.4}$$

Equation 4.1.1 becomes

$$\frac{df(x)}{dx} = \left(\hat{A} + \hat{B} + \left[\hat{A}, \hat{B}\right]\right) f(x). \tag{2.4.5}$$

Since $\left[\hat{A}, \hat{B}\right]$ commutes with $\hat{A}$ and $\hat{B}$, we can solve equation 2.4.5 as an ordinary equation. The solution is simply

$$f(x) = \exp\left[\left(\hat{A} + \hat{B}\right) x\right] \exp\left(\frac{1}{2}\left[\hat{A}, \hat{B}\right] x^2\right) \tag{2.4.6}$$

If we take $x = 1$ we will have

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}\left[\hat{A}, \hat{B}\right]} \tag{2.4.7}$$

2.5 problem 2.5

$$|\Psi(0)\rangle = \frac{1}{\sqrt{2}} \left(|n\rangle + e^{i\varphi}|n+1\rangle\right). \tag{2.5.1}$$

$$
\begin{aligned}
|\Psi(t)\rangle &= e^{-i\frac{\hat{H}t}{\hbar}}|\Psi(0)\rangle \\
&= \frac{1}{\sqrt{2}}\left(e^{-i\frac{\hat{H}t}{\hbar}}|n\rangle + e^{-i\frac{\hat{H}t}{\hbar}}|n+1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(e^{-in\omega t}|n\rangle + e^{i\varphi}e^{-i(n+1)\omega t}|n+1\rangle\right),
\end{aligned}
$$

where we have used $\frac{E}{\hbar} = \omega$

$$
\begin{aligned}
\hat{n}|\Psi(t)\rangle &= \hat{a}^\dagger \hat{a}|\Psi(t)\rangle \\
&= \frac{1}{\sqrt{2}}\left(e^{-in\omega t}n|n\rangle + e^{i\varphi}e^{-i(n+1)\omega t}(n+1)|n+1\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
\langle \hat{n} \rangle &= \langle \Psi(t)|\hat{n}|\Psi(t)\rangle \\
&= \frac{1}{2}(n+n+1) \\
&= n + \frac{1}{2}
\end{aligned}
$$

the same way

$$
\begin{aligned}
\left\langle \hat{n}^2 \right\rangle &= \langle \Psi(t)|\hat{n}\hat{n}|\Psi(t)\rangle \\
&= \frac{1}{2}\left(n^2 + (n+1)^2\right) \\
&= n^2 + n + \frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \ (\Delta\hat{n})^2 \right\rangle &= \left\langle \hat{n}^2 \right\rangle - \langle \hat{n} \rangle^2 \\
&= \frac{1}{4}
\end{aligned}
$$

$$
\begin{aligned}
\hat{E}|\Psi(t)\rangle &= \mathcal{E}_0 \sin(kz)\left(\hat{a}^\dagger + \hat{a}\right)|\Psi(t)\rangle \\
&= \frac{1}{\sqrt{2}}\mathcal{E}_0 \sin(kz)\left(\hat{a}^\dagger + \hat{a}\right)\left(e^{-in\omega t}|n\rangle + e^{i\varphi}e^{-i(n+1)\omega t}|n+1\rangle\right) \\
&= \frac{1}{\sqrt{2}}\mathcal{E}_0 \sin(kz)\Big[e^{-in\omega t}\left(\sqrt{n+1}|n+1\rangle + \sqrt{n}|n-1\rangle\right) \\
&\quad + e^{i\varphi}e^{-i(n+1)\omega t}\left(\sqrt{n+2}|n+2\rangle + \sqrt{n+1}|n\rangle\right)\Big]
\end{aligned}
$$

$$\begin{aligned}\langle\Psi(t)|\hat{E}|\Psi(t)\rangle &= \frac{1}{2}\mathcal{E}_0\sin(kz)\left(e^{i\omega t}\sqrt{n+1}+e^{i\varphi}e^{-i\omega t}\sqrt{n+1}\right)\\ &= \sqrt{n+1}\mathcal{E}_0\sin(kz)\cos(\varphi-\omega t)\end{aligned}$$

$$\begin{aligned}\left\langle\hat{E}^2\right\rangle &= \langle\Psi(t)|\hat{E}\hat{E}|\Psi(t)\rangle\\ &= 2(n+1)\mathcal{E}_0^2\sin^2(kz)\end{aligned}$$

$$\left\langle\left(\Delta\hat{E}\right)^2\right\rangle = (n+1)\mathcal{E}_0^2\sin^2(kz)\left[2-\cos^2(\varphi-\omega t)\right]$$

$$\begin{aligned}(\hat{a}^\dagger-\hat{a})|\Psi(t)\rangle &= \frac{1}{\sqrt{2}}\left[e^{-in\omega t}\left(\sqrt{n+1}|n+1\rangle-\sqrt{n}|n-1\rangle\right)\right.\\ &\quad\left.+\,e^{i\varphi}e^{-i(n+1)\omega t}\left(\sqrt{n+2}|n+2\rangle-\sqrt{n+1}|n\rangle\right)\right]\\ \langle(\hat{a}^\dagger-\hat{a})\rangle &= -i\sqrt{n+1}\sin(\varphi-\omega t)\end{aligned}$$

Finally we have the following quantities

$$\begin{aligned}\Delta n &= \frac{1}{2}\\ \Delta E &= \mathcal{E}_0|\sin(kz)|\sqrt{2(n+1)\left[2-\cos^2(\varphi-\omega t)\right]}\\ \left|\langle(\hat{a}^\dagger-\hat{a})\rangle\right| &= \sqrt{n+1}|\sin(\varphi-\omega t)|.\end{aligned}$$

Certainly the inequality in (2.49) holds true since

$$\sqrt{2\left(2-\cos^2(\varphi-\omega t)\right)} > |\sin(\varphi-\omega t)|.$$

2.6 problem 2.6

$$\begin{aligned}\hat{X}_1 &= \frac{1}{2}\left(\hat{a}+\hat{a}^\dagger\right)\\ \hat{X}_2 &= \frac{1}{2i}\left(\hat{a}-\hat{a}^\dagger\right)\\ \hat{X}_1^2 &= \frac{1}{4}\left(\hat{a}^{\dagger 2}+\hat{a}^2+2\hat{a}^\dagger\hat{a}+1\right)\\ \hat{X}_2^2 &= -\frac{1}{4}\left(\hat{a}^{\dagger 2}+\hat{a}^2-2\hat{a}^\dagger\hat{a}-1\right)\end{aligned}$$

$$|\Psi_{01}\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $|\alpha|^2 + |\beta|^2 = 1$. So we can rewrite $\beta = \sqrt{1-|\alpha|^2}e^{i\phi}$ and $\alpha^2 = |\alpha|^2$ without any loss of generality.

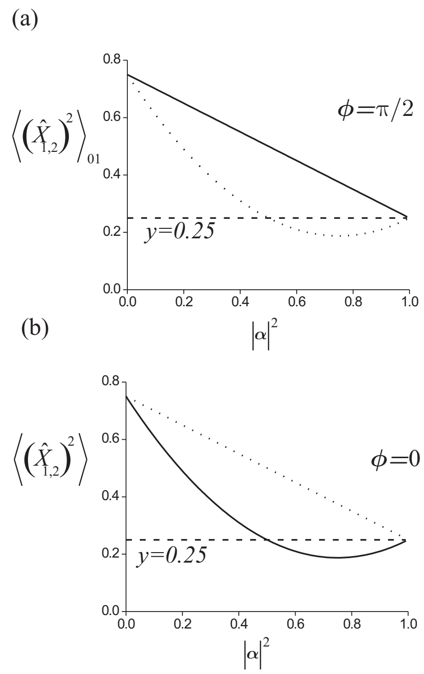
$$\begin{aligned}
\left\langle \hat{X}_1 \right\rangle_{01} &= \frac{1}{2}\left(\alpha^*\beta + \alpha\beta^*\right) \\
\left\langle \hat{X}_2 \right\rangle_{01} &= \frac{1}{2i}\left(\alpha^*\beta - \alpha\beta^*\right)
\end{aligned}$$

$$\begin{aligned}
\left\langle \hat{a}^{\dagger 2} \right\rangle_{01} &= 0 \\
\left\langle \hat{a}^{2} \right\rangle_{01} &= 0 \\
\langle \hat{a}^{\dagger}\hat{a} \rangle_{01} &= |\beta|^2
\end{aligned}$$

$$\begin{aligned}
\left\langle \hat{X}_1^2 \right\rangle_{01} &= \frac{1}{4}\left(2|\beta|^2 + 1\right) \\
\left\langle \hat{X}_2^2 \right\rangle_{01} &= \frac{1}{4}\left(2|\beta|^2 + 1\right)
\end{aligned}$$

$$\begin{aligned}
\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle_{01} &= \frac{1}{4}\left[2|\beta|^2 + 1 - (\alpha^*\beta)^2 - (\alpha\beta^*)^2 - 2|\alpha|^2|\beta|^2\right] \\
&= \frac{1}{4}\left[3 - 4|\alpha|^2 + 2|\alpha|^4 - 2|\alpha|^2(1-|\alpha|^2)\cos(2\phi)\right] \\
\left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle_{01} &= \frac{1}{4}\left[2|\beta|^2 + 1 + (\alpha^*\beta)^2 + (\alpha\beta^*)^2 - 2|\alpha|^2|\beta|^2\right] \\
&= \frac{1}{4}\left[3 - 4|\alpha|^2 + 2|\alpha|^4 + 2|\alpha|^2(1-|\alpha|^2)\cos(2\phi)\right]
\end{aligned}$$

In figures a and b below we plot $\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle_{01}$ (solid line) for $\phi = \pi/2$ and $\left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle_{01}$ (doted line) for $\phi = 0$, respectively. Clearly the quadratures in hands go below the quadrature variances of the vacuum in more than one occasion.



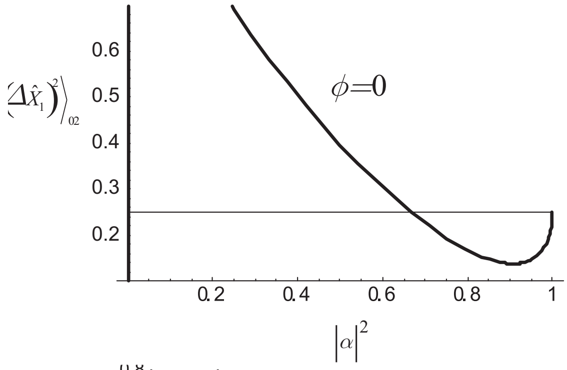
$$|\Psi_{02}\rangle = \alpha|0\rangle + \beta|2\rangle. \tag{2.6.1}$$

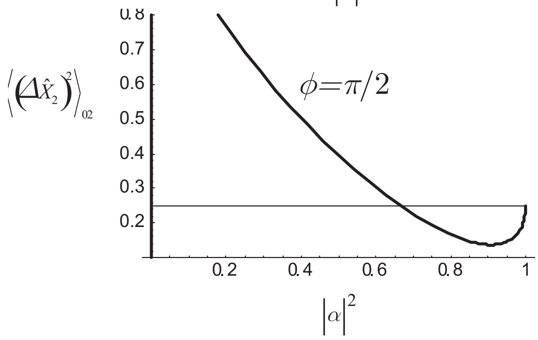
Again, where $|\alpha|^2 + |\beta|^2 = 1$. So we can rewrite $\beta = \sqrt{1-|\alpha|^2}e^{i\phi}$ and $\alpha^2 = |\alpha|^2$ without any loss of generality.

$$\begin{aligned}
\left\langle \hat{X}_1 \right\rangle_{02} &= 0 = \left\langle \hat{X}_2 \right\rangle_{02} \\
\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle_{02} &= \left\langle \hat{X}_1^2 \right\rangle_{02} \\
&= \frac{1}{4}\left(|\alpha + \sqrt{2}\beta|^2 + 3|\beta|^2\right) \\
&= \frac{1}{4}\left[5 - 4|\alpha|^2 + 2\sqrt{2|\alpha|^2(1-|\alpha|^2)}\cos\phi\right] \\
\left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle_{02} &= \left\langle \hat{X}_2^2 \right\rangle_{02} \\
&= \frac{1}{4}\left(|\alpha - \sqrt{2}\beta|^2 + 3|\beta|^2\right) \\
&= \frac{1}{4}\left[5 - 4|\alpha|^2 - 2\sqrt{2|\alpha|^2(1-|\alpha|^2)}\cos\phi\right]
\end{aligned}$$

In figures c and d below we plot $\left\langle \left(\Delta \hat{X}_1\right)^2 \right\rangle_{02}$ for $\phi = 0$ and $\left\langle \left(\Delta \hat{X}_2\right)^2 \right\rangle_{02}$

for $\phi = \pi/2$, respectively. Clearly the quadratures in hands go below the quadrature variances of the vacuum in more than one occasion.





2.7 Problem 2.7

$$|\Psi'\rangle = \mathcal{N}\hat{a}\,|\Psi\rangle$$

$$\begin{aligned}|\mathcal{N}|^2 &= \langle\hat{n}\rangle \\ &= \bar{n} \\ \mathcal{N} &= \frac{1}{\sqrt{\bar{n}}}\end{aligned}$$

$$|\Psi'\rangle = \frac{1}{\sqrt{\bar{n}}}\hat{a}\,|\Psi\rangle$$

$$\begin{aligned}
\overline{n'} &= \langle \Psi' | \hat{n} | \Psi' \rangle \\
&= \frac{1}{\bar{n}} \langle \Psi | \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a} | \Psi \rangle \\
&= \frac{1}{\bar{n}} \left(\langle \Psi | \hat{n}^2 | \Psi \rangle - \langle \Psi | \hat{n} | \Psi \rangle \right) \\
&= \frac{\langle \Psi | \hat{n}^2 | \Psi \rangle}{\bar{n}} - 1 \\
&= \frac{\langle \hat{n}^2 \rangle}{\langle \hat{n} \rangle} - 1.
\end{aligned}$$

Notice that $\overline{n'} \neq \overline{n} - 1$ in general, but for the number state $|n\rangle$, and only of this state we have $\overline{n'} = \overline{n} - 1$.

2.8 Problem 2.8

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |10\rangle) \tag{2.8.1}$$

The average photon number, $\bar{n}$, of this state is

$$\bar{n} = \langle \Psi | \hat{a}^\dagger \hat{a} | \Psi \rangle, \tag{2.8.2}$$

which can be easily calculated to be

$$\bar{n} = \frac{1}{2}(0 + 10) = 5. \tag{2.8.3}$$

If we assume that a single photon is absorbed, our <u>normalized</u> state will become

$$|\Psi\rangle = |9\rangle, \tag{2.8.4}$$

then the average photon becomes

$$\bar{n} = 9. \tag{2.8.5}$$

2.9 Problem 2.9

$$\begin{aligned}
\mathbf{E}(\mathbf{r}, t) &= i \sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} \right] \\
\mathbf{B}(\mathbf{r}, t) &= \frac{i}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s} \right) \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r} - \omega_k t)} \right]
\end{aligned}$$

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r}, t) &= i\nabla \cdot \left(\sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \right) \\
&= i \sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \cdot \left[A_{\mathbf{k}s} \nabla \left(e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right) - A^*_{\mathbf{k}s} \nabla \left(e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right) \right] \\
&= i \sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \cdot \left[i\mathbf{k} A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + i\mathbf{k} A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= -\sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \cdot \mathbf{k} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= 0
\end{aligned}$$

where we have used the vector identity

$$\nabla \cdot (f\mathbf{A}) = f\left(\nabla \cdot \mathbf{A}\right) + \mathbf{A} \cdot \left(\nabla f\right), \tag{2.9.1}$$

and

$$\mathbf{e}_{\mathbf{k}s} \cdot \mathbf{k} = 0. \tag{2.9.2}$$

$$\begin{aligned}
\nabla \cdot \mathbf{B}(\mathbf{r}, t) &= \frac{i}{c}\nabla \cdot \left(\sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \right) \\
&= \frac{i}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \cdot \left[A_{\mathbf{k}s} \nabla \left(e^{i(\mathbf{k}\mathbf{r}-\omega_k t)} \right) - A^*_{\mathbf{k}s} \nabla \left(e^{-i(\mathbf{k}\mathbf{r}-\omega_k t)} \right) \right] \\
&= -\frac{1}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \cdot \mathbf{k} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\mathbf{r}-\omega_k t)} \right] \\
&= 0
\end{aligned}$$

$$
\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r},t) &= i\nabla \times \left(\sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \right) \\
&= i \sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \times \left[A_{\mathbf{k}s} \nabla \left(e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right) - A^*_{\mathbf{k}s} \nabla \left(e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right) \right] \\
&= i \sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \times \left[i\mathbf{k} A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + i\mathbf{k} A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= -\sum_{\mathbf{k},s} \omega_k \mathbf{e}_{\mathbf{k}s} \times \mathbf{k} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= -\sum_{\mathbf{k},s} \frac{\omega_k^2}{c} \mathbf{e}_{\mathbf{k}s} \times \kappa \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= \sum_{\mathbf{k},s} \frac{\omega_k^2}{c} \kappa \times \mathbf{e}_{\mathbf{k}s} \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right]
\end{aligned}
$$

where we have used the vector identity

$$
\nabla \times (f\mathbf{A}) = f\left(\nabla \times \mathbf{A}\right) + \mathbf{A} \times \left(\nabla f\right), \tag{2.9.3}
$$

and

$$
\mathbf{k} = \frac{\omega_k}{c} \kappa. \tag{2.9.4}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= \frac{i}{c} \sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s} \right) \left[A_{\mathbf{k}s} \frac{\partial e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}}{\partial t} - A^*_{\mathbf{k}s} \frac{\partial e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}}{\partial t} \right] \\
&= \frac{1}{c} \sum_{\mathbf{k},s} \omega_k^2 \left(\kappa \times \mathbf{e}_{\mathbf{k}s} \right) \left[A_{\mathbf{k}s} e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s} e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} \right] \\
&= -\nabla \times \mathbf{E}
\end{aligned}
$$

$$\begin{aligned}
\nabla \times \mathbf{B}(\mathbf{r},t) &= \frac{i}{c}\nabla \times \left(\sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right)\left[A_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} - A^*_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right]\right) \\
&= \frac{i}{c}\sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \times \left[A_{\mathbf{k}s}\nabla\left(e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right) - A^*_{\mathbf{k}s}\nabla\left(e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right)\right] \\
&= \frac{i}{c}\sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \times \left[i\mathbf{k}A_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + i\mathbf{k}A^*_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] \\
&= -\frac{1}{c}\sum_{\mathbf{k},s} \omega_k \left(\kappa \times \mathbf{e}_{\mathbf{k}s}\right) \times \mathbf{k}\left[A_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] \\
&= -\sum_{\mathbf{k},s} \frac{\omega_k^2}{c^2}\mathbf{e}_{\mathbf{k}s}\left[A_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] \\
&= \mu_0\epsilon_0\sum_{\mathbf{k},s} \omega_k^2\mathbf{e}_{\mathbf{k}s}\left[A_{\mathbf{k}s}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)} + A^*_{\mathbf{k}s}e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega_k t)}\right] \\
&= \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}
\end{aligned}$$

2.10 Problem 2.10

For thermal light

$$P_n = \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \tag{2.10.1}$$

$$\begin{aligned}
\sum_n n(n-1)...(n-r+1)P_n &= \sum_n n(n-1)...(n-r+1)\frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \\
&= \frac{1}{\bar{n}}(\frac{\bar{n}}{1+\bar{n}})^{r+1}\sum_n n(n-1)...(n-r+1)(\frac{\bar{n}}{1+\bar{n}})^{n-r}
\end{aligned}$$

To simplify the last expression, let's define $x = \frac{\bar{n}}{1+\bar{n}}$, for which $x < 1$,

$$\begin{aligned}
\sum_n n(n-1)...(n-r+1)P_n &= \frac{1}{\bar{n}} x^{r+1} \sum_n n(n-1)...(n-r+1)x^{n-r} \\
&= \frac{1}{\bar{n}} x^{r+1} \frac{\partial^r}{\partial x^r} \sum x^n \\
&= \frac{1}{\bar{n}} x^{r+1} \frac{\partial^r}{\partial x^r} \frac{1}{1-x} \\
&= \frac{1}{\bar{n}} x^{r+1} r! \frac{1}{(1-x)^{r+1}} \\
\langle \hat{n}(\hat{n}-1)(\hat{n}-1)\cdots(\hat{n}-r+1) \rangle &= r!\bar{n}^r
\end{aligned} \tag{2.10.2}$$

2.11 Problem 2.11

$$\begin{aligned}
\left[\hat{C}, \hat{S}\right] &= -\frac{i}{4} \left[\hat{E} + \hat{E}^\dagger, \hat{E} - \hat{E}^\dagger\right] \\
&= \frac{i}{2} \left[\hat{E}, \hat{E}^\dagger\right] \\
&= \frac{i}{2} \left(\hat{E}\hat{E}^\dagger - \hat{E}^\dagger\hat{E}\right) \\
&= \frac{i}{2} \left(1 - 1 + |0\rangle\langle 0|\right) \\
&= \frac{i}{2} |0\rangle\langle 0|
\end{aligned}$$

$$\langle m| \left[\hat{C}, \hat{S}\right] |n\rangle = \frac{i}{2} \delta_{m,0}\delta_{n,0}.$$

Obviously, only the diagonal matrix elements are nonzero.

2.12 Problem 2.12

Using equation (2.229) for

$$\hat{\rho} = \frac{1}{2} \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \tag{2.12.1}$$

we have

$$\begin{aligned}\mathcal{P}(\varphi) &= \frac{1}{2\pi}\langle\varphi|\hat{\rho}|\varphi\rangle \\ &= \frac{1}{2\pi}\sum_n\sum_{n'}\langle n'|e^{-in'\varphi}\hat{\rho}e^{in\varphi}|n\rangle \\ &= \frac{1}{2\pi}\left(1+e^{i\varphi}e^{-i\varphi}\right) \\ &= \frac{1}{\pi}.\end{aligned}$$

This is similar to a thermal state. On the other hand using equation (2.227) for $|\psi\rangle = \frac{1}{2}(|0\rangle + e^{i\theta}|1\rangle)$ we have

$$\begin{aligned}\mathcal{P}(\phi) &= \frac{1}{2\pi}|\langle\phi|\psi\rangle|^2 \\ &= \frac{1}{2\pi}\left[1+\cos(\phi-\theta)\right].\end{aligned}$$

As expected, it is different than a statistical mixture state, the one for the pure state exhibiting a phase dependence.

2.13 Problem 2.13

$$\hat{\rho}_{th} = \sum_{n=0}^{\infty} P_n|n\rangle\langle n| \tag{2.13.1}$$

$$\begin{aligned}\mathcal{P}(\varphi) &= \frac{1}{2\pi}\langle\varphi|\hat{\rho}|\varphi\rangle \\ &= \frac{1}{2\pi}\sum_n\sum_{n'}\langle n'|e^{-in'\varphi}\hat{\rho}e^{in\varphi}|n\rangle \\ &= \frac{1}{2\pi}\sum_n\sum_{n'}\sum_k P_k\langle n'|e^{-in'\varphi}|k\rangle\langle k|e^{in\varphi}|n\rangle \\ &= \frac{1}{2\pi}\sum_k P_k \\ &= \frac{1}{2\pi}\end{aligned}$$

Chapter 3

Coherent States

3.1 Problem 3.1

Let assume that the eigenvector of the creation operator $\hat{a}^\dagger$ exists. So we can write

$$\hat{a}^\dagger|\beta\rangle = \beta|\beta\rangle. \tag{3.1.1}$$

Now let's write $|\beta\rangle$ as a superposition of the number states, namely

$$|\beta\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \tag{3.1.2}$$

Now let's plug the last expression in equation 3.1.1:

$$\hat{a}^\dagger|\beta\rangle = \sum_{n=0}^{\infty} c_n\sqrt{n+1}|n+1\rangle \tag{3.1.3}$$

$$= \beta\sum_{n=0}^{\infty} c_n|n\rangle. \tag{3.1.4}$$

From the last express we deduce that

$$c_0 = 0, \tag{3.1.5}$$

$$c_{n+1} = \frac{1}{\beta}c_n\sqrt{n+1}, \tag{3.1.6}$$

which means all c_n's $= 0$.

3.2 Problem 3.2

Using equation (3.29), we can determine $\Delta\phi$ for large $|\alpha|$.

$$(\Delta\phi)^2 = \left\langle \phi^2 \right\rangle - (\langle \phi \rangle)^2 \tag{3.2.1}$$

For large α

$$\mathcal{P}(\phi) = \left(\frac{2|\alpha|^2}{\pi}\right)^{\frac{1}{2}} \exp\left[-2|\alpha|^2(\phi-\theta)^2\right]$$

$$\begin{aligned}
\left\langle \phi^2 \right\rangle &= \int_{-\pi}^{\pi} \phi^2 \mathcal{P}(\phi) d\phi \\
&= \int_{-\infty}^{\infty} \left(\frac{2|\alpha|^2}{\pi}\right)^{\frac{1}{2}} \phi^2 \exp\left[-2|\alpha|^2(\phi-\theta)^2\right] d\phi \\
&= \left(\frac{2|\alpha|^2}{\pi}\right)^{\frac{1}{2}} \frac{\sqrt{\pi}}{2(2|\alpha|^2)^{3/2}} \\
&= \frac{1}{2|\alpha|^2}
\end{aligned}$$

$$\begin{aligned}
\langle \phi \rangle &= \int_{-\pi}^{\pi} \phi \mathcal{P}(\phi) d\phi \\
&= \int_{-\pi}^{\pi} \left(\frac{2|\alpha|^2}{\pi}\right)^{\frac{1}{2}} \phi \exp\left[-2|\alpha|^2(\phi-\theta)^2\right] d\phi \\
&= \int_{-\infty}^{\infty} \left(\frac{2|\alpha|^2}{\pi}\right)^{\frac{1}{2}} \phi \exp\left[-2|\alpha|^2(\phi-\theta)^2\right] d\phi \\
&= 0
\end{aligned}$$

$$\Delta\phi = \frac{1}{\sqrt{2|\alpha|^2}},$$

where taking the limit of integration to $\pm\infty$ is justified.

3.3 Problem 3.3

We know that the generating function of the Hermite polynomials is defined as (see for example Arfken):

$$e^{-t^2+2tx} = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \tag{3.3.1}$$

Eq.(3.46) reads

$$\psi_\alpha(q) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\frac{\alpha}{\sqrt{2}})^n}{n!} H_n(\xi). \tag{3.3.2}$$

Replacing x, by ξ and t by $\frac{\alpha}{\sqrt{2}}$, we'll get

$$\psi_\alpha(q) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-|\alpha|^2}{2}} e^{-(\frac{\alpha}{\sqrt{2}})^2+2(\frac{\alpha}{\sqrt{2}})\xi}. \tag{3.3.3}$$

Completing the square in the last exponent by adding and subtracting $\frac{\xi^2}{2}$ we would get the needed result:

$$\psi_\alpha(q) = \left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{\frac{-|\alpha|^2}{2}} e^{\frac{\xi^2}{2}} e^{-(\xi-\frac{\alpha}{\sqrt{2}})^2}. \tag{3.3.4}$$

3.4 Problem 3.4

First, we expand $|\alpha\rangle\langle\alpha|$ in number states as

$$|\alpha\rangle\langle\alpha| = \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m|, \tag{3.4.1}$$

so now we can calculate

$$\hat{a}^\dagger|\alpha\rangle\langle\alpha| = \hat{a}^\dagger \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \tag{3.4.2}$$

$$\begin{aligned}
\hat{a}^\dagger|\alpha\rangle\langle\alpha| &= \hat{a}^\dagger \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle\langle m| \\
&= \sum_{n,m} e^{-|\alpha|^2} \frac{\alpha^n}{\sqrt{(n+1)!}} \frac{\alpha^{*m}}{\sqrt{m!}} (n+1)|n+1\rangle\langle m| \\
&= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} e^{-|\alpha|^2} \frac{\alpha^{n-1}}{\sqrt{(n)!}} \frac{\alpha^{*m}}{\sqrt{m!}} (n)|n\rangle\langle m|.
\end{aligned}$$

On the other hand

$$
\begin{aligned}
\left(\alpha^* + \frac{\partial}{\partial\alpha}\right)|\alpha\rangle\langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right)\sum_{n,m} e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}|n\rangle\langle m| \\
= \alpha^*\sum_{n,m} e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}|n\rangle\langle m| - \alpha^*\sum_{n,m} e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}|n\rangle\langle m| \\
+ \sum_{n,m} e^{-|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}\sqrt{n+1}|n+1\rangle\langle m| \\
= \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} e^{-|\alpha|^2}\frac{\alpha^{n-1}}{\sqrt{n!}}\frac{\alpha^{*m}}{\sqrt{m!}}n|n\rangle\langle m|.
\end{aligned}
$$

Notice that we have used $|\alpha|^2 = \alpha\alpha^*$. Also α and α^* are treated linearly independent. The same way, we can prove the other identity.

3.5 Problem 3.5

The quadrature operators are defined in equations (2.52) and (2.53) as

$$
\begin{aligned}
\hat{X}_1 &= \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right) \\
\hat{X}_2 &= \frac{1}{2i}\left(\hat{a} - \hat{a}^\dagger\right)
\end{aligned}
$$

Using the following properties of the coherent state

$$
\begin{aligned}
\hat{a}|\alpha\rangle &= \alpha|\alpha\rangle, \\
\langle\alpha|\hat{a}^\dagger &= \alpha^*\langle\alpha|,
\end{aligned}
$$

$$\langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2}(\alpha + \alpha^*) \tag{3.5.1}$$

$$\langle\alpha|\hat{X}_1|\alpha\rangle = \frac{1}{2i}(\alpha - \alpha^*) \tag{3.5.2}$$

$$
\begin{aligned}
\langle\alpha|\hat{X}_1|\alpha\rangle^2 &= \frac{1}{4}\left(\alpha^2 + \alpha^{*2} + 2|\alpha|^2\right) \\
\langle\alpha|\hat{X}_2|\alpha\rangle^2 &= \frac{-1}{4}\left(\alpha^2 + \alpha^{*2} - 2|\alpha|^2\right)
\end{aligned}
$$

$$\begin{aligned}
\hat{X}_1^2 &= \frac{1}{4}(\hat{a} + \hat{a}^\dagger)(\hat{a} + \hat{a}^\dagger) \\
&= \frac{1}{4}(\hat{a}^2 + \hat{a}^{\dagger 2} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}) \\
\hat{X}_1^2 &= \frac{1}{4}(\hat{a}^2 + \hat{a}^{\dagger 2} + 2\hat{a}^\dagger\hat{a} + 1) \\
\hat{X}_2^2 &= \frac{-1}{4}(\hat{a}^2 + \hat{a}^{\dagger 2} - 2\hat{a}^\dagger\hat{a} - 1)
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\hat{X}_1^2|\alpha\rangle &= \frac{1}{4}(\alpha^2 + \alpha^{*2} + 2|\alpha|^2 + 1) \\
\langle\alpha|\hat{X}_2^2|\alpha\rangle &= \frac{-1}{4}(\alpha^2 + \alpha^{*2} - 2|\alpha|^2 - 1).
\end{aligned}$$

Quantum fluctuations of the quadrature operators can be characterized by the variance

$$\left\langle \left(\triangle \hat{X}_{\substack{2\\1}}\right)^2 \right\rangle = \left\langle \hat{X}_{\substack{2\\1}}^2 \right\rangle - \left\langle \hat{X}_{\substack{2\\1}} \right\rangle^2 . \tag{3.5.3}$$

From the previous equations we will have

$$\left\langle \left(\triangle \hat{X}_1\right)^2 \right\rangle_\alpha = \frac{1}{4} = \left\langle \left(\triangle \hat{X}_2\right)^2 \right\rangle_\alpha , \tag{3.5.4}$$

which is exactly the same fluctuations for the quadrature operators for the vacuum.

3.6 Problem 3.6

In order to calculate the factorial moments,

$$\langle \hat{n}(\hat{n}-1)(\hat{n}-2)...(\hat{n}-r+1) \rangle , \tag{3.6.1}$$

for a coherent state $|\alpha\rangle$, one needs to write the operator $\hat{n}(\hat{n}-1)(\hat{n}-2)...(\hat{n}-r+1)$ in the normal order (all $\hat{a}^\dagger$'s on the left). The claim is that

$$\hat{n}(\hat{n}-1)(\hat{n}-2)...(\hat{n}-r+1) = \hat{a}^{\dagger r}\hat{a}^r, \tag{3.6.2}$$

which can be proved using the boson commutation rule, $[\hat{a}, \hat{a}^\dagger] = 1$, and mathematical induction. Now it is easy to the calculate the factorial moments for a coherent state.

$$\langle \hat{n}(\hat{n}-1)(\hat{n}-2)...(\hat{n}-r+1) \rangle = |\alpha|^{2r} \tag{3.6.3}$$

3.7 Problem 3.7

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{3.7.1}$$

$$\alpha = |\alpha|e^{i\theta} \tag{3.7.2}$$

$$\hat{C} = \frac{1}{2}\left(\hat{E} + \hat{E}^\dagger\right),\ \text{and}\ \hat{S} = \frac{1}{2i}\left(\hat{E} - \hat{E}^\dagger\right)$$

$$\begin{aligned}
\langle\alpha|\,\hat{E}\,|\alpha\rangle &= e^{-|\alpha|^2}\sum_{n,n'}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{n'}}{\sqrt{n'!}}\langle n|\,\hat{E}\,|n'\rangle \\
&= e^{-|\alpha|^2}\sum_{n,n'}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{n'}}{\sqrt{n'!}}\langle n|m\rangle\langle m+1|n'\rangle \\
&= \alpha e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}}
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\,\hat{C}\,|\alpha\rangle &= \frac{1}{2}\langle\alpha|\left(\hat{E} + \hat{E}^\dagger\right)|\alpha\rangle \\
&= \frac{1}{2}\left(\langle\alpha|\,\hat{E}\,|\alpha\rangle + \langle\alpha|\,\hat{E}^\dagger\,|\alpha\rangle\right) \\
&= \frac{1}{2}\left(\langle\alpha|\,\hat{E}\,|\alpha\rangle + \langle\alpha|\,\hat{E}\,|\alpha\rangle^*\right) \\
&= \frac{1}{2}\left(\alpha + \alpha^*\right)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \\
&= \Re(\alpha)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \\
&= \cos(\theta)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+1}}{n!\sqrt{n+1}}
\end{aligned}$$

$$
\begin{aligned}
\langle\alpha|\,\hat{S}\,|\alpha\rangle &= \frac{1}{2i}\,\langle\alpha|\left(\hat{E}-\hat{E}^\dagger\right)|\alpha\rangle \\
&= \frac{1}{2i}\left(\langle\alpha|\,\hat{E}\,|\alpha\rangle - \langle\alpha|\,\hat{E}^\dagger\,|\alpha\rangle\right) \\
&= \frac{1}{2i}\left(\langle\alpha|\,\hat{E}\,|\alpha\rangle - \langle\alpha|\,\hat{E}\,|\alpha\rangle^*\right) \\
&= \frac{1}{2i}\left(\alpha-\alpha^*\right)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \\
&= \Im(\alpha)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}} \\
&= \sin(\theta)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+1}}{n!\sqrt{n+1}}
\end{aligned}
$$

$$
\begin{aligned}
\hat{C}^2 &= \frac{1}{4}\left(\hat{E}+\hat{E}^\dagger\right)\left(\hat{E}+\hat{E}^\dagger\right) \\
&= \frac{1}{4}\left(\hat{E}^2+\hat{E}^{\dagger 2}+\hat{E}\hat{E}^\dagger+\hat{E}^\dagger\hat{E}\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{S}^2 &= \frac{-1}{4}\left(\hat{E}-\hat{E}^\dagger\right)\left(\hat{E}-\hat{E}^\dagger\right) \\
&= \frac{-1}{4}\left(\hat{E}^2+\hat{E}^{\dagger 2}-\hat{E}\hat{E}^\dagger-\hat{E}^\dagger\hat{E}\right)
\end{aligned}
$$

$$
\begin{aligned}
\hat{E}^2 &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|n\rangle\langle n+1|m\rangle\langle m+1| \\
&= \sum_{n=0}^{\infty}|n\rangle\langle n+2|, \\
\hat{E}^{\dagger 2} &= \sum_{n=0}^{\infty}|n+2\rangle\langle n| \\
\hat{E}\hat{E}^\dagger &= 1, \\
\hat{E}^\dagger\hat{E} &= 1-|0\rangle\langle 0| \\
\hat{E}\hat{E}^\dagger+\hat{E}^\dagger\hat{E} &= 2-|0\rangle\langle 0|
\end{aligned}
$$

$$\begin{aligned}
\langle\alpha|\,\hat{E}^2\,|\alpha\rangle &= e^{-|\alpha|^2}\sum_{n,n'}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{n'}}{\sqrt{n'!}}\,\langle n|\,\hat{E}^2\,|n'\rangle \\
&= e^{-|\alpha|^2}\sum_{n,n'}^{\infty}\sum_{m=0}^{\infty}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\alpha^{n'}}{\sqrt{n'!}}\langle n|m\rangle\langle m+2|n'\rangle \\
&= \alpha^2 e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}}
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\,\hat{C}^2\,|\alpha\rangle &= \frac{1}{4}\,\langle\alpha|\left(\hat{E}^2+\hat{E}^{\dagger 2}+\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\right)|\alpha\rangle \\
&= \frac{1}{4}\left(\langle\alpha|\,\hat{E}^2\,|\alpha\rangle+\langle\alpha|\,\hat{E}^{\dagger 2}\,|\alpha\rangle+\langle\alpha|\,\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\,|\alpha\rangle\right) \\
&= \frac{1}{4}\left(\langle\alpha|\,\hat{E}^2\,|\alpha\rangle+\langle\alpha|\,\hat{E}^2\,|\alpha\rangle^*+\langle\alpha|\,\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\,|\alpha\rangle\right) \\
&= \frac{1}{4}\left(2\Re(\alpha^2)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}}+2+e^{-|\alpha|^2}\right) \\
&= \frac{1}{4}\left(2\cos(2\theta)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+2}}{n!\sqrt{(n+1)(n+2)}}+2+e^{-|\alpha|^2}\right)
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\,\hat{S}^2\,|\alpha\rangle &= \frac{-1}{4}\,\langle\alpha|\left(\hat{E}^2+\hat{E}^{\dagger 2}-\hat{E}\hat{E}^{\dagger}-\hat{E}^{\dagger}\hat{E}\right)|\alpha\rangle \\
&= \frac{-1}{4}\left(\langle\alpha|\,\hat{E}^2\,|\alpha\rangle+\langle\alpha|\,\hat{E}^{\dagger 2}\,|\alpha\rangle-\langle\alpha|\,\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\,|\alpha\rangle\right) \\
&= \frac{-1}{4}\left(\langle\alpha|\,\hat{E}^2\,|\alpha\rangle+\langle\alpha|\,\hat{E}^2\,|\alpha\rangle^*-\langle\alpha|\,\hat{E}\hat{E}^{\dagger}+\hat{E}^{\dagger}\hat{E}\,|\alpha\rangle\right) \\
&= \frac{-1}{4}\left(2\Re(\alpha^2)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{(n+1)(n+2)}}-2-e^{-|\alpha|^2}\right) \\
&= \frac{-1}{4}\left(2\cos(2\theta)e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n+2}}{n!\sqrt{(n+1)(n+2)}}-2-e^{-|\alpha|^2}\right)
\end{aligned}$$

As $|\alpha| \to \infty$

$$
\begin{aligned}
\lim_{|\alpha|\to\infty} e^{-|\alpha|^2} &= 0 \\
\lim_{|\alpha|\to\infty} e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n+1}}{n!\sqrt{n+1}} &= 1 \\
\lim_{|\alpha|\to\infty} e^{-|\alpha|^2} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n+2}}{n!\sqrt{(n+1)(n+2)}} &= 1 \\
\langle\alpha|\,\hat{C}\,|\alpha\rangle &= \cos\theta \\
\langle\alpha|\,\hat{S}\,|\alpha\rangle &= \sin\theta
\end{aligned}
$$

and

$$
\begin{aligned}
\langle\alpha|\,\hat{C}^2\,|\alpha\rangle &= \frac{1}{2}\left(\cos(2\theta)+1\right) = \cos^2(\theta) \\
\langle\alpha|\,\hat{S}^2\,|\alpha\rangle &= \frac{1}{2}\left(\cos(2\theta)-1\right) = \sin^2(\theta)
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \alpha \middle| \left(\Delta\hat{C}\right)^2 \middle| \alpha \right\rangle &= \left\langle \alpha \middle| \hat{C}^2 \middle| \alpha \right\rangle - \left\langle \alpha \middle| \hat{C} \middle| \alpha \right\rangle^2 = 0 \\
\left\langle \alpha \middle| \left(\Delta\hat{S}\right)^2 \middle| \alpha \right\rangle &= \left\langle \alpha \middle| \hat{S}^2 \middle| \alpha \right\rangle - \left\langle \alpha \middle| \hat{S} \middle| \alpha \right\rangle^2 = 0
\end{aligned}
$$

The uncertainty products of Eqs. (2.215) and (2.216) equalize as $|\alpha| \to \infty$.

3.8 Problem 3.8

a. Let define $|z\rangle$ as

$$
|z\rangle = \sum_{n=0}^{\infty} c_n |n\rangle. \tag{3.8.1}
$$

The eigenvalue equation

$$\begin{aligned}
\hat{E}|z\rangle = z|z\rangle &= \sum_{n=0}^{\infty} z c_n |n\rangle \\
\frac{1}{\sqrt{\hat{n}+1}}\hat{a}|z\rangle &= \sum_{n=0}^{\infty} c_n \frac{1}{\sqrt{\hat{n}+1}}\sqrt{n}|n-1\rangle \\
&= \sum_{n=0}^{\infty} c_n \frac{1}{\sqrt{n}}\sqrt{n}|n-1\rangle \\
&= \sum_{n=0}^{\infty} c_n |n-1\rangle \\
&= \sum_{n=0}^{\infty} c_{n+1} |n\rangle
\end{aligned}$$

leads to

$$c_n = c_{n-1} z = \dots = c_0 z^n. \tag{3.8.2}$$

Thus the eigenstate has the the expansion

$$|z\rangle = \sum_{n=0}^{\infty} c_0 z^n |n\rangle. \tag{3.8.3}$$

The state of Eq. 3.8.3 is normalized for any z, such that $|z| < 1$. For such a case, c_0 can be determined as

$$1 = |c_0|^2 \sum_{n=0}^{\infty} |z|^{2n} = |c_0|^2 \frac{1}{1-|z|^2}, \tag{3.8.4}$$

where we have used the properties of the geometric series. Finally, c_0 and $|z\rangle$ can be defined respectively as

$$\begin{aligned}
c_0 &= \sqrt{1-|z|^2} \\
|z\rangle &= \sqrt{1-|z|^2} \sum_{n=0}^{\infty} z^n |n\rangle.
\end{aligned}$$

Notice that $|z| < 1$, otherwise the state will not be normalized.

b.

$$\begin{aligned}
\int d^2z|z\rangle\langle z| &= \int d^2z\left(1-|z|^2\right)\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} z^n z^{n'}|n\rangle\langle n'| \\
&= \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\int_0^1 d|z|^2\int_0^{2\pi} d\phi\left(1-|z|^2\right)|z|^{2(n+n')}e^{i\phi(n-n')}|n\rangle\langle n'| \\
&= \sum_{n=0}^{\infty}\sum_{n'=0}^{\infty}\int_0^1 dr\,(1-r)\,r^{(n+n')/2}2\pi\delta_{n,n'}|n\rangle\langle n'| \\
&= 2\pi\sum_{n=0}^{\infty}\int_0^1 dr\left(r^n-r^{n+1}\right)|n\rangle\langle n| \\
&= 2\pi\sum_{n=0}^{\infty}\frac{1}{(n+1)(n+2)}|n\rangle\langle n|,
\end{aligned}$$

It does not resolve unity.

c. We have proved that the state is not normalized for $|z|<1$. Thus we drop the normalization constant and we write $z=e^{i\phi}$ and we obtain the phase states $|\phi\rangle$ of Eq. (2.221). Obviously the the last states resolve unity as in Eq. (2.223).

d. The average photon number

$$\begin{aligned}
\bar{n} &= \langle z|\hat{n}|z\rangle \\
&= \left(1-|z|^2\right)\sum_{n=0}^{\infty} n|z|^{2n} \\
&= \left(1-|z|^2\right)\frac{\partial}{\partial|z|^2}\sum_{n=0}^{\infty}|z|^{2n} \\
&= \left(1-|z|^2\right)\frac{\partial}{\partial|z|^2}\left(\frac{1}{1-|z|^2}\right) \\
&= \frac{1}{1-|z|^2}
\end{aligned}$$

The photon number distribution for $|z\rangle$ is

$$\begin{aligned}
P_n = |\langle n|z\rangle|^2 &= \left(1-|z|^2\right)|z|^{2n} \\
&= \frac{1}{\bar{n}}\left(\frac{\bar{n}-1}{\bar{n}}\right)^n .
\end{aligned}$$

This distribution resembles the thermal light distribution.

e.

$$\begin{aligned}
\mathcal{P}(\phi) &= |\langle\phi|z\rangle|^2 \\
&= \left(1-|z|^2\right)\left|\sum_{n=0}^{\infty} e^{in\phi} z^n\right|^2 \\
&= \left(1-|z|^2\right)\sum_{n=0}^{\infty}\sum_{n'=0}^{\infty} e^{i(n-n')\phi}|z|^{n+n'}
\end{aligned}$$

3.9 Problem 3.9

$$\begin{aligned}
\left\langle : (\Delta\hat{n})^2 : \right\rangle &= \left\langle : \hat{n}^2 : \right\rangle - \langle : \hat{n} : \rangle^2 \\
&= \mathbf{Tr}\left(: \hat{n}^2 : \hat{\rho}\right) - \left[\mathbf{Tr}\left(: \hat{n} : \hat{\rho}\right)\right]^2 \\
&= \mathbf{Tr}\int \left(: \hat{n}^2 : \right) P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha - \left[\mathbf{Tr}\int \left(: \hat{n} : \right) P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha\right]^2 \\
&= \int P(\alpha)\langle\alpha| : \hat{n}^2 : |\alpha\rangle d^2\alpha - \left[\int P(\alpha)\langle\alpha| : \hat{n} : |\alpha\rangle d^2\alpha\right]^2 \\
&= \int P(\alpha)\,|\alpha|^4\, d^2\alpha - \left[\int P(\alpha)\,|\alpha|^2\, d^2\alpha\right]^2
\end{aligned}$$

For a coherent state $|\beta\rangle$ we have $P(\alpha) = \delta^2(\alpha-\beta)$, so obviously

$$\begin{aligned}
\left\langle : (\Delta\hat{n})^2 : \right\rangle &= \int \delta^2(\alpha-\beta)\,|\alpha|^4\, d^2\alpha - \left[\int \delta^2(\alpha-\beta)\,|\alpha|^2\, d^2\alpha\right]^2 \\
&= |\beta|^4 - |\beta|^4 = 0
\end{aligned}$$

3.10 Problem 3.10

$$
\begin{aligned}
\left\langle : \left(\Delta \hat{X}\right)_i^2 : \right\rangle &= \mathbf{Tr}\left(: \hat{X}_i^2 : \hat{\rho}\right) - \left[\mathbf{Tr} : \hat{X}_i : \hat{\rho}\right]^2 \\
&= \mathbf{Tr} \int : \hat{X}_i : P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha - \left[\mathbf{Tr} \int : \hat{X}_i : P(\alpha)|\alpha\rangle\langle\alpha| d^2\alpha\right]^2 \\
&= \int \langle\alpha| : \hat{X}_i : |\alpha\rangle P(\alpha) d^2\alpha - \left[\int \langle\alpha| : \hat{X}_i : |\alpha\rangle P(\alpha) d^2\alpha\right]^2 \\
&= \frac{1}{4}\int \langle\alpha| : (\hat{a} \pm \hat{a}^\dagger)^2 : |\alpha\rangle P(\alpha) d^2\alpha - \frac{1}{4}\left[\int \langle\alpha| : (\hat{a} \pm \hat{a}^\dagger) : |\alpha\rangle P(\alpha) d^2\alpha\right]^2 \\
&= \frac{1}{4}\int \langle\alpha|(\hat{a}^2 + \hat{a}^{\dagger 2} \pm 2\hat{a}^\dagger\hat{a})|\alpha\rangle P(\alpha) d^2\alpha - \frac{1}{4}\left[\int \langle\alpha|(\hat{a}^\dagger \pm \hat{a})|\alpha\rangle P(\alpha) d^2\alpha\right]^2 \\
&= \frac{1}{4}\int (\alpha^{*2} + \alpha^2 \pm 2|\alpha|^2) P(\alpha) d^2\alpha - \frac{1}{4}\left[\int (\alpha \pm \alpha^*) P(\alpha) d^2\alpha\right]^2
\end{aligned}
$$

Where it is clear that $+(-)$ stands for $i = 1(2)$. Again for a coherent state $|\beta\rangle$

$$
\left\langle : \left(\Delta \hat{X}\right)_i^2 : \right\rangle = 0
$$

3.11 Problem 3.11

$$
\begin{aligned}
W(q,p) &= \frac{1}{\pi^2}\int d^2\lambda e^{\lambda^*\alpha - \lambda\alpha^*} C_W(\lambda) \\
&= \frac{1}{\pi^2}\int d^2\lambda e^{\lambda^*\alpha - \lambda\alpha^*} \mathrm{Tr}\left(e^{\lambda\hat{a}^\dagger - \lambda^*\hat{a}}\hat{\rho}\right)
\end{aligned}
$$

Let define the following

$$
\begin{aligned}
\alpha &= \frac{1}{\sqrt{2}}(q + ip) \qquad\qquad \lambda = \frac{1}{\sqrt{2}}(x + iy) \\
\hat{a} &= \frac{1}{\sqrt{2}}(\hat{q} + i\hat{p}).
\end{aligned}
$$

$$
W(q,p) = \frac{1}{4\pi^2}\int dxdy e^{-i(xp+yq)} \mathrm{Tr}\left(e^{-i(x\hat{p} - y\hat{q})}\hat{\rho}\right),
$$

where we have used $\lambda^*\alpha - \lambda\alpha^* = -i(x\hat{p} - y\hat{q})$ and $\lambda^*\alpha - \lambda\alpha^* = -i(x\hat{p} - y\hat{q})$. Using the identity in Eq.2.4.7 we can rewrite the Wigner function as

$$\begin{aligned}
W(q,p) &= \frac{1}{4\pi^2}\int dxdye^{-i(xp+yq)}\mathrm{Tr}\left(e^{-ix\hat{p}}e^{iy\hat{q}}e^{\frac{ixy}{2}}\hat{\rho}\right) \\
&= \frac{1}{4\pi^2}\int dxdye^{-i(xp+yq)}e^{-\frac{ixy}{2}}\mathrm{Tr}\left(e^{-i\frac{x}{2}\hat{p}}e^{iy\hat{q}}\hat{\rho}e^{-i\frac{x}{2}\hat{p}}\right) \\
&= \frac{1}{2\pi^2}\int dxdydq'e^{-i(xp+yq)}e^{-\frac{ixy}{2}}\left\langle q'\right| e^{-i\frac{x}{2}\hat{p}}e^{iy\hat{q}}\hat{\rho}e^{-i\frac{x}{2}\hat{p}}\left|q'\right\rangle \\
&= \frac{1}{2\pi^2}\int dxdydq'e^{-i(xp+yq)}e^{-\frac{ixy}{2}}\left\langle q'+\frac{x}{2}\right| e^{iy\hat{q}}\hat{\rho}\left|q'-\frac{x}{2}\right\rangle \\
&= \frac{1}{2\pi^2}\int dxdydq'e^{-i(xp+yq)}e^{-\frac{ixy}{2}}\left\langle q'+\frac{x}{2}\right| e^{iy\left(q'+\frac{x}{2}\right)}\hat{\rho}\left|q'-\frac{x}{2}\right\rangle \\
&= \frac{1}{2\pi^2}\int dxdydq'e^{-ixp}e^{iy(q'-q)}\left\langle q'+\frac{x}{2}\right| \hat{\rho}\left|q'-\frac{x}{2}\right\rangle \\
&= \frac{1}{\pi}\int dxdq'e^{-ixp}\delta(q'-q)\left\langle q'+\frac{x}{2}\right| \hat{\rho}\left|q'-\frac{x}{2}\right\rangle \\
&= \frac{1}{\pi}\int dxe^{-ixp}\left\langle q+\frac{x}{2}\right| \hat{\rho}\left|q-\frac{x}{2}\right\rangle,
\end{aligned}$$

where we have used the following

$$\begin{aligned}
\delta(q'-q) &= \frac{1}{2\pi^2}\int dye^{iy(q'-q)}, \\
e^{-i\frac{x}{2}\hat{p}}\left|q'\right\rangle &= \left|q'-\frac{x}{2}\right\rangle.
\end{aligned}$$

3.12 Problem 3.12

In general

$$\begin{aligned}
W(\alpha) &= \frac{1}{\pi^2}\int \exp(\lambda^*\alpha - \lambda\alpha^*)C_W(\lambda)d^2\lambda \\
&= \frac{1}{\pi^2}\int \exp(\lambda^*\alpha - \lambda\alpha^*)C_N(\lambda)e^{-|\lambda|^2}d^2\lambda
\end{aligned}$$

For $|\Psi\rangle = |\beta\rangle$

$$\begin{aligned}
C_N(\lambda) &= \langle\beta|e^{\lambda\hat{a}^\dagger}e^{-\lambda^*\hat{a}}|\beta\rangle \\
&= e^{\lambda\beta^* - \lambda^*\beta}
\end{aligned}$$

$$\begin{aligned}
W(\alpha) &= \frac{1}{2\pi^2}\int \exp(\lambda^*\alpha - \lambda\alpha^*)C_N(\lambda)e^{-|\lambda|^2}d^2\lambda \\
&= \frac{1}{\pi^2}\int \exp(\lambda^*\alpha - \lambda\alpha^*)e^{\lambda\beta^*-\lambda^*\beta}e^{-|\lambda|^2/2}d^2\lambda \\
&= \frac{1}{\pi^2}\int \exp\left[\lambda^*(\alpha-\beta) - \lambda(\alpha^*-\beta^*) - |\lambda|^2/2\right]d^2\lambda
\end{aligned}$$

Using the following identity we can compute the last integral

$$\int \exp(\lambda x + \lambda^* y - z|\lambda|^2)d^2\lambda = \pi z^{-1}\exp(z^{-1}xy), \tag{3.12.1}$$

by identifying

$$x = \alpha-\beta,\ y = -(\alpha^*-\beta^*),\ \text{and}\ z = \frac{1}{2}.$$

$$W(\alpha) = \frac{2}{\pi}e^{-2|\alpha-\beta|^2} \tag{3.12.2}$$

For $|\Psi\rangle = |N\rangle$
Using Eq. (3.128a) we have

$$\begin{aligned}
C_W(\lambda) &= \langle N|\hat{D}(\lambda)|N\rangle \\
&= e^{-|\lambda|^2/2}\langle N|e^{\lambda\hat{a}^\dagger}e^{-\lambda\hat{a}}|N\rangle \\
&= e^{-|\lambda|^2/2}\sum_{n'=0}^{N}\sum_{n=0}^{N}\langle N|\frac{\lambda^{n'}\hat{a}^{\dagger n'}}{n'!}\frac{(-1)^n\lambda^n\hat{a}^n}{n!}|N\rangle \\
&= e^{-|\lambda|^2/2}\sum_{n=0}^{N}\frac{(-1)^n|\lambda|^{2n}}{n!n!}\langle N|\hat{a}^{\dagger n}\hat{a}^n|N\rangle \\
&= e^{-|\lambda|^2/2}\sum_{n=0}^{N}\frac{(-1)^n|\lambda|^{2n}}{n!n!}\frac{N!}{(N-n)!} \\
&= (-1)^N e^{-|\lambda|^2/2}L_N(|\lambda|^2),
\end{aligned} \tag{3.12.3}$$

where we have used the Laguerre polynomials expansion. The Wigner function is given by

$$W(\alpha) = (-1)^N\frac{1}{\pi^2}\int e^{\lambda^*\alpha-\lambda\alpha^*}e^{\frac{-|\lambda|^2}{2}}L_N(|\lambda|^2)d^2\lambda \tag{3.12.4}$$

Using the following identity

$$\int f(\alpha)e^{\alpha^* y - z|\alpha|^2}\pi^{-1}d^2\alpha = z^{-1}f(z^{-1}y), \tag{3.12.5}$$

we compute the integral in Eq. 3.12.4

$$W(\alpha) = (-1)^N \frac{2}{\pi} e^{-2|\alpha|^2} L_N(4|\alpha|^2).$$

3.13 Problem 3.13

a. For the state

$$|\psi\rangle = \mathcal{N}(|\beta\rangle + |-\beta\rangle)$$

$\langle\psi|\psi\rangle = 1 = |\mathcal{N}|^2[\langle\beta|\beta\rangle + \langle-\beta|-\beta\rangle + \langle-\beta|\beta\rangle + \langle\beta|-\beta\rangle] = |\mathcal{N}|^2[2 + 2e^{-2|\beta|^2}]$
For large β, $e^{-2|\beta|^2} \approx 0$ so this state is normalized for:

$$\mathcal{N} = \frac{1}{\sqrt{2}}.$$

b.

$$\langle n|\psi\rangle = \frac{1}{\sqrt{2}}\frac{\beta^n}{\sqrt{n!}}[1 + (-1)^n], \tag{3.13.1}$$

thus

$$\begin{cases} P_n = e^{-|\beta|^2}\frac{|\beta|^{2n}}{n!} & \text{n is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{3.13.2}$$

c.

$$\langle\phi|\psi\rangle = \frac{1}{\sqrt{2}}\sum_{n=0}^{\infty} e^{-|\beta|^2/2}e^{i\phi n}\frac{\beta^n}{\sqrt{n!}}[1 + (-1)^n] \tag{3.13.3}$$

$$P(\phi) = \frac{e^{-|\beta|^2}}{2}\sum_{n,n'}^{\infty}\frac{\beta^n}{\sqrt{n!}}\frac{\beta^{*n'}}{\sqrt{n'!}}e^{i\phi(n-n')}[1 + (-1)^n][1 + (-1)^{n'}] \tag{3.13.4}$$

d. The Q function is given by

$$Q(\alpha) = \frac{1}{2\pi}\langle\alpha|\rho|\alpha\rangle$$
$$Q(\alpha) = \frac{1}{4\pi}\left|\langle\alpha|\beta\rangle + \langle\alpha|-\beta\rangle\right|^2$$
$$Q(\alpha) = \frac{1}{4\pi}e^{-|\alpha|^2-|\beta|^2}\left|e^{\alpha^*\beta} + e^{-\alpha^*\beta}\right|^2.$$

The Wigner function is given by

$$W(\alpha) = \frac{1}{2\pi^2} \int d^2\lambda \exp\left(\lambda^* \alpha - \lambda \alpha^*\right) C_W(\lambda). \tag{3.13.5}$$

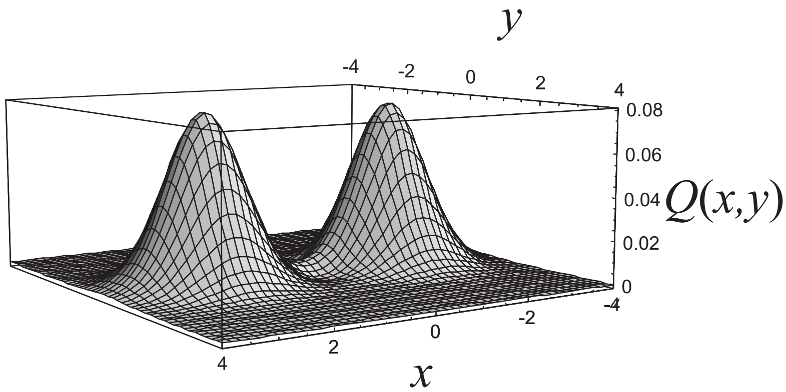
First we calculate

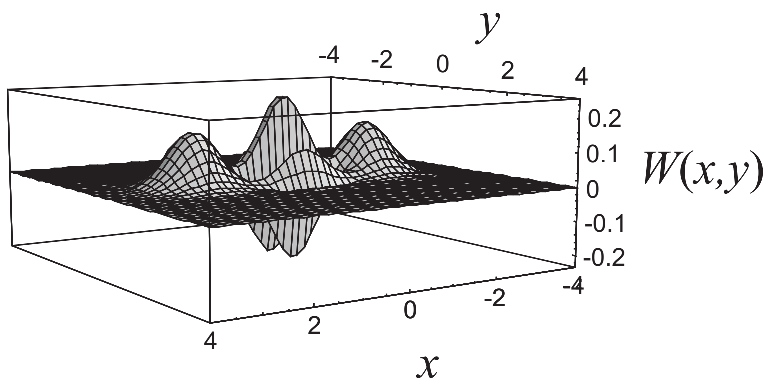
$$\begin{aligned}
C_W(\lambda) &= \mathrm{Tr}\left[\hat{\rho}\hat{D}(\lambda)\right] \\
&= \frac{1}{2}(\langle\beta| + \langle-\beta|)\hat{D}(\lambda)(|\beta\rangle + |-\beta\rangle) \\
&= \frac{1}{2}(\langle\beta| + \langle-\beta|)\left(e^{i\Im(\lambda\beta^*)}|\lambda+\beta\rangle + e^{-i\Im(\lambda\beta^*)}|\lambda-\beta\rangle\right) \\
&= \frac{1}{2}e^{-|\beta|^2}e^{-|\lambda|^2/2}\left[e^{-\lambda^*\beta}\left(e^{-|\beta|^2-\beta^*\lambda} + e^{|\beta|^2+\beta^*\lambda}\right) + e^{\lambda^*\beta}\left(e^{|\beta|^2-\beta^*\lambda} + e^{-|\beta|^2+\beta^*\lambda}\right)\right].
\end{aligned}$$

Back into Eq. 3.13.5

$$\begin{aligned}
W(\alpha) = \frac{1}{4\pi^2}e^{-|\beta|^2}\Bigg\{ & e^{-|\beta|^2}\int d^2\lambda e^{-|\lambda|^2/2}e^{\lambda^*(\alpha-\beta)}e^{-\lambda(\alpha^*+\beta^*)} \\
& + e^{|\beta|^2}\int d^2\lambda e^{-|\lambda|^2/2}e^{\lambda^*(\alpha-\beta)}e^{-\lambda(\alpha^*-\beta^*)} \\
& + e^{|\beta|^2}\int d^2\lambda e^{-|\lambda|^2/2}e^{\lambda^*(\alpha+\beta)}e^{-\lambda(\alpha^*+\beta^*)} \\
& + e^{-|\beta|^2}\int d^2\lambda e^{-|\lambda|^2/2}e^{\lambda^*(\alpha+\beta)}e^{-\lambda(\alpha^*-\beta^*)}\Bigg\} \\
= \frac{1}{2\pi}\Big[& e^{-2|\alpha-\beta|^2} + e^{-2|\alpha+\beta|^2}e^{-2|\alpha|^2}\left(e^{-2(\beta\alpha^*-\alpha\beta^*)} + e^{-2(-\beta\alpha^*+\alpha\beta^*)}\right)\Big],
\end{aligned}$$

where we have used the identity in Eq. 3.12 to carry out the integrals. The Q and Wigner functions are displayed in Graphs below. Obviously the state $|\Psi\rangle$ is not a classical state as the Wigner function is negative.





3.14 Problem 3.14

First of all we have to prove the following identity

$$\int_0^\infty dr|-r\rangle\langle r| = \sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n|$$

$$\begin{aligned}\int dr|-r\rangle\langle r| &= \sum_{n=0}^{\infty}|n\rangle\langle n|\int dr|-r\rangle\langle r|\sum_{m=0}^{\infty}|m\rangle\langle m| \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int dr|n\rangle\langle n|-r\rangle\langle r|m\rangle\langle m| \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\int dr(-1)^n|n\rangle\langle n|r\rangle\langle r|m\rangle\langle m| \\ &= \sum_{n=0}^{\infty}(-1)^n|n\rangle\langle n| \qquad (3.14.1)\end{aligned}$$

Also we have for

$$\begin{aligned}\hat{D}(\alpha)|-r\rangle &= \exp(ip\hat{q}-iq\hat{p})|-r\rangle \\ &= e^{-pq\frac{[\hat{q},\hat{p}]}{2}}e^{ip\hat{q}}e^{-iq\hat{p}}|-r\rangle \\ &= e^{-i\frac{pq}{2}}e^{ip\hat{q}}|q-r\rangle \\ &= e^{-i\frac{pq}{2}}e^{ip(q-r)}|q-r\rangle, \qquad (3.14.2)\end{aligned}$$

where we assume that $\hbar = 1$, also

$$\langle r|\hat{D}^{\dagger}(\alpha) = e^{i\frac{pq}{2}}e^{-ip(q+r)}\langle q+r|. \qquad (3.14.3)$$

Now we use the Wigner function definition as in Eq.3.116

$$W(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\langle q+\frac{x}{2}\right|\hat{\rho}\left|q-\frac{x}{2}\right\rangle e^{ipx}dx \qquad (3.14.4)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty}\langle q+r|\hat{\rho}|q-r\rangle e^{i2pr}dr \qquad (3.14.5)$$

Using Eqs. 3.14.2 and 3.14.3 we can rewrite the Wigner function as

$$W(\alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty}dr\langle r|\hat{D}^{\dagger}(\alpha)\hat{\rho}\hat{D}(\alpha)|-r\rangle$$

For $\hat{\rho} = |\Psi\rangle\langle\Psi|$ we have

$$\begin{aligned}W(\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\infty}dr\langle r|\hat{D}^{\dagger}(\alpha)|\Psi\rangle\langle\Psi|\hat{D}(\alpha)|-r\rangle \\ &= \frac{2}{\pi}\int_{0}^{\infty}dr\langle\Psi|\hat{D}(\alpha)|-r\rangle\langle r|\hat{D}^{\dagger}(\alpha)|\Psi\rangle \\ &= \frac{2}{\pi}\sum_{n=0}^{\infty}(-1)^n\langle\Psi|\hat{D}(\alpha)|n\rangle\langle n|\hat{D}^{\dagger}(\alpha)|\Psi\rangle\end{aligned}$$

Chapter 4

Emission and Absorption of Radiation by Atoms

4.1 Problem 4.1

We still can use equation (4.78)

$$C_e(t) = A_+ e^{i\lambda_+ t} + A_- e^{i\lambda_- t} \tag{4.1.1}$$

where

$$\lambda_\pm = \frac{1}{2}\left\{\Delta \pm \left[\Delta^2 + \frac{\mathcal{V}^2}{\hbar^2}\right]^{1/2}\right\}. \tag{4.1.2}$$

From the initial conditions

$$C_e(0) = 1 \tag{4.1.3}$$

$$C_g(0) = 0, \tag{4.1.4}$$

we can determine $A_\pm$, explicitly

$$C_e(0) = 1 = A_+ + A_-, \tag{4.1.5}$$

so

$$A_- = 1 - A_+. \tag{4.1.6}$$

Equation 4.1.1 becomes

$$C_e(t) = A_+ e^{i\lambda_+ t} + (1 - A_+)e^{i\lambda_- t}. \tag{4.1.7}$$

Equation (4.71) can be used to find the following

$$C_g(t) = \frac{i2\hbar}{\mathcal{V}} \exp[i(\omega-\omega_0)t] \left[i\lambda_+ A_+ e^{i\lambda_+ t} + i\lambda_-(1-A_+)e^{i\lambda_- t} \right] \tag{4.1.8}$$

for $t=0$, we can solve for A_+ in the last equation

$$A_+ = \frac{\lambda_-}{\lambda_- - \lambda_+} = \frac{1}{2}\left(1 - \frac{\Delta}{\Omega_R}\right), \tag{4.1.9}$$

which leads to

$$\begin{aligned} C_e(t) &= \frac{1}{2}\left\{ \left[1 - \frac{\Delta}{\Omega_R}\right] e^{i\lambda_+ t} + \left[1 + \frac{\Delta}{\Omega_R}\right] e^{i\lambda_- t} \right\} \\ C_g(t) &= \frac{-\hbar}{\mathcal{V}} \exp\left[i(\omega-\omega_0)t\right] \left(1 - \frac{\Delta}{\Omega_R}\right)(\Delta + \Omega_R) e^{i\frac{1}{2}\Delta t} \sin(\Omega_R t/2). \end{aligned}$$

Finally, we have

$$\begin{aligned} C_e(t) &= e^{\frac{i\Delta t}{2}} \left[\cos(\Omega_R t/2) - i\frac{\Delta}{\Omega_R} \sin\left(\Omega_R t/2\right) \right] \\ C_g(t) &= \frac{-\hbar\Omega_R}{\mathcal{V}} e^{i(\omega-\omega_0)t} \left[1 - \left(\frac{\Delta}{\Omega_R}\right)^2 \right] e^{i\Delta t/2} \sin(\Omega_R t/2). \end{aligned}$$

$$\begin{aligned} W(t) &= |C_e(t)|^2 - |c_g(t)|^2 \\ &= \cos^2\left(\Omega_R t/2\right) + \left\{ \frac{\Delta^2}{\Omega_R^2} - \frac{\hbar^2\Omega_R^2}{\mathcal{V}^2} \left[1 - \left(\frac{\Delta}{\Omega_R}\right)^2 \right]^2 \right\} \sin^2\left(\Omega_R t/2\right) \\ &= \cos^2\left(\Omega_R t/2\right) + \left[\frac{\Delta^2}{\Omega_R^2} - \frac{\mathcal{V}^2}{\hbar^2\Omega_R^2} \right] \sin^2\left(\Omega_R t/2\right) \\ &= \cos^2\left(\Omega_R t/2\right) + \left[\frac{\Delta^2 - \mathcal{V}^2/\hbar^2}{\Delta^2 + \mathcal{V}^2/\hbar^2} \right] \sin^2\left(\Omega_R t/2\right) \end{aligned}$$

4.2 Problem 4.2

Equation 4.67 gives the exact solution to the evolving state

$$|\Psi(t)\rangle = C_g(t)e^{-i\frac{E_g t}{\hbar}}|g\rangle + C_e(t)e^{-i\frac{E_e t}{\hbar}}|e\rangle. \tag{4.2.1}$$

Using Eq. (4.91) as definition of the dipole operator

$$\begin{aligned}
\hat{d} &= d(\hat{\sigma}_+ + \hat{\sigma}_-) \\
\hat{d}|\Psi(t)\rangle &= d\{C_g(t)e^{-i\frac{E_g t}{\hbar}}|e\rangle + C_e(t)e^{-i\frac{E_e t}{\hbar}}|g\rangle\}. \\
\langle\hat{d}\rangle &= \langle\Psi(t)|\hat{d}|\Psi(t)\rangle \\
&= d\left\{C_g C_e^* e^{-i\frac{E_g - E_e}{\hbar}t} + C_g^* C_e e^{i\frac{E_g - E_e}{\hbar}t}\right\}. \qquad (4.2.2)
\end{aligned}$$

Using results from the previous problem, we obtain

$$C_e C_g^* e^{i\frac{E_g - E_e}{\hbar}t} = \frac{\hbar\Omega_R e^{-i\omega t}}{\mathcal{V}}\left[\cos\left(\Omega_R t/2\right) - i\frac{\Delta}{\Omega_R}\sin\left(\Omega_R t/2\right)\right]\left[1 - \left(\frac{\Delta}{\Omega_R}\right)^2\right]\sin\left(\Omega_R t/2\right),$$

where we have used $E_g - E_e = -\omega_0$. After algebra we also can rewrite equation 4.2.2 as

$$\begin{aligned}
\left\langle\hat{d}\right\rangle =& -2d\frac{\hbar\Omega_R}{\mathcal{V}}\left[1 - \left(\frac{\Delta}{\Omega_R}\right)^2\right]\sin\left(\Omega_R t/2\right) \\
&\times\left[\cos\left(\Omega_R t/2\right)\cos\left(\omega t\right) - \frac{\Delta}{\Omega_R}\sin\left(\Omega_R t/2\right)\sin\left(\omega t\right)\right].
\end{aligned}$$

For the case of exact resonance, $\Delta = 0$, we have

$$\left\langle\hat{d}\right\rangle = -d\sin\left(\mathcal{V}t/\hbar\right)\cos\left(\omega t\right).$$

4.3 Problem 4.3

The state is already solved in equation (4.109)

$$|\Psi(t)\rangle = \cos(\lambda t\sqrt{n+1}|e\rangle|n\rangle - i\sin(\lambda t\sqrt{n+1})|g\rangle|n+1\rangle.$$

Now we can evaluate the following

$$\hat{d}|\Psi(t)\rangle = \cos(\lambda t\sqrt{n+1}|g\rangle|n\rangle - i\sin(\lambda t\sqrt{n+1})|e\rangle|n+1\rangle,$$

which simply means that

$$\langle\hat{d}\rangle = 0.$$

This is a consequence of the entanglement between the atom and field number state.

4.4 Problem 4.4

Let's the field initial state be

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}},$$

and the atom initial state

$$|\Psi_a\rangle = |e\rangle. \tag{4.4.1}$$

$$\begin{aligned} |\Psi_i\rangle &= |\alpha\rangle|e\rangle \\ &= e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle|e\rangle. \end{aligned}$$

For $t > 0$

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} \left(c_{e,n}(t)|n\rangle|e\rangle + c_{g,n}(t)|n+1\rangle|g\rangle\right)$$

$$i\hbar \frac{d\,|\Psi(t)\rangle}{dt} = \hat{H}_{II}\,|\Psi(t)\rangle\,.$$

$$i\hbar \frac{d\,|\Psi(t)\rangle}{dt} = i\hbar \sum \left(\dot{c}_{e,n}(t)|n\rangle|e\rangle + \dot{c}_{g,n}(t)|n+1\rangle|g\rangle\right)$$

$$\hat{H}_{II}\,|\Psi(t)\rangle = \hbar\lambda \sum \left(\sqrt{n+1}c_{e,n}(t)|n+1\rangle|g\rangle + \sqrt{n+1}c_{g,n}(t)|n\rangle|e\rangle\right)$$

$$\begin{aligned} \dot{c}_{e,n}(t) &= -i\lambda\sqrt{n+1}c_{g,n}(t) \\ \dot{c}_{g,n}(t) &= -i\lambda\sqrt{n+1}c_{e,n}(t). \end{aligned}$$

Similar coupled differential equations have lead to the equation of the form

$$\ddot{c}_{e,n}(t) + \lambda^2(n+1)c_{e,n}(t) = 0, \tag{4.4.2}$$

which has a solution of the form

$$c_{e,n}(t) = A_n \cos\left(\lambda\sqrt{n+1}t\right) + B_n \sin\left(\lambda\sqrt{n+1}t\right) \tag{4.4.3}$$

also

$$
\begin{aligned}
c_{g,n}(t) &= \frac{i}{\lambda\sqrt{n+1}}\dot{c}_{e,n}(t) \\
&= -A_n \sin\left(\lambda\sqrt{n+1}t\right) + B_n \cos\left(\lambda\sqrt{n+1}t\right).
\end{aligned}
$$

From initial conditions

$$
\begin{aligned}
A_n &= c_{e,n}(0) = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}, \\
B_n &= 0.
\end{aligned}
$$

Thus

$$
\begin{aligned}
c_{e,n}(t) &= e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\cos\left(\lambda\sqrt{n+1}t\right) \\
c_{g,n}(t) &= -ie^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\sin\left(\lambda\sqrt{n+1}t\right),
\end{aligned}
$$

$$
|\Psi(t)\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left[\cos\left(\lambda\sqrt{n+1}t\right)|n\rangle|e\rangle - i\sin\left(\lambda\sqrt{n+1}t\right)|n+1\rangle|g\rangle\right]
$$

$$
\hat{d}\,|\Psi(t)\rangle = de^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left[\cos\left(\lambda\sqrt{n+1}t\right)|n\rangle|g\rangle - i\sin\left(\lambda\sqrt{n+1}t\right)|n+1\rangle|e\rangle\right]
$$

$$
\langle\Psi(t)|\,\hat{d}\,|\Psi(t)\rangle = -2\Im(\alpha)de^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^{2n}}{n!\sqrt{n+1}}\cos\left(\lambda\sqrt{n+2}t\right)\sin\left(\lambda\sqrt{n+1}t\right),
$$

where $\Im(\alpha)$ represents the imaginary part of the complex number α.

4.5 Problem 4.5

Let

$$
|\Psi\rangle = c_i(t)|i\rangle + c_f(t)|f\rangle
$$

$$i\frac{d}{dt}|\Psi\rangle = \hat{H}_{II}|\Psi\rangle. \tag{4.5.1}$$

Given that

$$\begin{aligned}\hat{H}_{II}|i\rangle &= \left[-\Delta|g\rangle\langle g| + \lambda\left(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger\right)\right]|i\rangle \\ &= \lambda\sqrt{n+1}|f\rangle\end{aligned}$$

and

$$\begin{aligned}\hat{H}_{II}|f\rangle &= \left[-\Delta|g\rangle\langle g| + \lambda\left(\sigma_+\hat{a} + \sigma_-\hat{a}^\dagger\right)\right]|f\rangle \\ &= -\Delta|f\rangle + \lambda\sqrt{n+1}|i\rangle,\end{aligned}$$

we have

$$\begin{aligned}\hat{H}_{II}|\Psi\rangle &= \hat{H}_{II}(c_i(t)|i\rangle + c_f(t)|f\rangle) \\ &= \left(\lambda\sqrt{n+1}c_i(t) - \Delta c_f(t)\right)|f\rangle + \lambda\sqrt{n+1}c_f(t)|i\rangle.\end{aligned}$$

On the other hand, we have

$$\frac{d}{dt}|\Psi\rangle = \dot{c}_i(t)|i\rangle + \dot{c}_f(t)|f\rangle$$

into equation 4.5.1 we obtain the following coupled equations

$$\begin{aligned}i\dot{c}_i(t) &= \lambda\sqrt{n+1}c_f(t), \\ i\dot{c}_f(t) &= \left(\lambda\sqrt{n+1}c_i(t) - \Delta c_f(t)\right).\end{aligned}$$

Which we can rewrite as

$$\begin{aligned}\dot{c}_i(t) &= -i\lambda\sqrt{n+1}c_f(t), \\ \dot{c}_f(t) &= -i\left(\lambda\sqrt{n+1}c_i(t) - \Delta c_f(t)\right).\end{aligned} \tag{4.5.2}$$

Taking the time derivative of the last equation we will obtain

$$\ddot{c}_f(t) = -i\left(\lambda\sqrt{n+1}\dot{c}_i(t) - \Delta\dot{c}_f(t)\right).$$

Using equation 4.5.2 we end up by getting a second order differential equation

$$\ddot{c}_f(t) - i\Delta\dot{c}_f(t) + \lambda^2(n+1)c_f(t) = 0$$

Assume that $c_f(t) = e^{Xt}$, and plug it into the differential equation we obtain the following quadratic equation

$$X^2 - i\Delta(n+1) + \lambda^2(n+1) = 0,$$

whose solutions are

$$\begin{aligned} X &= \frac{1}{2}(i\Delta \pm \sqrt{-\Delta^2 - 4\lambda^2(n+1)}) \\ &= \frac{i}{2}(\Delta \pm \sqrt{\Delta^2 + 4\lambda^2(n+1)}). \end{aligned}$$

The general solution then is

$$c_f(t) = e^{\frac{i}{2}\Delta t}\left(Ae^{i\Omega_n t} + Be^{-i\Omega_n t}\right),$$

where $\Omega_n = \sqrt{\frac{\Delta^2}{4} + \lambda^2(n+1)}$.

From initial conditions, we have $B = -A$, so

$$\begin{aligned} c_f(t) &= Ae^{\frac{i}{2}\Delta t}\left(e^{i\Omega_n t} - e^{-i\Omega_n t}\right) \\ &= i2Ae^{\frac{i}{2}\Delta t}\sin\left(\Omega_n t\right) \\ &= A'e^{\frac{i}{2}\Delta t}\sin\left(\Omega_n t\right), \end{aligned}$$

where A' is just a constant. Also

$$\dot{c}_f(t) = A'e^{\frac{i}{2}\Delta t}\left(\frac{i}{2}\Delta\sin\left(\Omega_n t\right) + \Omega_n\cos\left(\Omega_n t\right)\right),$$

Back to equation 4.5.2

$$\begin{aligned} c_i(t) &= (i\dot{c}_f(t) + \Delta c_f(t))/\left(\lambda\sqrt{n+1}\right) \\ &= \frac{A'e^{i\Delta t/2}}{\lambda\sqrt{n+1}}\left(-\frac{\Delta}{2}\sin(\Omega_n t) + \Omega_n\cos(\Omega_n t) + \Delta\sin(\Omega_n t)\right) \\ &= \frac{A'e^{i\Delta t/2}}{\lambda\sqrt{n+1}}\left(\frac{\Delta}{2}\sin(\Omega_n t) + \Omega_n\cos(\Omega_n t)\right) \end{aligned}$$

Using the second initial condition, $c_i(0) = 1$, we obtain

$$A' = \frac{\lambda\sqrt{n+1}}{\Omega_n}.$$

And finally we have

$$\begin{aligned}
c_i(t) &= \frac{e^{i\Delta t/2}}{\Omega_n}\left(\frac{\Delta}{2}\sin(\Omega_n t) + \Omega_n\cos(\Omega_n t)\right) \\
c_f(t) &= \frac{\lambda\sqrt{n+1}}{\Omega_n} e^{i\Delta t/2}\sin(\Omega_n t).
\end{aligned}$$

$$|\Psi(t)\rangle = \frac{e^{i\Delta t/2}}{\Omega_n}\left(\frac{\Delta}{2}\sin(\Omega_n t) + \Omega_n\cos(\Omega_n t)\right)|i\rangle + \frac{\lambda\sqrt{n+1}}{\Omega_n} e^{i\Delta t/2}\sin(\Omega_n t)|f\rangle$$

The atomic inversion is given by

$$\begin{aligned}
W(t) &= |c_i(t)|^2 - |c_f(t)|^2 \\
&= \left|\frac{e^{i\Delta t/2}}{\Omega_n}\left(\frac{\Delta}{2}\sin(\Omega_n t) + \Omega_n\cos(\Omega_n t)\right)\right|^2 - \left|\frac{\lambda\sqrt{n+1}}{\Omega_n} e^{i\Delta t/2}\sin(\Omega_n t)\right|^2 \\
&= \frac{1}{\Omega_n^2}\left[\left(\frac{\Delta}{2}\sin(\Omega_n t) + \Omega_n\cos(\Omega_n t)\right)^2 - \lambda^2(n+1)\sin^2(\Omega_n t)\right]. \quad (4.5.3)
\end{aligned}$$

For a general case where we have the sum of n-photon inversions of Eq. 4.5.3 weighted with photon number distribution of the initial fields state we have

$$W(t) = \sum_{n=0}^{\infty} |c_n|^2 \frac{1}{\Omega_n^2}\left[\left(\frac{\Delta}{2}\sin(\Omega_n t) + \Omega_n\cos(\Omega_n t)\right)^2 - \lambda^2(n+1)\sin^2(\Omega_n t)\right]. \tag{4.5.4}$$

Notice that the last equation is in agreement with Eq. (4.123) for $\Delta = 0$.

4.6 Problem 4.6

For an atom initially in the excited state and the cavity field initially in a thermal state the atomic inversion is

$$W(t) = \frac{1}{1+\bar{n}}\sum_{n=0}^{\infty}\left(\frac{\bar{n}}{1+\bar{n}}\right)^n \cos(2\lambda t\sqrt{n+1})$$

Let

$$\Omega(n) = 2\lambda\sqrt{n+1}.$$

The collapse time is given by

$$t_c \left[\Omega(\bar{n}+\Delta n) - \Omega(\bar{n}-\Delta n)\right] \backsimeq 1.$$

For thermal light $\Delta n = \left(\bar{n}^2+\bar{n}\right)^{1/2}$ so rather generally

$$t_c \simeq \left[2\lambda\sqrt{\bar{n}+1+\left(\bar{n}^2+\bar{n}\right)^{1/2}} - 2\lambda\sqrt{\bar{n}+1-\left(\bar{n}^2+\bar{n}\right)^{1/2}}\right]^{-1}.$$

We can exam two limiting cases : $\bar{n} >> 1$ and $\bar{n} << 1$.
For $\bar{n} >> 1$, $\Delta n = \bar{n}+1/2 \simeq \bar{n}$, $\bar{n}+1 \to \bar{n}$, and thus

$$t_c \simeq \frac{1}{2\sqrt{2}\lambda\sqrt{\bar{n}}}$$

For the case where $\bar{n} << 1$, $\Delta n = \sqrt{\bar{n}}$, $\bar{n}+1 \to 1$

$$\begin{aligned} t_c &\simeq \left[2\lambda(1+\sqrt{\bar{n}})^{1/2} - 2\lambda(1-\sqrt{\bar{n}})^{1/2}\right]^{-1} \\ &\simeq \left[2\lambda(1+\frac{\sqrt{\bar{n}}}{2}) - 2\lambda(1-\frac{\sqrt{\bar{n}}2}{)}\right]^{-1} \\ &\simeq \left[2\lambda\sqrt{n}\right]^{-1} \\ &\simeq \frac{1}{2\lambda\sqrt{2}}. \end{aligned}$$

In both cases we get $t_c \sim \frac{1}{\sqrt{\bar{n}}}$. **a.** Here we consider the following Hamiltonian

$$\hat{H} = \frac{1}{2}\hbar\omega_0\hat{\sigma}_3 + \hbar\omega\hat{a}^\dagger\hat{a} + \hbar\lambda\hat{a}^\dagger\hat{a}(\hat{\sigma}_+ + \hat{\sigma}_-).$$

Also we define the following "bare" states

$$\begin{aligned} |\psi_{1n}\rangle &= |e\rangle|n\rangle \\ |\psi_{2n}\rangle &= |g\rangle|n\rangle. \end{aligned}$$

Clearly $\langle\psi_{1n}|\psi_{2n}\rangle = 0$. Using these basis we obtain the matrix elements of $\hat{H}$.

$$\begin{aligned} \hat{H}|\psi_{1n}\rangle &= \frac{1}{2}\hbar\omega_0|e\rangle|n\rangle + \hbar\omega n|e\rangle|n\rangle + \hbar\lambda n|g\rangle|n\rangle, \\ \hat{H}|\psi_{2n}\rangle &= -\frac{1}{2}\hbar\omega_0|g\rangle|n\rangle + \hbar\omega n|g\rangle|n\rangle + \hbar\lambda n|e\rangle|n\rangle \end{aligned}$$

$$
\begin{aligned}
\langle \psi_{1n} | \hat{H} | \psi_{1n} \rangle &= \hbar \left(\frac{1}{2}\omega_0 + n\omega \right), \\
\langle \psi_{2n} | \hat{H} | \psi_{2n} \rangle &= \hbar \left(-\frac{1}{2}\omega_0 + n\omega \right), \\
\langle \psi_{2n} | \hat{H} | \psi_{1n} \rangle &= \hbar n\lambda, \\
\langle \psi_{1n} | \hat{H} | \psi_{2n} \rangle &= \hbar n\lambda.
\end{aligned}
$$

$\hat{H}$ can be written in the matrix form as

$$
\hat{H} = \begin{pmatrix} \hbar \left(\frac{1}{2}\omega_0 + n\omega \right) & \hbar n\lambda \\ \hbar n\lambda & \hbar \left(-\frac{1}{2}\omega_0 + n\omega \right) \end{pmatrix}. \tag{4.6.1}
$$

It is easy to find the energy eigenvalues by solving the following secular equation

$$
\left(\frac{1}{2}\hbar\omega_0 + \hbar n\omega - E \right) \left(-\frac{1}{2}\hbar\omega_0 + \hbar n\omega - E \right) - \hbar^2\lambda^2 n^2 = 0. \tag{4.6.2}
$$

After some arrangements, we find two solutions for E, which we label as E_{n+} and E_{n-}.

$$
\begin{aligned}
E_{n\pm} &= \hbar n\omega \pm \hbar \left(\frac{1}{4}\omega_0^2 + \lambda^2 n^2 \right)^{1/2} \\
&= \hbar n\omega \pm \hbar\Omega_n,
\end{aligned}
$$

where $\Omega_n = \left(\frac{1}{4}\omega_0^2 + \lambda^2 n^2 \right)^{1/2}$. The eigenstates associated with the energy eigenvalues are given by

$$
\begin{aligned}
|n, +\rangle &= \quad \cos(\Phi_n/2) |\psi_{1n}\rangle + \sin(\Phi_n/2) |\psi_{2n}\rangle, \\
|n, -\rangle &= -\sin(\Phi_n/2) |\psi_{1n}\rangle + \cos(\Phi_n/2) |\psi_{2n}\rangle,
\end{aligned} \tag{4.6.3}
$$

where

$$
\begin{aligned}
\cos(\Phi_n/2) &= \frac{n\lambda}{\sqrt{2\Omega_n(\Omega_n - \omega_0/2)}}, \\
\sin(\Phi_n/2) &= \frac{\Omega_n - \omega_0/2}{\sqrt{2\Omega_n(\Omega_n - \omega_0/2)}}.
\end{aligned}
$$

b. For coherent states as an initial field state,

$$|\psi_f\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle,$$

and the initial atomic state at the ground state, $|g\rangle$, we have

$$\begin{aligned}
|\Psi(0)\rangle &= |\psi_f\rangle|g\rangle \\
&= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle|g\rangle \\
&= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|\psi_{2n}\rangle \\
&= e^{-|\alpha|^2/2}\sum_{n}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left[\sin(\Phi_n/2)|n,+\rangle + \cos(\Phi_n/2)|n,-\rangle\right],
\end{aligned}$$

where we have used Eqs. 4.6.3.

From Eq. (4.155) we have

$$\begin{aligned}
|\Psi(t)\rangle &= \exp\left[-\frac{i}{\hbar}\hat{H}t\right]|\Psi(0)\rangle \\
&= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left[\sin(\Phi_n/2)e^{-iE_{n+}t/\hbar}|n,+\rangle + \cos(\Phi_n/2)e^{-iE_{n-}t/\hbar}|n,-\rangle\right] \\
&= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\left[\sin(\Phi_n/2)\cos(\Phi_n/2)\left(e^{-iE_{n+}t/\hbar} - e^{-iE_{n-}t/\hbar}\right)|\psi_{1n}\rangle\right. \\
&\qquad\left. + \left(\sin^2(\Phi_n/2)e^{-iE_{n+}t/\hbar} + \cos^2(\Phi_n/2)e^{-iE_{n-}t/\hbar}\right)|\psi_{2n}\rangle\right] \\
&= e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-in\omega t} \\
&\qquad\times\left[i\sin(\Omega_n t)\sin(\Phi_n)|\psi_{1n}\rangle + \left(\cos(\Omega_n t) + i\sin(\Omega_n t)\cos(\Phi_n)\right)|\psi_{2n}\rangle\right]
\end{aligned}$$

Using Eq. (4.123) we found the atomic inversion to be

$$W(t) = e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{|\alpha|^n}{n!}\left[\sin^2(\Omega_n t)\sin^2(\Phi_n) - \cos^2(\Omega_n t) - \sin^2(\Omega_n t)\cos^2(\Phi_n)\right]$$

c. For the case of an initial thermal field state and the initial atomic state in the ground state, the initial density operator is given by

$$\begin{aligned}\hat{\rho}(0) &= \hat{\rho}_{\mathrm{a}}(0)\hat{\rho}_{\mathrm{Th}}(0) \\ &= \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} |\psi_{1n}\rangle\langle\psi_{1n}|\end{aligned}$$

For $t > 0$, the density operator becomes

$$\hat{\rho}(t) = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]\hat{\rho}(0)\exp\left[\frac{i}{\hbar}\hat{H}t\right]$$

Using results of part b, we easily find that the atomic inversion is given by

$$W(t) = \sum_{n=0}^{\infty} \frac{\bar{n}^n}{(1+\bar{n})^{n+1}} \left[\sin^2(\Omega_n t)\sin^2(\Phi_n) - \cos^2(\Omega_n t) - \sin^2(\Omega_n t)\cos^2(\Phi_n)\right].$$

4.7 Problem 4.7

a.

$$\hat{H}_{eff} = \hbar\eta\left(\hat{a}^2\hat{\sigma}_+^\dagger + \hat{a}^{2\dagger}\hat{\sigma}_-\right). \tag{4.7.1}$$

Let define the following states

$$\begin{aligned}|i\rangle &= |e\rangle|n\rangle \\ |f\rangle &= |g\rangle|n+2\rangle\end{aligned}$$

$$\begin{aligned}\langle i|\hat{H}_{eff}|i\rangle &= 0 \\ \langle f|\hat{H}_{eff}|i\rangle &= \hbar\eta\sqrt{(n+2)(n+1)} \\ \langle f|\hat{H}_{eff}|f\rangle &= 0 \\ \langle i|\hat{H}_{eff}|f\rangle &= \hbar\eta\sqrt{(n+2)(n+1)}\end{aligned}$$

$$\mathbf{H}^{(n)} = \begin{pmatrix} 0 & \hbar\eta\sqrt{(n+2)(n+1)} \\ \hbar\eta\sqrt{(n+2)(n+1)} & 0 \end{pmatrix}$$

$$\begin{aligned}
|n,+\rangle &= \frac{1}{\sqrt{2}}\left(|i\rangle + |f\rangle\right)\\
|n,-\rangle &= \frac{1}{\sqrt{2}}\left(|i\rangle - |f\rangle\right)\\
E_{n,\pm} &= \pm\hbar\eta\sqrt{(n+2)(n+1)}
\end{aligned}$$

b.
Initial field at a number state

$$\begin{aligned}
|\Psi_{af}(0)\rangle &= |g\rangle|n+2\rangle\\
&= |f\rangle\\
&= \frac{1}{\sqrt{2}}\left(|n,+\rangle - |n,-\rangle\right)
\end{aligned}$$

$$\begin{aligned}
|\Psi_{af}(t)\rangle &= e^{-i\hat{H}t/\hbar}|\Psi_{af}(0)\rangle\\
&= e^{-i\hat{H}t/\hbar}\frac{1}{\sqrt{2}}\left(|n,+\rangle - |n,-\rangle\right)\\
&= \frac{1}{\sqrt{2}}\left(e^{-iE_+t/\hbar}|n,+\rangle - e^{-iE_-t/\hbar}|n,-\rangle\right)\\
&= \frac{1}{\sqrt{2}}\left(e^{-i\eta\sqrt{(n+2)(n+1)}}|n,+\rangle - e^{i\eta\sqrt{(n+2)(n+1)}t}|n,-\rangle\right)\\
&= i\sin\left(\eta\sqrt{(n+2)(n+1)}t\right)|i\rangle + \cos\left(\eta\sqrt{(n+2)(n+1)}t\right)|f\rangle
\end{aligned}$$

$$\begin{aligned}
W(t) &= \sin^2\left(\eta\sqrt{(n+2)(n+1)}t\right) - \cos^2\left(\eta\sqrt{(n+2)(n+1)}t\right)\\
&= -\cos\left(2\eta\sqrt{(n+2)(n+1)}t\right)
\end{aligned}$$

Initial field at a coherent state

$$
\begin{aligned}
|\Psi_{af}(0)\rangle &= |g\rangle|\alpha\rangle \\
&= \sum_{n=0}^{\infty} c_n |g\rangle|n\rangle \\
&= |g\rangle(c_0|0\rangle + c_1|1\rangle) + \sum_{n=0}^{\infty} c_{n+2}|g\rangle|n+2\rangle \\
&= |g\rangle(c_0|0\rangle + c_1|1\rangle) + \sum_{n=0}^{\infty} c_{n+2}|f_n\rangle \\
&= |g\rangle(c_0|0\rangle + c_1|1\rangle) + \sum_{n=0}^{\infty} c_{n+2}\frac{1}{\sqrt{2}}\left(|n,+\rangle - |n,-\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
|\Psi_{af}(t)\rangle &= e^{-i\hat{H}t/\hbar}|\Psi_{af}(0)\rangle \\
&= |g\rangle(c_0|0\rangle + c_1|1\rangle) + \sum_{n=0}^{\infty} c_{n+2}\frac{1}{\sqrt{2}}\left(e^{-iE_{n,+}t}|n,+\rangle - e^{-iE_{n,-}t}|n,-\rangle\right) \\
&= |g\rangle(c_0|0\rangle + c_1|1\rangle) \\
&+ \sum_{n=0}^{\infty} c_{n+2}\left(i\sin\left(\eta\sqrt{(n+2)(n+1)}t\right)|i\rangle + \cos\left(\eta\sqrt{(n+2)(n+1)}t\right)|f\rangle\right) \\
&= |g\rangle\left(c_0|0\rangle + c_1|1\rangle + i\sum_{n=0}^{\infty} c_{n+2}\sin\left(\eta\sqrt{(n+2)(n+1)}t\right)|n+2\rangle\right) \\
&+ |e\rangle\sum_{n=0}^{\infty} c_{n+2}\cos\left(\eta\sqrt{(n+2)(n+1)}t\right)|n\rangle
\end{aligned}
$$

$$
\begin{aligned}
W(t) &= \langle\Psi_{af}(t)|\hat{\sigma}_3|\Psi_{af}(t)\rangle \\
&= \left[|c_0|^2 + |c_1|^2 + \sum_{n=0}^{\infty}|c_{n+2}|^2\sin^2\left(\eta\sqrt{(n+2)(n+1)}t\right)\right] \\
&\quad - \left[\sum_{n=0}^{\infty}|c_{n+2}|^2\cos^2\left(\eta\sqrt{(n+2)(n+1)}t\right)\right] \\
&= |c_0|^2 + |c_1|^2 - \sum_{n=0}^{\infty}|c_{n+2}|^2\cos\left(2\eta\sqrt{(n+2)(n+1)}t\right)
\end{aligned}
$$

c.

$$\begin{aligned}
\hat{\rho}_{af}(0) &= \hat{\rho}_a(0) \otimes \hat{\rho}_f(0) \\
&= \sum_{n=0}^{\infty} P_n |g\rangle|n\rangle\langle g|\langle n| \\
&= P_0|g\rangle|0\rangle\langle g|\langle 0| + P_1|g\rangle|1\rangle\langle g|\langle 1| + \sum_{n=2}^{\infty} P_n|g\rangle|n\rangle\langle g|\langle n|
\end{aligned}$$

$$\begin{aligned}
\hat{\rho}_{af}(t) &= \hat{U}(t)\hat{\rho}_{af}(0)\hat{U}^\dagger(t) \\
&= \hat{U}(t)\left(\sum_{n=0}^{\infty} P_n|g\rangle|n\rangle\langle g|\langle n|\right)\hat{U}^\dagger(t) \\
&= P_0|g\rangle|0\rangle\langle g|\langle 0| + P_1|g\rangle|1\rangle\langle g|\langle 1| + \hat{U}(t)\left(\sum_{n=2}^{\infty} P_n|g\rangle|n\rangle\langle g|\langle n|\right)\hat{U}^\dagger(t) \\
&= P_0|g\rangle|0\rangle\langle g|\langle 0| + P_1|g\rangle|1\rangle\langle g|\langle 1| + \sum_{n=2}^{\infty} P_n\hat{U}(t)|g\rangle|n\rangle\langle g|\langle n|\hat{U}^\dagger(t) \\
&= P_0|g\rangle|0\rangle\langle g|\langle 0| + P_1|g\rangle|1\rangle\langle g|\langle 1| \\
&+ \sum_{n=2}^{\infty} P_n\left(i\sin(\Omega_n t)|i\rangle + \cos(\Omega_n t)|f\rangle\right)\left(-i\sin(\Omega_n t)\langle i| + \cos(\Omega_n t)\langle f|\right)
\end{aligned}$$

$$\begin{aligned}
W(t) &= \mathbf{Tr}\left(\hat{\sigma}_3\hat{\rho}_{af}(t)\right) \\
&= \sum_{n=2}^{\infty} P_n\sin^2(\Omega_n t) - \sum_{n=2}^{\infty} P_n\cos^2(\Omega_n t) - P_0 - P_2 \\
&= -P_0 - P_2 - \sum_{n=2}^{\infty} P_n\cos(2\Omega_n t)
\end{aligned}$$

4.8 Problem 4.8

a.

$$\hat{H}_{eff} = \hbar\eta\left(\hat{a}^2\hat{\sigma}_+^\dagger + \hat{a}^{2\dagger}\hat{\sigma}_-\right). \tag{4.8.1}$$

Let define the following states

$$|i\rangle = |e\rangle|n\rangle$$
$$|f\rangle = |g\rangle|n+2\rangle$$

$$\begin{aligned}
\langle i|\hat{H}_{eff}|i\rangle &= 0 \\
\langle f|\hat{H}_{eff}|i\rangle &= \hbar\eta\sqrt{(n+2)(n+1)} \\
\langle f|\hat{H}_{eff}|f\rangle &= 0 \\
\langle i|\hat{H}_{eff}|f\rangle &= \hbar\eta\sqrt{(n+2)(n+1)}
\end{aligned}$$

$$\mathbf{H}^{(n)} = \begin{pmatrix} 0 & \hbar\eta\sqrt{(n+2)(n+1)} \\ \hbar\eta\sqrt{(n+2)(n+1)} & 0 \end{pmatrix}$$

$$\begin{aligned}
|n,+\rangle &= \frac{1}{\sqrt{2}}\left(|i\rangle + |f\rangle\right) \\
|n,-\rangle &= \frac{1}{\sqrt{2}}\left(|i\rangle - |f\rangle\right) \\
E_{n,\pm} &= \pm\hbar\eta\sqrt{(n+2)(n+1)}
\end{aligned}$$

b.
Initial field at a number state

$$\begin{aligned}
|\Psi_{af}(0)\rangle &= |g\rangle|n+2\rangle \\
&= |f\rangle \\
&= \frac{1}{\sqrt{2}}\left(|n,+\rangle - |n,-\rangle\right)
\end{aligned}$$

$$\begin{aligned}
|\Psi_{af}(t)\rangle &= e^{-i\hat{H}t/\hbar}|\Psi_{af}(0)\rangle \\
&= e^{-i\hat{H}t/\hbar}\frac{1}{\sqrt{2}}\left(|n,+\rangle - |n,-\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(e^{-iE_+t/\hbar}|n,+\rangle - e^{-iE_-t/\hbar}|n,-\rangle\right) \\
&= \frac{1}{\sqrt{2}}\left(e^{-i\eta\sqrt{(n+2)(n+1)}}|n,+\rangle - e^{i\eta\sqrt{(n+2)(n+1)}t}|n,-\rangle\right) \\
&= i\sin\left(\eta\sqrt{(n+2)(n+1)}t\right)|i\rangle + \cos\left(\eta\sqrt{(n+2)(n+1)}t\right)|f\rangle
\end{aligned}$$

$$
\begin{aligned}
W(t) &= \sin^2\left(\eta\sqrt{(n+2)(n+1)}t\right) - \cos^2\left(\eta\sqrt{(n+2)(n+1)}t\right) \\
&= -\cos\left(2\eta\sqrt{(n+2)(n+1)}t\right)
\end{aligned}
$$

Initial field at a coherent state

$$
\begin{aligned}
|\Psi_{af}(0)\rangle &= |g\rangle|\alpha\rangle \\
&= \sum_{n=0}^{\infty} c_n|g\rangle|n\rangle \\
&= |g\rangle(c_0|0\rangle + c_1|1\rangle) + \sum_{n=0}^{\infty} c_{n+2}|g\rangle|n+2\rangle \\
&= |g\rangle(c_0|0\rangle + c_1|1\rangle) + \sum_{n=0}^{\infty} c_{n+2}|f_n\rangle \\
&= |g\rangle(c_0|0\rangle + c_1|1\rangle) + \sum_{n=0}^{\infty} c_{n+2}\frac{1}{\sqrt{2}}\left(|n,+\rangle - |n,-\rangle\right)
\end{aligned}
$$

$$
\begin{aligned}
|\Psi_{af}(t)\rangle &= e^{-i\hat{H}t/\hbar}|\Psi_{af}(0)\rangle \\
&= |g\rangle(c_0|0\rangle + c_1|1\rangle) + \sum_{n=0}^{\infty} c_{n+2}\frac{1}{\sqrt{2}}\left(e^{-iE_{n,+}t}|n,+\rangle - e^{-iE_{n,-}t}|n,-\rangle\right) \\
&= |g\rangle(c_0|0\rangle + c_1|1\rangle) \\
&+ \sum_{n=0}^{\infty} c_{n+2}\left(i\sin\left(\eta\sqrt{(n+2)(n+1)}t\right)|i\rangle + \cos\left(\eta\sqrt{(n+2)(n+1)}t\right)|f\rangle\right) \\
&= |g\rangle\left(c_0|0\rangle + c_1|1\rangle + i\sum_{n=0}^{\infty} c_{n+2}\sin\left(\eta\sqrt{(n+2)(n+1)}t\right)|n+2\rangle\right) \\
&+ |e\rangle\sum_{n=0}^{\infty} c_{n+2}\cos\left(\eta\sqrt{(n+2)(n+1)}t\right)|n\rangle
\end{aligned}
$$

$$\begin{aligned}
W(t) &= \langle \Psi_{af}(t)|\hat{\sigma}_3|\Psi_{af}(t)\rangle \\
&= \left[|c_0|^2 + |c_1|^2 + \sum_{n=0}^{\infty} |c_{n+2}|^2 \sin^2\left(\eta\sqrt{(n+2)(n+1)}t\right)\right] \\
&\quad - \left[\sum_{n=0}^{\infty} |c_{n+2}|^2 \cos^2\left(\eta\sqrt{(n+2)(n+1)}t\right)\right] \\
&= |c_0|^2 + |c_1|^2 - \sum_{n=0}^{\infty} |c_{n+2}|^2 \cos\left(2\eta\sqrt{(n+2)(n+1)}t\right)
\end{aligned}$$

c.

$$\begin{aligned}
\hat{\rho}_{af}(0) &= \hat{\rho}_a(0) \otimes \hat{\rho}_f(0) \\
&= \sum_{n=0}^{\infty} P_n |g\rangle|n\rangle\langle g|\langle n| \\
&= P_0|g\rangle|0\rangle\langle g|\langle 0| + P_1|g\rangle|1\rangle\langle g|\langle 1| + \sum_{n=2}^{\infty} P_n|g\rangle|n\rangle\langle g|\langle n|
\end{aligned}$$

$$\begin{aligned}
\hat{\rho}_{af}(t) &= \hat{U}(t)\hat{\rho}_{af}(0)\hat{U}^\dagger(t) \\
&= \hat{U}(t)\left(\sum_{n=0}^{\infty} P_n|g\rangle|n\rangle\langle g|\langle n|\right)\hat{U}^\dagger(t) \\
&= P_0|g\rangle|0\rangle\langle g|\langle 0| + P_1|g\rangle|1\rangle\langle g|\langle 1| + \hat{U}(t)\left(\sum_{n=2}^{\infty} P_n|g\rangle|n\rangle\langle g|\langle n|\right)\hat{U}^\dagger(t) \\
&= P_0|g\rangle|0\rangle\langle g|\langle 0| + P_1|g\rangle|1\rangle\langle g|\langle 1| + \sum_{n=2}^{\infty} P_n\hat{U}(t)|g\rangle|n\rangle\langle g|\langle n|\hat{U}^\dagger(t) \\
&= P_0|g\rangle|0\rangle\langle g|\langle 0| + P_1|g\rangle|1\rangle\langle g|\langle 1| \\
&+ \sum_{n=2}^{\infty} P_n\left(i\sin(\Omega_n t)|i\rangle + \cos(\Omega_n t)|f\rangle\right)\left(-i\sin(\Omega_n t)\langle i| + \cos(\Omega_n t)\langle f|\right)
\end{aligned}$$

$$
\begin{aligned}
W(t) &= \mathbf{Tr}\left(\hat{\sigma}_3 \hat{\rho}_{af}(t)\right) \\
&= \sum_{n=2}^{\infty} P_n \sin^2(\Omega_n t) - \sum_{n=2}^{\infty} P_n \cos^2(\Omega_n t) - P_0 - P_2 \\
&= -P_0 - P_2 - \sum_{n=2}^{\infty} P_n \cos(2\Omega_n t)
\end{aligned}
$$

4.9 Problem 4.9

$$
\hat{H}_{eff} = \hbar\eta \left(\hat{a}\hat{b}\hat{\sigma}_+^\dagger + \hat{a}^\dagger \hat{b}^\dagger \hat{\sigma}_- \right).
$$

Let define the following states

$$
\begin{aligned}
|f_{n,m}\rangle &= |e\rangle |n\rangle_a |m\rangle_b \\
|i_{n,m}\rangle &= |g\rangle |n+1\rangle_a |m+1\rangle_b
\end{aligned}
$$

$$
\begin{aligned}
\langle i_{n,m}|\hat{H}_{eff}|i_{n,m}\rangle &= 0 \\
\langle f_{n,m}|\hat{H}_{eff}|i_{n,m}\rangle &= \hbar\eta\sqrt{(m+1)(n+1)} = \hbar\Omega_{n,m} \\
\langle f_{n,m}|\hat{H}_{eff}|f_{n,m}\rangle &= 0 \\
\langle i_{n,m}|\hat{H}_{eff}|f_{n,m}\rangle &= \hbar\eta\sqrt{(m+1)(n+1)} = \hbar\Omega_{n,m}
\end{aligned}
$$

where we have defined $\Omega_{n,m} = \eta\sqrt{(m+1)(n+1)}$.

$$
\mathbf{H}^{(n,m)} = \begin{pmatrix} 0 & \hbar\Omega_{n,m} \\ \hbar\Omega_{n,m} & 0 \end{pmatrix}
$$

$$
\begin{aligned}
|n,m,+\rangle &= \frac{1}{\sqrt{2}} \left(|i_{n,m}\rangle + |f_{n,m}\rangle \right) \\
|n,m,-\rangle &= \frac{1}{\sqrt{2}} \left(|i_{n,m}\rangle - |f_{n,m}\rangle \right) \\
E_{n,m,\pm} &= \pm\hbar\Omega_{n,m}
\end{aligned}
$$

Now for an initial state with the atom at the excited state and the two fields at coherent states, we have

$$\begin{aligned}
|\Psi(0)\rangle &= |e\rangle|\alpha\rangle_a|\beta\rangle_b \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{a,n}c_{b,m}|f_{n,m}\rangle \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{a,n}c_{b,m}\frac{1}{\sqrt{2}}\left(|n,m,+\rangle - |n,m,-\rangle\right)
\end{aligned}$$

$$\begin{aligned}
|\Psi(t)\rangle &= e^{-i\hat{H}_{eff}t/\hbar}|\Psi(0)\rangle \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{a,n}c_{b,m}\frac{1}{\sqrt{2}}\left(e^{-i\hat{H}_{eff}t/\hbar}|n,m,+\rangle - e^{-i\hat{H}_{eff}t/\hbar}|n,m,-\rangle\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{a,n}c_{b,m}\frac{1}{\sqrt{2}}\left(e^{-i\Omega_{n,m}t}|n,m,+\rangle - e^{i\Omega_{n,m}t}|n,m,-\rangle\right) \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{a,n}c_{b,m}\left(-i\sin(\Omega_{n,m}t)|f_{n,m}\rangle + \cos(\Omega_{n,m}t)|i_{n,m}\rangle\right)
\end{aligned}$$

$$\begin{aligned}
W(t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |c_{a,n}c_{b,m}|^2\left(\sin^2(\Omega_{n,m}t) - \cos^2(\Omega_{n,m}t)\right) \\
&= -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |c_{a,n}c_{b,m}|^2\cos(2\Omega_{n,m}t)
\end{aligned}$$

4.10 Problem 4.10

Somehow, the book has no Problem 4.10.

4.11 Problem 4.11

a. From equation (4.120) we have

$$\begin{aligned}|\Psi(t)\rangle &= |\Psi_g(t)\rangle|g\rangle + |\Psi_e(t)\rangle|e\rangle \\ &= -i\sum_{n=0}^{\infty} e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\sin(\lambda t\sqrt{n+1})|n+1\rangle|g\rangle \\ &\quad + \sum_{n=0}^{\infty} e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}\cos(\lambda t\sqrt{n+1})|n\rangle|e\rangle \\ &= \sum_{N=0}^{\infty} c_{gN}|N+1\rangle|g\rangle + c_{eN}|N\rangle|e\rangle,\end{aligned}$$

where obviously we have

$$\begin{aligned}c_{gN} &= -ie^{-|\alpha|^2/2}\frac{\alpha^N}{\sqrt{N!}}\sin(\lambda t\sqrt{N+1}) \\ c_{eN} &= e^{-|\alpha|^2/2}\frac{\alpha^N}{\sqrt{N!}}\cos(\lambda t\sqrt{N+1}).\end{aligned}$$

The density operator is then

$$\hat{\rho} = |\Psi(t)\rangle\langle\Psi(t)|\,.$$

Tracing over the atomic states we obtain

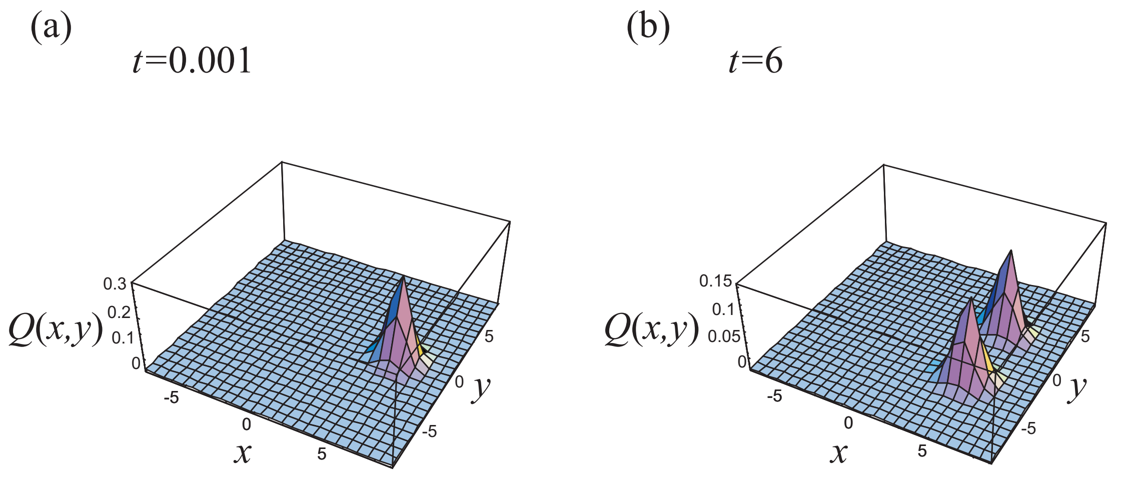
$$\begin{aligned}\hat{\rho}_f &= \mathrm{Tr}_A\hat{\rho} \\ &= \sum_{M=0}^{\infty}\sum_{N=0}^{\infty}\left(c_{gN}c_{gM}^*|N+1\rangle\langle M+1| + c_{eN}c_{eM}^*|N\rangle\langle M|\right) \\ &= \sum_{M=0}^{\infty}\sum_{N=0}^{\infty}\left(c_{gN-1}c_{gM-1}^* + c_{eN}c_{eM}^*\right)|N\rangle\langle M|\end{aligned}$$

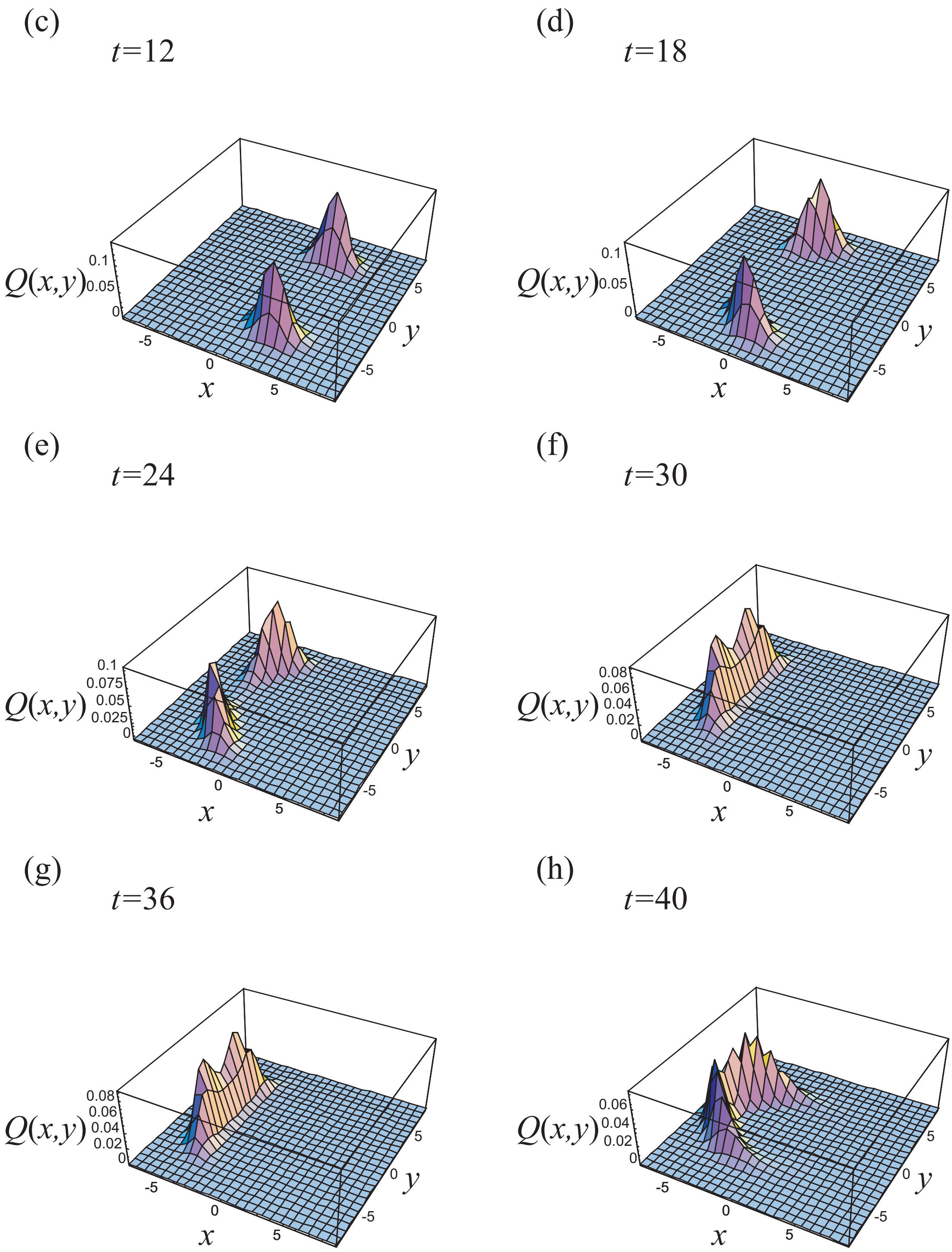
Obviously $\langle N|\hat{\rho}_f|M\rangle = c_{gN-1}c^*_{gM-1} + c_{eN}c^*_{eM}$

$$
\begin{aligned}
s(t) &= 1 - \mathrm{Tr}(\hat{\rho}_f^2) \\
&= 1 - \sum_N \langle N|\hat{\rho}_f^2|N\rangle \\
&= 1 - \sum_N \sum_M \langle N|\hat{\rho}_f|M\rangle\langle M|\hat{\rho}_f|N\rangle \\
&= 1 - \sum_N \sum_M |\langle N|\hat{\rho}_f|M\rangle|^2 \\
&= 1 - \sum_N \sum_M \frac{e^{-2|\alpha|^2}|\alpha|^{2(N+M)}}{N!M!} \\
&\qquad \times \left| \frac{\sqrt{NM}}{|\alpha|^2} \sin\left(\lambda t\sqrt{N}\right) \sin\left(\lambda t\sqrt{M}\right) + \cos\left(\lambda t\sqrt{N+1}\right) \cos\left(\lambda t\sqrt{M+1}\right) \right|^2
\end{aligned}
$$

b.

$$
\begin{aligned}
Q(\beta) &= \langle\beta|\hat{\rho}_f|\beta\rangle/\pi \\
&= \frac{1}{\pi}\sum_N \sum_M \frac{e^{-|\alpha|^2-|\beta|^2}\,(\alpha\beta^*)^N\,(\alpha^*\beta)^M}{N!M!} \left| \frac{|\beta|^2}{\sqrt{(N+1)(M+1)}} \right. \\
&\times \left. \sin\left(\lambda t\sqrt{N+1}\right) \sin\left(\lambda t\sqrt{M+1}\right) + \cos\left(\lambda t\sqrt{N+1}\right) \cos\left(\lambda t\sqrt{M+1}\right) \right|^2
\end{aligned}
$$




(c)
t=12
(d)
t=18
(e)
t=24
(f)
t=30
(g)
t=36
(h)
t=40
Q(x,y)
x
y

4.12 Problem 4.12

a. Equation (4.190) is of the form

$$\left|\Psi\left(\frac{\pi}{2\chi}\right)\right\rangle = \frac{1}{\sqrt{2}}\left(|g\rangle|-\alpha\rangle + |f\rangle|\alpha\rangle\right), \tag{4.12.1}$$

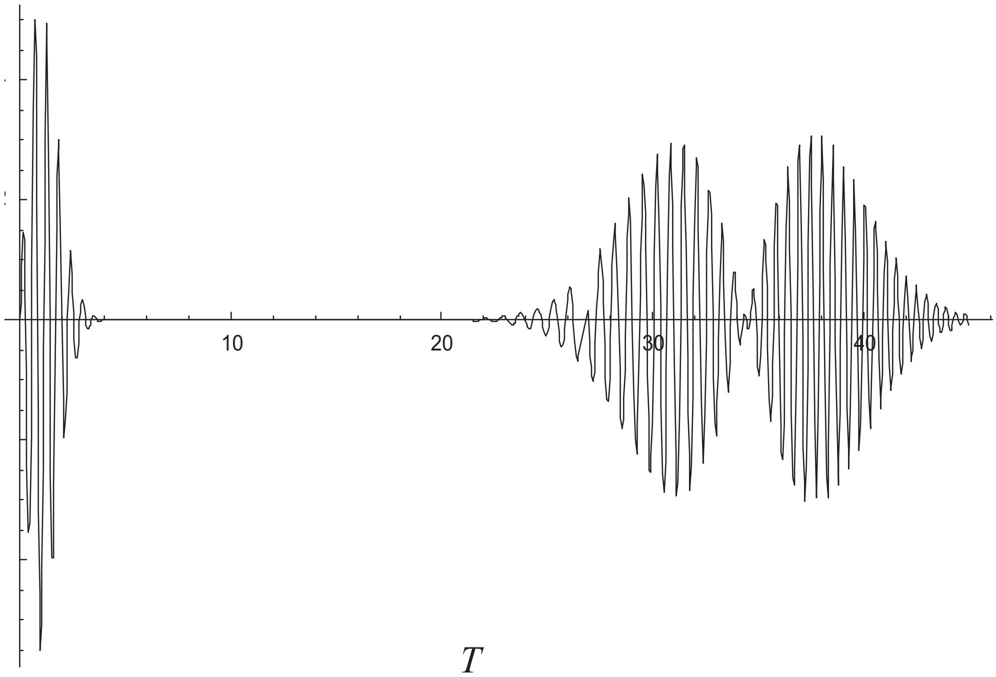
where we take $\phi = 0$.

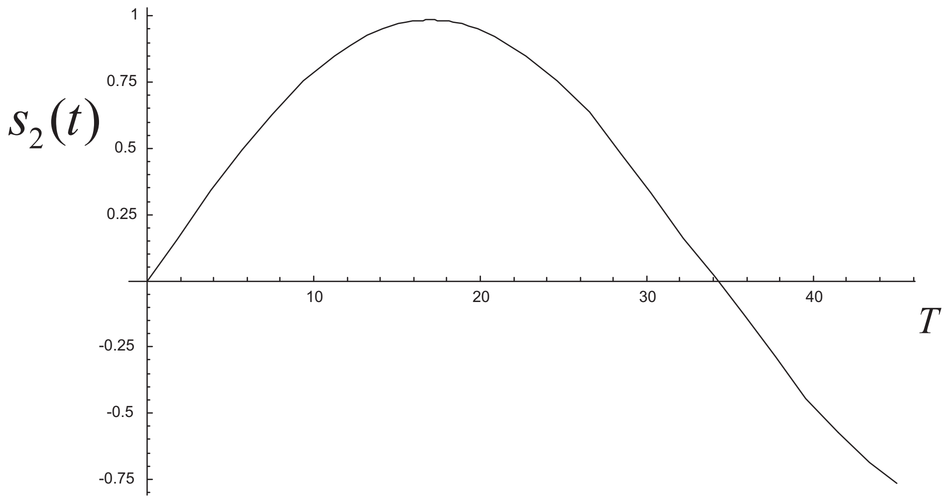
A detection of a superposition of the atomic states of the form $|S_\pm\rangle = (|g\rangle \pm |f\rangle)/\sqrt{2}$ would collapse the state in equation 4.12.1 to

$$\begin{aligned}\mathcal{N}_\pm \langle S_\pm|\Psi\rangle &= \mathcal{N}_\pm(\langle g| \pm \langle f|)(|g\rangle|-\alpha\rangle + |f\rangle|\alpha\rangle) \\ &= \mathcal{N}_\pm(|-\alpha\rangle \pm |\alpha\rangle),\end{aligned}$$

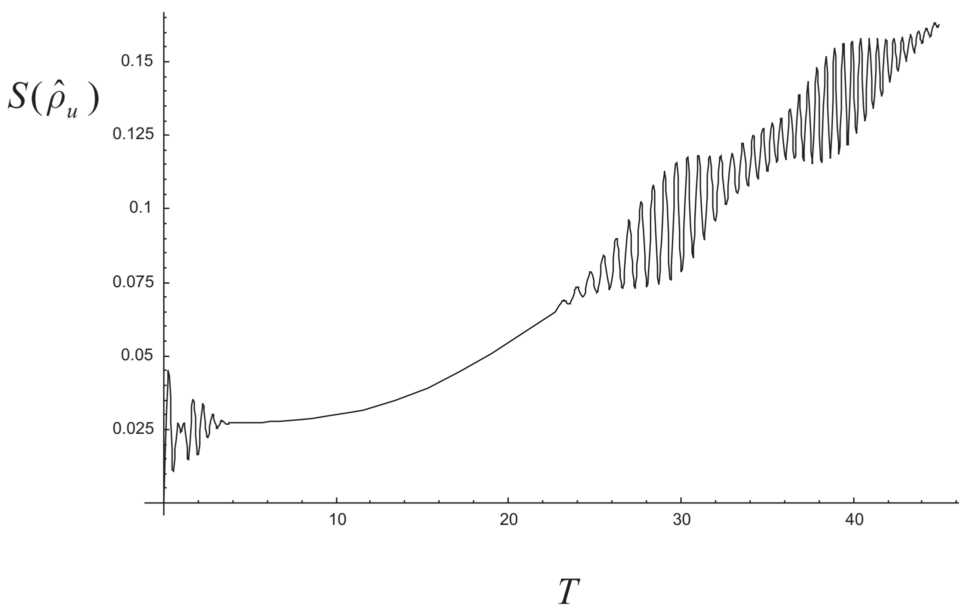
where $\mathcal{N}_\pm$ is the normalization factor. Notice that the obtained states are just the famous Schrödinger states.

4.13 Problem 4.13





$s_2(t)$
1
0.75
0.5
0.25
-0.25
-0.5
-0.75
10
20
30
40
T



$S(\hat{\rho}_u)$
0.15
0.125
0.1
0.075
0.05
0.025
10
20
30
40
T

Chapter 5

Quantum Coherence Functions

5.1 Problem 5.1

Eq. (5.55) reads

$$I(\mathbf{r},t) = |f(r)|^2 \left\{ \mathrm{Tr}(\hat{\rho}\hat{a}_1^\dagger \hat{a}_1) + \mathrm{Tr}(\hat{\rho}\hat{a}_2^\dagger \hat{a}_2) + 2|\mathrm{Tr}(\hat{\rho}\hat{a}_1^\dagger \hat{a}_2)| \cos\Phi \right\} \tag{5.1.1}$$

For an incident field n-photon state $|n\rangle_a|0\rangle_b = \frac{1}{\sqrt{n!}}(\frac{1}{\sqrt{2}})^n(\hat{a}_1^\dagger + \hat{a}_2^\dagger)^n|0\rangle_1|0\rangle_2$, as mentioned in equation (5.60). Also can be written as

$$\begin{aligned}
|n\rangle_a|0\rangle_b &= \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\right)^n (\hat{a}_1^\dagger + \hat{a}_2^\dagger)^n|0\rangle_1|0\rangle_2 \\
&= \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\right)^n \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \hat{a}_1^{\dagger k}\hat{a}_2^{\dagger n-k}|0\rangle_1|0\rangle_2 \\
&= \frac{1}{\sqrt{n!}}\left(\frac{1}{\sqrt{2}}\right)^n \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \sqrt{k!}\sqrt{(n-k)!}|k\rangle_1|n-k\rangle_2 \\
&= \left(\frac{1}{\sqrt{2}}\right)^n \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)^{\frac{1}{2}} |k\rangle_1|n-k\rangle_2
\end{aligned}$$

It is easy to see that

$$\begin{aligned}
\mathrm{Tr}(\hat{\rho}\hat{a}_1^\dagger \hat{a}_1) &= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n}\sum_{k'=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)^{\frac{1}{2}} \left(\begin{array}{c} n \\ k' \end{array} \right)^{\frac{1}{2}} \langle k', n-k'|\hat{a}_1^\dagger \hat{a}_1|k, n-k\rangle \\
&= \frac{1}{2^n}\sum_{k=0}^{n} k \left(\begin{array}{c} n \\ k \end{array} \right) \qquad (5.1.2)
\end{aligned}$$

To carry out the last sum, let's consider the following function

$$f_n(x) = \sum_{k=0}^{n} e^{kx} \left(\begin{array}{c} n \\ k \end{array} \right).$$

Using the binomial expansion we can write

$$\begin{aligned} f_n(x) &= (1+e^x)^n \\ f_n'(x) &= ne^x(1+e^x)^{n-1} \\ f_n'(0) &= n2^{n-1} = \sum_{k=0}^{n} k \left(\begin{array}{c} n \\ k \end{array} \right) \end{aligned}$$

Obviously, Eq. 5.1.2 now can be written as

$$\mathrm{Tr}(\hat{\rho}\hat{a}_1^\dagger \hat{a}_1) = \frac{n}{2} \qquad (5.1.3)$$

with the same technique we can calculate

$$\mathrm{Tr}(\hat{\rho}\hat{a}_2^\dagger \hat{a}_2) = \frac{n}{2}. \qquad (5.1.4)$$

we still have to determine $\mathrm{Tr}(\hat{\rho}\hat{a}_1^\dagger \hat{a}_2)$

$$\begin{aligned} \mathrm{Tr}(\hat{\rho}\hat{a}_1^\dagger \hat{a}_2) &= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \sum_{k'=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)^{\frac{1}{2}} \left(\begin{array}{c} n \\ k' \end{array} \right)^{\frac{1}{2}} \langle k', n-k' | \hat{a}_1^\dagger \hat{a}_2 | k, n-k \rangle \\ &= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \sum_{k'=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)^{\frac{1}{2}} \left(\begin{array}{c} n \\ k' \end{array} \right)^{\frac{1}{2}} \sqrt{k+1}\sqrt{n-k}\,\delta_{k',k+1} \\ &= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right)^{\frac{1}{2}} \left(\begin{array}{c} n \\ k+1 \end{array} \right)^{\frac{1}{2}} \sqrt{k+1}\sqrt{n-k} \\ &= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \sqrt{\frac{n!n!(k+1)(n-k)}{k!(n-k)!(k+1)!(n-k-1)!}} \\ &= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} \frac{n!}{k!(n-k-1)!} \\ &= \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} (n-k) \left(\begin{array}{c} n \\ k \end{array} \right) \\ &= \frac{n}{2} \end{aligned}$$

Finally

$$I(\mathbf{r},t) = n|f(\mathbf{r})|^2[1+\cos\Phi]. \tag{5.1.5}$$

5.2 Problem 5.2

Again we use equation (5.55) for thermal light

$$I(\mathbf{r},t) = |f(r)|^2 \left\{ \mathrm{Tr}\left(\hat{\rho}_{\mathrm{th}}\hat{a}_1^\dagger\hat{a}_1\right) + \mathrm{Tr}\left(\hat{\rho}_{\mathrm{th}}\hat{a}_2^\dagger\hat{a}_2\right) + 2\left|\mathrm{Tr}\left(\hat{\rho}_{\mathrm{th}}\hat{a}_1^\dagger\hat{a}_2\right)\right|\cos\Phi \right\}.$$

Before we compute the traces in the previous equation we need to find what what is the form of $\hat{\rho}_{\mathrm{th}}$ after the pinholes.

$$\begin{aligned}\hat{\rho}_{\mathrm{th}} &= \sum P_n|n\rangle\langle n| \\ &= \sum P_n|n\rangle_1|0\rangle_2\ {}_1\langle n|_2\langle 0|.\end{aligned}$$

From the previous problem we have

$$|n\rangle_a|0\rangle_b = \frac{1}{\sqrt{n!2^n}}(\hat{a}_1^\dagger + \hat{a}_2^\dagger)^n|0\rangle_1|0\rangle_2,$$

which helps us to rewrite $\hat{\rho}_{\mathrm{th}}$ after the pinholes as

$$\hat{\rho}_{\mathrm{th}} = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{k'=0}^{n}\frac{1}{2^n}P_n\left[\binom{n}{k}\binom{n}{k'}\right]^{1/2}|k\rangle_1|n-k\rangle_2\ {}_1\langle k'|_2\langle n-k'|$$

$$
\begin{aligned}
\mathrm{Tr}(\hat{\rho}_{\mathrm{th}}\hat{a}_1^\dagger\hat{a}_1) &= \mathrm{Tr}\left(\hat{a}_1\hat{\rho}_{\mathrm{th}}\hat{a}_1^\dagger\right) \\
&= \sum_{N,M}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{k'=0}^{n}\frac{1}{2^n}P_n\left[\binom{n}{k}\binom{n}{k'}\right]^{1/2} \\
&\quad \times {}_1\langle N|_2\langle M|\hat{a}_1^\dagger|k\rangle_1|n-k\rangle_2\ {}_1\langle k'|_2\langle n-k'|\hat{a}_1|N\rangle_1|M\rangle_2 \\
&= \sum_{N,M}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{k'=0}^{n}\frac{1}{2^n}P_n\left[\binom{n}{k}\binom{n}{k'}\right]^{1/2} \\
&\quad \times\ {}_1\langle k'|_2\langle n-k'|\hat{a}_1|N\rangle_1|M\rangle_2{}_1\langle N|_2\langle M|\hat{a}_1^\dagger|k\rangle_1|n-k\rangle_2 \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{k'=0}^{n}\frac{1}{2^n}P_n\left[\binom{n}{k}\binom{n}{k'}\right]^{1/2}\ {}_1\langle k'|_2\langle n-k'|\hat{a}_1\hat{a}_1^\dagger|k\rangle_1|n-k\rangle_2 \\
&= \sum_{n=0}^{\infty}P_n\frac{1}{2^n}\sum_{k=0}^{n}k\binom{n}{k} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}nP_n \\
&= \frac{\bar{n}}{2}.
\end{aligned}
$$

The same procedure would lead to

$$\mathrm{Tr}(\hat{\rho}_{\mathrm{th}}\hat{a}_2^\dagger\hat{a}_2) = \frac{\bar{n}}{2},$$

and

$$\mathrm{Tr}(\hat{\rho}_{\mathrm{th}}\hat{a}_1^\dagger\hat{a}_2) = \frac{\bar{n}}{2}.$$

Finally, we find that

$$I(\mathbf{r},t) = \bar{n}|f(\mathbf{r})|^2[1+\cos\Phi]. \tag{5.2.1}$$

5.3 Problem 5.3

For thermal light we have

$$
\begin{aligned}
G^{(1)}(x,x) &= \mathrm{Tr}\left(\hat{\rho}_{\mathrm{Th}}\hat{E}^{(-)}(x)\hat{E}^{(+)}(x)\right) \\
&= K^2\mathrm{Tr}\left(\hat{\rho}\hat{a}^\dagger\hat{a}\right) \\
&= K^2\bar{n},
\end{aligned}
$$

also

$$G^{(1)}(x_1, x_2) = K^2 \bar{n} e^{i[\mathbf{k}(\mathbf{r}_1 - \mathbf{r}_2) - \omega(t_2 - t_1)]}.$$

So we obtain $|g^{(1)}(x_1, x_2)| = 1$. Thus the thermal light is first-order coherent. Using Eq. (5.92)

$$g^{(2)}(\tau) = \frac{\langle \hat{n}(\hat{n} - 1) \rangle}{\langle \hat{n} \rangle^2}. \tag{5.3.1}$$

For a thermal state, the factorial moments have already been calculated in Eq. 2.10.2. So the second order coherence for the thermal state is

$$g^{(2)}(\tau) = \frac{2\bar{n}^2}{\bar{n}^2} = 2. \tag{5.3.2}$$

Clearly the thermal light is not second-order coherent. It is straightforward to show that thermal light is not higher-order coherent. Using Eq. (5.101) and Eq. (5.102) and the result of Eq. 2.10.2 we can show that

$$\left| g^{(n)}(x_1, \cdots x_n; x_n, \cdots x_1) \right| = n!.$$

5.4 Problem 5.4

$$|\Psi\rangle = C_0|0\rangle + C_1|1\rangle.$$

$$G^{(1)}(x_1, x_2) = \langle \Psi | \hat{E}^{(-)}(x_1) \hat{E}^{(+)}(x_2) | \Psi \rangle,$$

where

$$\hat{E}^{(+)}(x) = iK\hat{a}e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}.$$

$$\begin{aligned} \hat{E}^{(+)}(x)|\Psi\rangle &= iK\hat{a}e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}|\Psi\rangle \\ &= iKC_1 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}|0\rangle \end{aligned}$$

$$\begin{aligned} G^{(1)}(x_1, x_2) &= \langle \Psi | \hat{E}^{(-)}(x_1) \hat{E}^{(+)}(x_2) | \Psi \rangle \\ &= |C_1|^2 K^2 e^{i[\mathbf{k}\cdot(\mathbf{r}_2 - \mathbf{r}_1) - \omega(t_2 - t_1)]}. \end{aligned}$$

Also

$$G^{(1)}(x, x) = |C_1|^2 K^2.$$

$$
\begin{aligned}
g^{(1)}(x_1,x_2) &= \frac{G^{(1)}(x_1,x_2)}{\sqrt{G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)}} \\
&= e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]}.
\end{aligned}
$$

Clearly

$$
\left|g^{(1)}(x_1,x_2)\right| = 1.
$$

Since

$$
\hat{E}^{(+)}(x_2)\hat{E}^{(+)}(x_1)|\Psi\rangle = 0,
$$

we have

$$
G^{(2)}(x_1,x_2;x_2,x_1) = \langle\Psi|\hat{E}^{(-)}(x_1)\hat{E}^{(-)}(x_2)\hat{E}^{(+)}(x_2)\hat{E}^{(+)}(x_1)|\Psi\rangle = 0.
$$

So the second order coherence function vanishes for $|\Psi\rangle$.

On the other hand, we can study the statistical mixture of the vacuum and one photon number state,

$$
\hat{\rho} = |C_0|^2|0\rangle\langle 0| + |C_1|^2|1\rangle\langle 1|.
$$

$$
\begin{aligned}
G^{(1)}(x_1,x_2) &= \mathrm{Tr}\left\{\hat{\rho}\hat{E}^{(-)}(x_1)\hat{E}^{(+)}(x_2)\right\} \\
&= K^2 e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]}\mathrm{Tr}\left(\hat{\rho}\hat{a}^\dagger\hat{a}\right) \\
&= |C_1|^2 K^2 e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]}.
\end{aligned}
$$

$$
\begin{aligned}
g^{(1)}(x_1,x_2) &= \frac{G^{(1)}(x_1,x_2)}{\sqrt{G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)}} \\
&= e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]}.
\end{aligned}
$$

Since

$$
\mathrm{Tr}\left\{\hat{\rho}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\right\} = 0,
$$

we have $G^{(2)} = 0$.

5.5 Problem 5.5

$$
|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\alpha\rangle + |-\alpha\rangle\right).
$$

$$G^{(1)}(x_1, x_2) = \langle\Psi|\hat{E}^{(-)}(x_1)\hat{E}^{(+)}(x_2)|\Psi\rangle,$$

where

$$\hat{E}^{(+)}(x) = iK\hat{a}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}.$$

$$\begin{aligned}\hat{E}^{(+)}(x)|\Psi\rangle &= iK\hat{a}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\frac{1}{\sqrt{2}}\left(|\alpha\rangle + |-\alpha\rangle\right) \\ &= iKe^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\frac{\alpha}{\sqrt{2}}\left(|\alpha\rangle - |-\alpha\rangle\right)\end{aligned}$$

$$\begin{aligned}G^{(1)}(x_1, x_2) &= \langle\Psi|\hat{E}^{(-)}(x_1)\hat{E}^{(+)}(x_2)|\Psi\rangle \\ &= |\alpha|^2K^2e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]},\end{aligned}$$

where we have used $\langle\alpha|-\alpha\rangle = 0$ for large α.

Also

$$G^{(1)}(x, x) = |\alpha|^2K^2.$$

$$\begin{aligned}g^{(1)}(x_1, x_2) &= \frac{G^{(1)}(x_1, x_2)}{\sqrt{G^{(1)}(x_1, x_1)G^{(1)}(x_2, x_2)}} \\ &= e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]}.\end{aligned}$$

Clearly

$$\left|g^{(1)}(x_1, x_2)\right| = 1.$$

$$\hat{E}^{(+)}(x_2)\hat{E}^{(+)}(x_1)|\Psi\rangle = -K^2e^{i[\mathbf{k}\cdot(\mathbf{r}_2+\mathbf{r}_1)-\omega(t_2+t_1)]}\hat{a}^2\frac{1}{\sqrt{2}}\left(|\alpha\rangle + |-\alpha\rangle\right),$$

we have

$$\begin{aligned}G^{(2)}(x_1, x_2; x_2, x_1) &= \langle\Psi|\hat{E}^{(-)}(x_1)\hat{E}^{(-)}(x_2)\hat{E}^{(+)}(x_2)\hat{E}^{(+)}(x_1)|\Psi\rangle \\ &= K^4|\alpha|^4.\end{aligned}$$

So the second order coherence function for $|\Psi\rangle$ is

$$g^{(2)} = \frac{\alpha^4}{|\alpha|^4}.$$

On the other hand, we can study the following statistical mixture,

$$\hat{\rho} = \frac{1}{2}\left(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|\right).$$

$$\begin{aligned}
G^{(1)}(x_1,x_2) &= \mathrm{Tr}\left\{\hat{\rho}\hat{E}^{(-)}(x_1)\hat{E}^{(+)}(x_2)\right\} \\
&= K^2 e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]}\mathrm{Tr}\left(\hat{\rho}\hat{a}^\dagger\hat{a}\right) \\
&= K^2 e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]}\mathrm{Tr}\left(\hat{a}\hat{\rho}\hat{a}^\dagger\right) \\
&= K^2 e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]}|\alpha|^2\mathrm{Tr}\left(\hat{\rho}\right) \\
&= |\alpha|^2 K^2 e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]}
\end{aligned}$$

$$G^{(1)}(x,x) = |\alpha|^2 K^2$$

$$\begin{aligned}
g^{(1)}(x_1,x_2) &= \frac{G^{(1)}(x_1,x_2)}{\sqrt{G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2)}} \\
&= e^{i[\mathbf{k}\cdot(\mathbf{r}_2-\mathbf{r}_1)-\omega(t_2-t_1)]}.
\end{aligned}$$

$$\left|g^{(1)}(x_1,x_2)\right| = 1.$$

$$\begin{aligned}
G^{(2)}(x_1,x_2) &= \mathrm{Tr}\left\{\hat{\rho}\hat{E}^{(-)}(x_1)\hat{E}^{(-)}(x_2)\hat{E}^{(+)}(x_2)\hat{E}^{(+)}(x_1)\right\} \\
&= K^4\mathrm{Tr}\left(\hat{\rho}\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}\right) \\
&= K^4\mathrm{Tr}\left(\hat{a}\hat{a}\hat{\rho}\hat{a}^\dagger\hat{a}^\dagger\right) \\
&= |\alpha|^4 K^4\mathrm{Tr}\left(\hat{\rho}\right) \\
&= |\alpha|^4 K^4
\end{aligned}$$

$$g^{(2)} = 1.$$

Chapter 6

Interferometry

6.1 Problem 6.1

$$\hat{U}^\dagger \hat{a}_0^\dagger \hat{U} = e^{-i\frac{\pi}{2}\hat{J}_1}\hat{a}_0 e^{i\frac{\pi}{2}\hat{J}_1} \tag{6.1.1}$$

Using the operator identity

$$e^{\xi\hat{A}}\hat{B}e^{-\xi\hat{A}} = \hat{B} + \xi[\hat{A},\hat{B}] + \frac{\xi^2}{2!}[\hat{A},[\hat{A},\hat{B}]] + ..., \tag{6.1.2}$$

and equation6.1.1 we'll have

$$\hat{U}^\dagger \hat{a}_0^\dagger \hat{U} = \hat{a}_0 - i\frac{\pi}{2}[\hat{J}_1,\hat{a}_0] + \frac{(-i\frac{\pi}{2})^2}{2!}[\hat{J}_1,[\hat{J}_1,\hat{a}_0]] + ... \tag{6.1.3}$$

It is easy to see that

$$[\hat{J}_1,\hat{a}_0] = -\frac{1}{2}\hat{a}_1 \tag{6.1.4}$$

and

$$[\hat{J}_1,\hat{a}_1] = -\frac{1}{2}\hat{a}_0. \tag{6.1.5}$$

Equation 6.1.3 now reads

$$\hat{U}^\dagger \hat{a}_0^\dagger \hat{U} = \cos\frac{\pi}{4}\hat{a}_0^\dagger + i\sin\frac{\pi}{4}\hat{a}_0^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_0^\dagger + i\hat{a}_1^\dagger) \tag{6.1.6}$$

The same procedure would lead us to

$$\hat{U}^\dagger \hat{a}_1^\dagger \hat{U} = \frac{1}{\sqrt{2}}(i\hat{a}_0^\dagger + \hat{a}_1^\dagger). \tag{6.1.7}$$

6.2 Problem 6.2

If we replace θ instead of $\frac{\pi}{2}$ in the equation 6.1.3 of the previous problem we will get

$$\hat{U}^{\dagger}\hat{a}_0^{\dagger}\hat{U} = \cos\frac{\theta}{2}\hat{a}_0^{\dagger} + i\sin\frac{\theta}{2}\hat{a}_0^{\dagger} \tag{6.2.1}$$

$$\hat{U}^{\dagger}\hat{a}_1^{\dagger}\hat{U} = \cos\frac{\theta}{2}i\hat{a}_0^{\dagger} + \sin\frac{\theta}{2}\hat{a}_1^{\dagger}. \tag{6.2.2}$$

From the previous equations, it is easy to identify the parameters r, t, r', and t' as: $r = \cos\frac{\theta}{2}$ t.

6.3 Problem 6.3

Again we repeat the procedure that we have used to solve problem 6.1

$$\begin{aligned}
\hat{a}_2 &= \hat{U}(\theta)\hat{a}_0\hat{U}^{\dagger}(\theta) \\
&= e^{i\theta\hat{J}_2}\hat{a}_0 e^{-i\theta\hat{J}_2} \\
&= \hat{a}_0 + i\theta\left[\hat{J}_2, \hat{a}_0\right] + \frac{(i\theta)^2}{2!}\left[\hat{J}_2, \left[\hat{J}_2, \hat{a}_0\right]\right] + \ldots \\
&= \hat{a}_0 - \frac{\theta}{2}\hat{a}_1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2\hat{a}_0 + \ldots \\
&= \cos\left(\frac{\theta}{2}\right)\hat{a}_0 - \sin\left(\frac{\theta}{2}\right)\hat{a}_1,
\end{aligned}$$

where we have used the identity of problem 2.3 and the usual Bosonic commutation rules.

$$\begin{aligned}
\hat{a}_3 &= \hat{U}(\theta)\hat{a}_1\hat{U}^{\dagger}(\theta) \\
&= e^{i\theta\hat{J}_2}\hat{a}_1 e^{-i\theta\hat{J}_2} \\
&= \hat{a}_0 + i\theta\left[\hat{J}_2, \hat{a}_1\right] + \frac{(i\theta)^2}{2!}\left[\hat{J}_2, \left[\hat{J}_2, \hat{a}_1\right]\right] + \ldots \\
&= \hat{a}_1 + \frac{\theta}{2}\hat{a}_1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2\hat{a}_0 + \ldots \\
&= \cos\left(\frac{\theta}{2}\right)\hat{a}_0 + \sin\left(\frac{\theta}{2}\right)\hat{a}_1.
\end{aligned}$$

Also,

$$\hat{a}_2^\dagger = \sin\left(\frac{\theta}{2}\right)\hat{a}_0^\dagger - \cos\left(\frac{\theta}{2}\right)\hat{a}_1^\dagger$$

and

$$\hat{a}_3^\dagger = \cos\left(\frac{\theta}{2}\right)\hat{a}_0^\dagger + \sin\left(\frac{\theta}{2}\right)\hat{a}_1^\dagger.$$

For the case of 50:50 beam splitter

$$\begin{aligned}
\hat{a}_2 &= \frac{1}{\sqrt{2}}\left(\hat{a}_0 - \hat{a}_1\right), \\
\hat{a}_3 &= \frac{1}{\sqrt{2}}\left(\hat{a}_0 + \hat{a}_1\right), \\
\hat{a}_2^\dagger &= \frac{1}{\sqrt{2}}\left(\hat{a}_0^\dagger - \hat{a}_1^\dagger\right), \\
\text{and} & \\
\hat{a}_3^\dagger &= \frac{1}{\sqrt{2}}\left(\hat{a}_0^\dagger + \hat{a}_1^\dagger\right).
\end{aligned}$$

6.4 Problem 6.4

It is straightforward to carry out the computations using the explicit formulae of the J's operators.

$$\hat{J}_1 = \frac{1}{2}(\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_0\hat{a}_1^\dagger) \qquad \hat{J}_2 = \frac{1}{2i}(\hat{a}_0^\dagger\hat{a}_2 - \hat{a}_0\hat{a}_1^\dagger)$$

$$\hat{J}_3 = \frac{1}{2}(\hat{a}_0^\dagger\hat{a}_0 - \hat{a}_1^\dagger\hat{a}_1) \qquad \hat{J}_0 = \frac{1}{2}(\hat{a}_0^\dagger\hat{a}_0 + \hat{a}_1^\dagger\hat{a}_1)$$

$$
\begin{aligned}
\left[\hat{J}_1, \hat{J}_2\right] &= \frac{1}{4i}[\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_0 \hat{a}_1^\dagger, \hat{a}_0^\dagger \hat{a}_2 - \hat{a}_0 \hat{a}_1^\dagger] \\
&= \frac{1}{4i}\left\{[\hat{a}_0 \hat{a}_1^\dagger, \hat{a}_0^\dagger \hat{a}_1] - [\hat{a}_0^\dagger \hat{a}_1, \hat{a}_0 \hat{a}_1^\dagger]\right\} \\
&= \frac{1}{2i}[\hat{a}_0 \hat{a}_1^\dagger, \hat{a}_0^\dagger \hat{a}_1] \\
&= \frac{1}{2i}\left\{\hat{a}_0[\hat{a}_1^\dagger, \hat{a}_0^\dagger \hat{a}_1] + [\hat{a}, \hat{a}_0^\dagger \hat{a}_1]\hat{a}_1^\dagger\right\} \\
&= \frac{1}{2i}(-\hat{a}_0 \hat{a}_0^\dagger + \hat{a}_1 \hat{a}_1^\dagger) \\
&= \frac{1}{2i}(-\hat{a}_0^\dagger \hat{a}_0 + \hat{a}_1^\dagger \hat{a}_1) \\
&= i\frac{1}{2}(\hat{a}_0^\dagger \hat{a}_0 - \hat{a}_1^\dagger \hat{a}_1) \\
&= i\hat{J}_3
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{J}_1, \hat{J}_3\right] &= \frac{1}{4}[\hat{a}_0^\dagger \hat{a}_1 + \hat{a}_0 \hat{a}_1^\dagger, \hat{a}_0^\dagger \hat{a}_0 - \hat{a}_1^\dagger \hat{a}_1] \\
&= \frac{1}{4}\left\{[\hat{a}_0^\dagger \hat{a}_1, \hat{a}_0^\dagger \hat{a}_0] - [\hat{a}_0^\dagger \hat{a}_1, \hat{a}_1^\dagger \hat{a}_1] + [\hat{a}_0 \hat{a}_1^\dagger, \hat{a}_0^\dagger \hat{a}_0] - [\hat{a}_0 \hat{a}_1^\dagger, \hat{a}_1^\dagger \hat{a}_1]\right\} \\
&= \frac{1}{4}\left\{[\hat{a}_0^\dagger, \hat{a}_0^\dagger \hat{a}_0]\hat{a}_1 - \hat{a}_0^\dagger[\hat{a}_1, \hat{a}_1^\dagger \hat{a}_1] + [\hat{a}_0, \hat{a}_0^\dagger \hat{a}_0]\hat{a}_1^\dagger - \hat{a}_0[\hat{a}_1^\dagger, \hat{a}_1^\dagger \hat{a}_1]\right\} \\
&= \frac{1}{4}(-2\hat{a}_0^\dagger \hat{a}_1 + 2\hat{a}_0 \hat{a}_1^\dagger) \\
&= -i\frac{1}{2i}(\hat{a}_0^\dagger \hat{a}_1 - \hat{a}_0 \hat{a}_1^\dagger) \\
&= -i\hat{J}_2
\end{aligned}
$$

$$
\begin{aligned}
\left[\hat{J}_2, \hat{J}_3\right] &= \frac{1}{4i}[\hat{a}_0^\dagger \hat{a}_1 - \hat{a}_0 \hat{a}_1^\dagger, \hat{a}_0^\dagger \hat{a}_0 - \hat{a}_1^\dagger \hat{a}_1] \\
&= \frac{1}{4i}\left\{[\hat{a}_0^\dagger \hat{a}_1, \hat{a}_0^\dagger \hat{a}_0] - [\hat{a}_0^\dagger \hat{a}_1, \hat{a}_1^\dagger \hat{a}_1] - [\hat{a}_0 \hat{a}_1^\dagger, \hat{a}_0^\dagger \hat{a}_0] + [\hat{a}_0 \hat{a}_1^\dagger, \hat{a}_1^\dagger \hat{a}_1]\right\} \\
&= \frac{1}{4i}\left\{[\hat{a}_0^\dagger, \hat{a}_0^\dagger \hat{a}_0]\hat{a}_1 - \hat{a}_0^\dagger[\hat{a}_1, \hat{a}_1^\dagger \hat{a}_1] - [\hat{a}_0, \hat{a}_0^\dagger \hat{a}_0]\hat{a}_1^\dagger + \hat{a}_0[\hat{a}_1^\dagger, \hat{a}_1^\dagger \hat{a}_1]\right\} \\
&= \frac{1}{4i}(-2\hat{a}_0^\dagger \hat{a}_1 - 2\hat{a}_0 \hat{a}_1^\dagger) \\
&= -i\frac{1}{2i}(\hat{a}_0^\dagger \hat{a}_1 - \hat{a}_0 \hat{a}_1^\dagger) \\
&= i\hat{J}_1.
\end{aligned}
$$

Thus

$$\left[\hat{J}_i, \hat{J}_j\right] = i\varepsilon_{ijk}\hat{J}_k. \tag{6.4.1}$$

$$\begin{aligned}
\left[\hat{J}_1, \hat{J}_0\right] &= \frac{1}{4}[\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_0\hat{a}_1^\dagger, \hat{a}_0^\dagger\hat{a}_0 + \hat{a}_1^\dagger\hat{a}_1] \\
&= \frac{1}{4}\left\{[\hat{a}_0^\dagger\hat{a}_1, \hat{a}_0^\dagger\hat{a}_0] + [\hat{a}_0^\dagger\hat{a}_1, \hat{a}_1^\dagger\hat{a}_1] + [\hat{a}_0\hat{a}_1^\dagger, \hat{a}_0^\dagger\hat{a}_0] + [\hat{a}_0\hat{a}_1^\dagger, \hat{a}_1^\dagger\hat{a}_1]\right\} \\
&= \frac{1}{4}\left\{[\hat{a}_0^\dagger, \hat{a}_0^\dagger\hat{a}_0]\hat{a}_1 + \hat{a}_0^\dagger[\hat{a}_1, \hat{a}_1^\dagger\hat{a}_1] + [\hat{a}_0, \hat{a}_0^\dagger\hat{a}_0]\hat{a}_1^\dagger + \hat{a}_0[\hat{a}_1^\dagger, \hat{a}_1^\dagger\hat{a}_1]\right\} \\
&= \frac{1}{4}\left\{\hat{a}_0^\dagger[\hat{a}_0^\dagger, \hat{a}_0]\hat{a}_1 + \hat{a}_0^\dagger[\hat{a}_1, \hat{a}_1^\dagger]\hat{a}_1 + [\hat{a}_0, \hat{a}_0^\dagger]\hat{a}_0\hat{a}_1^\dagger + \hat{a}_0\hat{a}_1^\dagger[\hat{a}_1^\dagger, \hat{a}_1]\right\} \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\left[\hat{J}_2, \hat{J}_0\right] &= \frac{1}{4i}[\hat{a}_0^\dagger\hat{a}_1 - \hat{a}_0\hat{a}_1^\dagger, \hat{a}_0^\dagger\hat{a}_0 + \hat{a}_1^\dagger\hat{a}_1] \\
&= \frac{1}{4i}\left\{[\hat{a}_0^\dagger\hat{a}_1, \hat{a}_0^\dagger\hat{a}_0] + [\hat{a}_0^\dagger\hat{a}_1, \hat{a}_1^\dagger\hat{a}_1] - [\hat{a}_0\hat{a}_1^\dagger, \hat{a}_0^\dagger\hat{a}_0] - [\hat{a}_0\hat{a}_1^\dagger, \hat{a}_1^\dagger\hat{a}_1]\right\} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
\left[\hat{J}_3, \hat{J}_0\right] &= \frac{1}{4}[\hat{a}_0^\dagger\hat{a}_0 - \hat{a}_1^\dagger\hat{a}_1, \hat{a}_0^\dagger\hat{a}_0 + \hat{a}_1^\dagger\hat{a}_1] \\
&= 0.
\end{aligned}$$

In fact, $\hat{J}_0$ commutes with all $\hat{J}_i$ for $i = 1, 2, 3$.

6.5 Problem 6.5

First we have to rewrite the input state as, $|in\rangle$

$$|in\rangle = |0\rangle_0|N\rangle_1 = \frac{\hat{a}^{\dagger N}}{\sqrt{N!}}|0\rangle_0|0\rangle_1. \tag{6.5.1}$$

Using the fact that the J_1 type beam splitters do the following transformations

$$|0\rangle_0|0\rangle_1 \Rightarrow |0\rangle_2|0\rangle_3 \tag{6.5.2}$$

and

$$\hat{a}_1^\dagger \Rightarrow \left(i\sin(\theta/2)\hat{a}_2^\dagger + \cos(\theta/2)\hat{a}_3^\dagger\right) \tag{6.5.3}$$

we will get the following for

$$\frac{{\hat{a}_1}^{\dagger N}}{\sqrt{N!}}|0\rangle_0|0\rangle_1 \Rightarrow \frac{1}{\sqrt{N!}}[(i\sin(\theta/2)\hat{a}_2^\dagger + \cos(\theta/2)\hat{a}_3^\dagger)]^N|0\rangle_2|0\rangle_3. \tag{6.5.4}$$

$$= \frac{1}{\sqrt{N!}}\left[i\sin(\theta/2)\hat{a}_2^\dagger + \cos(\theta/2)\hat{a}_3^\dagger\right]^N |0\rangle_2|0\rangle_3$$

$$= \frac{1}{\sqrt{N!}}\sum_{k=0}^{N}\begin{pmatrix} N \\ k \end{pmatrix} i^k \sin^k(\theta/2)\cos^{N-k}(\theta/2)\hat{a}_2^{\dagger k}\hat{a}_3^{\dagger N-k}|0\rangle_2|0\rangle_3$$

$$= \frac{1}{\sqrt{N!}}\sum_{k=0}^{N}\begin{pmatrix} N \\ k \end{pmatrix} i^k \sin^k(\theta/2)\cos^{N-k}(\theta/2)\sqrt{k!(N-k)!}|k\rangle_2|N-k\rangle_3$$

$$= \sum_{k=0}^{N}\begin{pmatrix} N \\ k \end{pmatrix}^{\frac{1}{2}} i^k \sin^k(\theta/2)\cos^{N-k}(\theta/2)|k\rangle_2|N-k\rangle_3$$

$$= \cos^N(\theta/2)\sum_{k=0}^{N}\begin{pmatrix} N \\ k \end{pmatrix}^{\frac{1}{2}} i^k \tan^k(\theta/2)|k\rangle_2|N-k\rangle_3$$

$$= \left[1+\tan^2(\theta/2)\right]^{-N/2}\sum_{k=0}^{N}\begin{pmatrix} N \\ k \end{pmatrix}^{\frac{1}{2}} i^k \tan^k(\theta/2)|k\rangle_2|N-k\rangle_3$$

6.6 Problem 6.6

$$|in\rangle = |\alpha\rangle_0|\beta\rangle_1 \tag{6.6.1}$$

$$= \hat{D}(\alpha,\hat{a}_0)\hat{D}(\beta,\hat{a}_1)|0\rangle, \tag{6.6.2}$$

where $\hat{D}(\hat{a},\alpha)$ is defined as

$$\hat{D}(\hat{a},\alpha) = \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a}). \tag{6.6.3}$$

Let $\hat{U}$ be the unitary transformation associated with the beam splitter of type $\hat{J}_1$. Using the solution to the problem 6.1, we know that for a 50:50 beam splitter

$$\hat{U}\hat{a}_0\hat{U}^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_2 - i\hat{a}_3)\ ,\ \hat{U}\hat{a}_1\hat{U}^\dagger = \frac{1}{\sqrt{2}}(-i\hat{a}_2 + \hat{a}_3)$$

and

$$\hat{U}\hat{a}_0^\dagger\hat{U}^\dagger = \frac{1}{\sqrt{2}}(\hat{a}_2^\dagger + i\hat{a}_3^\dagger)\ ,\ \hat{U}\hat{a}_1^\dagger\hat{U}^\dagger = \frac{1}{\sqrt{2}}(i\hat{a}_2^\dagger + \hat{a}_3^\dagger),$$

thus

$$
\begin{aligned}
\hat{U}\hat{D}(\hat{a}_0,\alpha)\hat{U}^\dagger &= \hat{U}\exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})\hat{U}^\dagger \\
&= \exp\left(\alpha\frac{1}{\sqrt{2}}(\hat{a}_2^\dagger + i\hat{a}_3^\dagger) - \alpha^*\frac{1}{\sqrt{2}}(i\hat{a}_2^\dagger + \hat{a}_3^\dagger)\right) \\
&= \exp\left(\frac{1}{\sqrt{2}}(\alpha\hat{a}_2^\dagger - \alpha^*\hat{a}_3^\dagger)\right)\exp\left(\frac{1}{\sqrt{2}}(i\alpha\hat{a}_2^\dagger - (i\alpha)^*\hat{a}_3^\dagger)\right) \\
&= \hat{D}(\hat{a}_2, \frac{1}{\sqrt{2}}\alpha)\hat{D}(\hat{a}_3, \frac{i}{\sqrt{2}}\alpha^*).
\end{aligned}
$$

and the same way we can prove that

$$
\hat{U}\hat{D}(\hat{a}_0,\beta)\hat{U}^\dagger = \hat{D}(\hat{a}_2, \frac{i}{\sqrt{2}}\beta)\hat{D}(\hat{a}_3, \frac{1}{\sqrt{2}}\beta^*). \tag{6.6.4}
$$

With the input state in equation 6.6.2, the state after the beam splitter should be

$$
\begin{aligned}
|out\rangle &= \hat{U}|in\rangle \\
&= \hat{U}\hat{D}(\hat{a}_0,\alpha)\hat{D}(\hat{a}_1,\beta)|0\rangle \\
&= \hat{U}\hat{D}(\hat{a}_0,\alpha)\hat{U}^\dagger\hat{U}\hat{D}(\hat{a}_1,\beta)\hat{U}^\dagger\hat{U}|0\rangle \\
&= \hat{D}(\hat{a}_2, \frac{1}{\sqrt{2}}\alpha)\hat{D}(\hat{a}_3, \frac{i}{\sqrt{2}}\alpha^*)\hat{D}(\hat{a}_2, \frac{i}{\sqrt{2}}\beta)\hat{D}(\hat{a}_3, \frac{1}{\sqrt{2}}\beta^*)|0\rangle \\
&= \hat{D}(\hat{a}_2, \frac{\alpha + i\beta}{\sqrt{2}})\hat{D}(\hat{a}_3, \frac{i\alpha + \beta}{\sqrt{2}})|0\rangle \\
&= \left|\frac{\alpha + i\beta}{\sqrt{2}}\right\rangle_2 \left|\frac{i\alpha + \beta}{\sqrt{2}}\right\rangle_3
\end{aligned}
$$

6.7 Problem 6.7

$$
\begin{aligned}
|N\rangle_0|N\rangle_1 = \frac{\hat{a}_0{}^{\dagger N}\hat{a}_1{}^{\dagger N}}{N!}|0\rangle_0|0\rangle_1 \Rightarrow &\frac{1}{N!}\left[\frac{1}{\sqrt{2}}(i\hat{a}_2^\dagger + \hat{a}_3^\dagger)\right]^N\left[\frac{1}{\sqrt{2}}(\hat{a}_2^\dagger - i\hat{a}_3^\dagger)\right]^N|0\rangle_2|0\rangle_3 \\
&= \frac{1}{N!2^N}(\hat{a}_2^\dagger + i\hat{a}_3^\dagger)^N(i\hat{a}_2^\dagger + \hat{a}_3^\dagger)^N|0\rangle_2|0\rangle_3 \\
&= \frac{i^N}{N!2^N}(\hat{a}_2^\dagger + i\hat{a}_3^\dagger)^N(\hat{a}_2^\dagger - i\hat{a}_3^\dagger)^N|0\rangle_2|0\rangle_3 \\
&= \frac{i^N}{N!2^N}\left[(\hat{a}_2^\dagger)^2 - (\hat{a}_3^\dagger)^2\right]^N|0\rangle_2|0\rangle_3.
\end{aligned}
$$

It is clear from the last equation that photon are created in pairs, so there will be no odd-numbered photon states in either of the output states.

6.8 Problem 6.8

Using the same technique used in the previous problem we write

$$
\begin{aligned}
|N\rangle_0|N\rangle_1 &= \frac{\hat{a_0}^{\dagger N}\hat{a_1}^{\dagger N}}{N!}|0\rangle_0|0\rangle_1 \\
&\Rightarrow \frac{1}{N!}\left[\frac{1}{\sqrt{2}}(i\hat{a}_2^\dagger + \hat{a}_3^\dagger)\right]^N \left[\frac{1}{\sqrt{2}}(\hat{a}_2^\dagger - i\hat{a}_3^\dagger)\right]^N |0\rangle_2|0\rangle_3 \\
&= \frac{1}{N!2^N}(\hat{a}_2^\dagger + i\hat{a}_3^\dagger)^N(i\hat{a}_2^\dagger + \hat{a}_3^\dagger)^N|0\rangle_2|0\rangle_3 \\
&= \frac{i^N}{N!2^N}(\hat{a}_2^\dagger + i\hat{a}_3^\dagger)^N(\hat{a}_2^\dagger - i\hat{a}_3^\dagger)^N|0\rangle_2|0\rangle_3 \\
&= \frac{i^N}{N!2^N}\left((\hat{a}_2^\dagger)^2 - (\hat{a}_3^\dagger)^2\right)^N|0\rangle_2|0\rangle_3 \\
&= \frac{i^N}{N!2^N}\sum_{k=0}^{N}\left(\begin{array}{c} N \\ k \end{array}\right)(\hat{a}_2^\dagger)^{2k}(\hat{a}_3^\dagger)^{2(N-k)}|0\rangle_2|0\rangle_3 \\
&= \frac{i^N}{N!2^N}\sum_{k=0}^{N}\frac{N!}{k!(N-k)!}\sqrt{2k!}\sqrt{2(N-k)!}|2k\rangle_2|2N-2k\rangle_3 \\
&= \frac{i^N}{2^N}\sum_{k=0}^{N}\sqrt{\frac{2k!}{k!k!}}\sqrt{\frac{2(N-k)!}{(N-k)!(N-k)!}}|2k\rangle_2|2N-2k\rangle_3 \\
&= i^N\sum_{k=0}^{N}\left[\left(\frac{1}{2}\right)^{2N}\left(\begin{array}{c} 2k \\ k \end{array}\right)\left(\begin{array}{c} 2N-2k \\ N-k \end{array}\right)\right]^{1/2}|2k\rangle_2|2N-2k\rangle_3
\end{aligned}
$$

6.9 Problem 6.9

Using the result of the problem 6.6 we have

$$
\begin{aligned}
|0\rangle_0|\alpha\rangle_1 &\Rightarrow \left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2\left|\frac{\alpha}{\sqrt{2}}\right\rangle_3 \\
|0\rangle_0\,|-\alpha\rangle_1 &\Rightarrow \left|\frac{-i\alpha}{\sqrt{2}}\right\rangle_2\left|\frac{-\alpha}{\sqrt{2}}\right\rangle_3,
\end{aligned}
$$

so for $|0\rangle_0(|\alpha\rangle_1 + |-\alpha\rangle_1)/\sqrt{2}$ an input state the output state would be

$$\frac{1}{\sqrt{2}}\left(\left|\frac{i\alpha}{\sqrt{2}}\right\rangle_2\left|\frac{\alpha}{\sqrt{2}}\right\rangle_3 + \left|\frac{-i\alpha}{\sqrt{2}}\right\rangle_2\left|\frac{-\alpha}{\sqrt{2}}\right\rangle_3\right) \tag{6.9.1}$$

For large α, we have

$$\langle -\alpha|\alpha\rangle = 0. \tag{6.9.2}$$

In other words, $|\alpha\rangle$ and $|-\alpha\rangle$ are orthogonal states. Thus, state in equation 6.9.1 is a Bell state, so it is entangled.

6.10 Problem 6.10

$$\begin{aligned}
|in\rangle &= \frac{1}{\sqrt{2}}|0\rangle\left[|\alpha\rangle + |-\alpha\rangle\right] \\
\hat{U}_{BS1}|in\rangle &= \frac{1}{\sqrt{2}}\left(\left|i\frac{\alpha}{\sqrt{2}}\right\rangle\left|\frac{\alpha}{\sqrt{2}}\right\rangle + \left|-i\frac{\alpha}{\sqrt{2}}\right\rangle\left|-\frac{\alpha}{\sqrt{2}}\right\rangle\right) \\
\hat{U}_{PS}\hat{U}_{BS1}|in\rangle &= \frac{1}{\sqrt{2}}\left(\left|i\frac{\alpha}{\sqrt{2}}\right\rangle\left|\frac{\alpha e^{i\theta}}{\sqrt{2}}\right\rangle + \left|-i\frac{\alpha}{\sqrt{2}}\right\rangle\left|-\frac{\alpha e^{i\theta}}{\sqrt{2}}\right\rangle\right) \\
|out\rangle &= \hat{U}_{BS2}\hat{U}_{PS}\hat{U}_{BS1}|in\rangle \\
&= \frac{1}{\sqrt{2}}\left(\left|i\frac{\alpha(1+e^{i\theta})}{2}\right\rangle\left|\frac{-\alpha(1-e^{i\theta})}{2}\right\rangle + \left|-i\frac{\alpha(1+e^{i\theta})}{2}\right\rangle\left|\frac{\alpha(1-e^{i\theta})}{2}\right\rangle\right)
\end{aligned}$$

Taking into account $|\alpha|$ very large, we have $\langle\alpha|-\alpha\rangle = 0$.

$$\begin{aligned}
\langle out|\hat{a}^\dagger\hat{a}|out\rangle &= \frac{1}{2}\left(\frac{|\alpha|^2}{2}|1+e^{i\theta}|^2\right) \\
&= \frac{|\alpha|^2}{2}(1+\cos\theta)
\end{aligned}$$

and

$$\begin{aligned}
\langle out|\hat{b}^\dagger\hat{b}|out\rangle &= \frac{1}{2}\left(\frac{|\alpha|^2}{2}|1-e^{i\theta}|^2\right) \\
&= \frac{|\alpha|^2}{2}(1-\cos\theta).
\end{aligned}$$

$$\begin{aligned}
\langle\hat{O}\rangle &= \left\langle \hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\right\rangle \\
&= |\alpha|^2\cos\theta
\end{aligned}$$

$$\begin{aligned}\hat{O}^2 &= \left(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\right)^2 \\ &= \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}\hat{b}^\dagger\hat{b} - 2\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b} \\ &= \hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b} + \hat{b}^\dagger\hat{b} - 2\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\end{aligned}$$

$$\begin{aligned}\langle out|\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a}|out\rangle &= \frac{|\alpha|^4}{16}\left|1+e^{i\theta}\right|^4 \\ &= \frac{|\alpha|^4}{4}\left(1+\cos^2\theta + 2\cos\theta\right)\end{aligned}$$

$$\begin{aligned}\langle out|\hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b}|out\rangle &= \frac{|\alpha|^4}{16}\left|1-e^{i\theta}\right|^4 \\ &= \frac{|\alpha|^4}{4}\left(1+\cos^2\theta - 2\cos\theta\right)\end{aligned}$$

$$\begin{aligned}\langle out|\hat{a}^\dagger\hat{a}^\dagger\hat{b}\hat{b}|out\rangle &= \frac{|\alpha|^4}{16}\left|1-e^{i\theta}\right|^2\left|1+e^{i\theta}\right|^2 \\ &= \frac{|\alpha|^4}{4}\left(1-\cos^2\theta\right)\end{aligned}$$

$$\left\langle \hat{O}^2 \right\rangle = |\alpha|^2$$

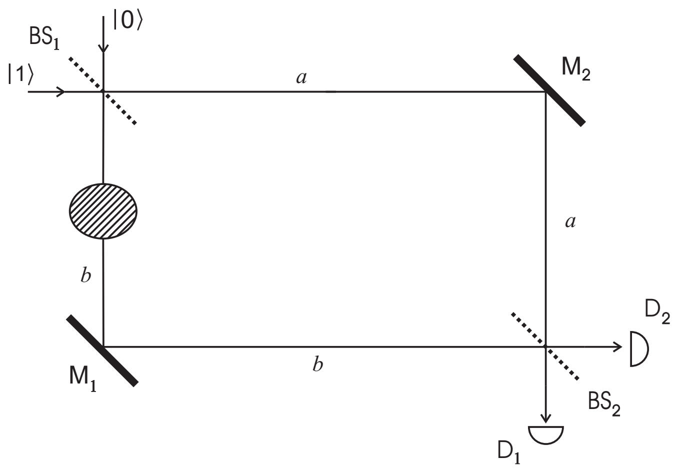
$$\begin{aligned}\Delta\theta &= \frac{\Delta O}{\left|\partial\left\langle\hat{O}\right\rangle/\partial\theta\right|} \\ &= \frac{1}{\sqrt{|\alpha|^2}|\sin\theta|}\end{aligned}$$

for $\theta \to \pi/2$ and large α we have

$$\Delta\theta = \frac{1}{\sqrt{|\alpha|^2}}.$$

It is exactly the standard quantum limit.

6.11 Problem 6.11



First, let assume that the transformations associated with beam splitters BS1 and BS2 can be described by $\hat{U}_{\text{BS1}} = e^{i\theta\hat{J}_1}$ and $\hat{U}_{\text{BS1}} = e^{i\theta'\hat{J}_1}$, respectively. We are faced with two possibilities in this situation: Either there is an object in the arm b or there is not, see figure above. In the former case the probability that the photon goes through arm a is $\cos^2(\theta/2)$ and the probability that detector D_1 clicks is $P_1(\theta, \theta') = \cos^2(\theta/2)\cos^2(\theta'/2)$. The second possibility, when there is no object, we have

$$\begin{aligned}
|in\rangle &= |1\rangle_a|0\rangle_b \\
|out\rangle &= \hat{U}_{\text{BS2}}\hat{U}_{\text{BS1}}|in\rangle \\
&= \hat{U}_{\text{BS2}}\hat{U}_{\text{BS1}}|1\rangle_a|0\rangle_b \\
&= \hat{U}_{\text{BS2}}(\cos(\theta/2)|1\rangle_a|0\rangle_b + i\cos(\theta'/2)|0\rangle_a|1\rangle_b) \\
&= \cos\left(\frac{\theta+\theta'}{2}\right)|1\rangle_a|0\rangle_b + i\sin\left(\frac{\theta+\theta'}{2}\right)|0\rangle_a|1\rangle_b
\end{aligned}$$

This time the probability that detector D_1 clicks is $P_2(\theta, \theta') = \cos^2\left(\frac{\theta+\theta'}{2}\right)$. An efficient detection would make $|P_1(\theta, \theta') - P_2(\theta, \theta')|$ a maximum. In fact, $P_1(\theta, \theta') - P_2(\theta, \theta') = 1$ for $\theta = \theta' = \pi$.

Chapter 7

Nonclassical Light

7.1 Problem 7.1

The general squeezed state of Eq. (7.80) is

$$|\alpha,\xi\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

$$c_n = \frac{1}{\sqrt{\cosh r}} \exp\left[-\frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^{*2} e^{i\theta}\tanh r\right] \frac{\left[\frac{1}{2}e^{i\theta}\tanh r\right]^{n/2}}{\sqrt{n!}} H_n\left[\gamma\left(e^{i\theta}\tanh 2r\right)^{-1/2}\right],$$

where $\gamma = \alpha\cosh r + \alpha^* e^{i\theta}\sinh r$ and H_n is the Hermite polynomials.

For $\alpha = 0$ we get the squeezed vacuum state. Ignoring the ZPE, the time evolving state vector is

$$|\alpha,\xi,t\rangle = \sum_{n=0}^{\infty} c_n e^{-i\omega n t}|n\rangle$$

and the wave packet is given by

$$\langle q|\alpha,\xi,t\rangle = \sum_{n=0}^{\infty} c_n e^{-i\omega n t}\langle q|n\rangle,$$

where

$$\begin{aligned}\langle q|n\rangle &= \psi_n(q)\\ &= (2^n n!)^{-1/2}\left(\frac{\omega}{\pi\hbar}\right)^{1/4} e^{-\xi^2/2} H_n(\xi),\end{aligned}$$

where $\xi = q\sqrt{\frac{\omega}{\hbar}}$. The evolution of the wave packet is given by the probability density

$$P(q,t) = |\langle q|\alpha,\xi,t\rangle|^2 .$$

For the case where $\alpha = 0$, we get the squeezed vacuum

$$|\xi\rangle = \sum_{m=0}^{\infty} B_{2m}|2m\rangle,$$

where

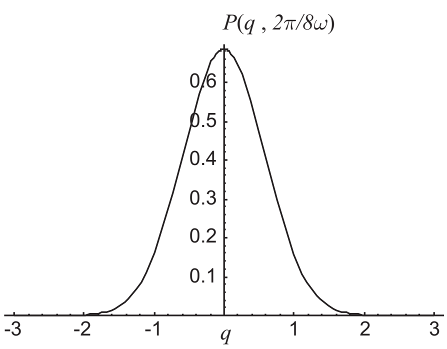
$$B_{2m} = \frac{1}{\sqrt{\cosh r}}(-1)^m \frac{\sqrt{(2m)!}}{2^m m!} e^{im\theta}(\tanh r)^m.$$

In time

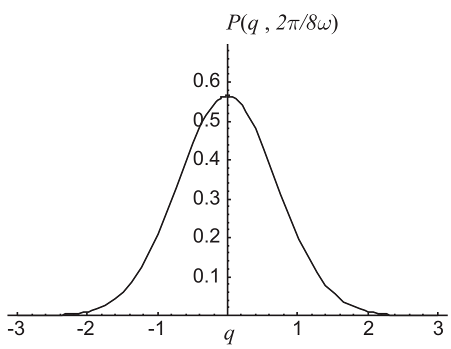
$$|\xi,t\rangle = \sum_{m=0}^{\infty} B_{2m} e^{-i2\omega m t}|2m\rangle.$$

Below, we have plotted $P(q,t)$ keeping $r = 0.2$ for different time. It is obvious from these graphs the centroid is stationary, but the width oscillates at twice the frequency of the harmonic oscillator.

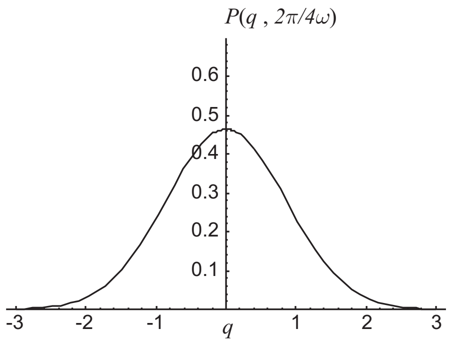
(a)



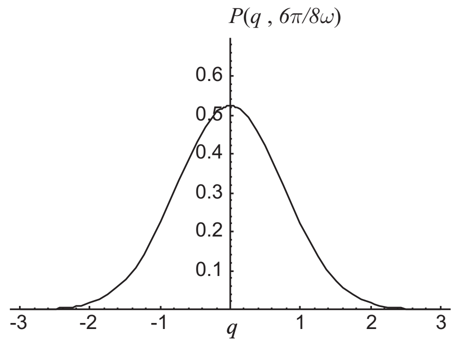
(b)



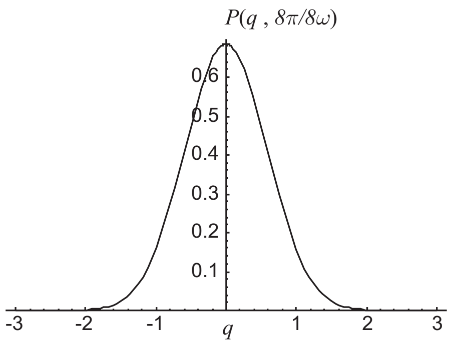
(c)


P(q , 2π/4ω)
0.6
0.5
0.4
0.3
0.2
0.1
-3
-2
-1
q
1
2
3

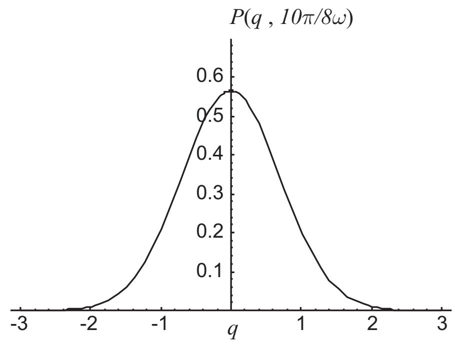
(d)


P(q , 6π/8ω)
0.6
0.5
0.4
0.3
0.2
0.1
-3
-2
-1
q
1
2
3

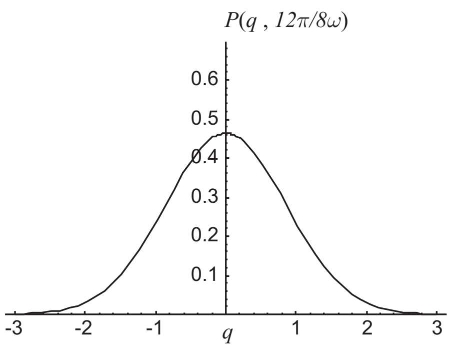
(e)


P(q , 8π/8ω)
0.6
0.5
0.4
0.3
0.2
0.1
-3
-2
-1
q
1
2
3

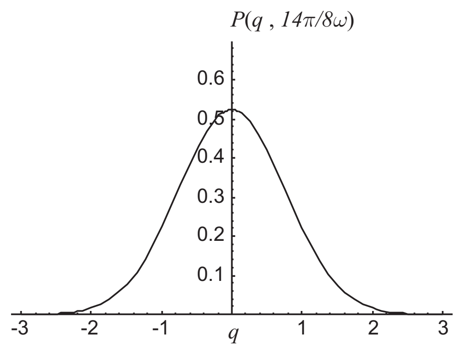
(f)


P(q , 10π/8ω)
0.6
0.5
0.4
0.3
0.2
0.1
-3
-2
-1
q
1
2
3

(g)


P(q , 12π/8ω)
0.6
0.5
0.4
0.3
0.2
0.1
-3
-2
-1
q
1
2
3

(h)


P(q , 14π/8ω)
0.6
0.5
0.4
0.3
0.2
0.1
-3
-2
-1
q
1
2
3

7.2 Problem 7.2

For vacuum squeezed state $\alpha = 0$ and $\theta = 0$

$$Q(\beta) = \frac{\exp\left\{-|\beta|^2 - \frac{\tanh r}{2}\left(\beta^{*2} + \beta^2\right)\right\}}{\pi \cosh r}$$

$$\begin{aligned}
C_A(\lambda) &= \int d^2\alpha Q(\alpha) e^{\lambda\alpha^* - \lambda^*\alpha} \\
&= \int d^2\alpha \frac{\exp\left[-|\alpha|^2 - \frac{\tanh r}{2}\left(\alpha^{*2} + \alpha^2\right)\right]}{\pi \cosh r} e^{\lambda\alpha^* - \lambda^*\alpha} \\
&= \frac{1}{\pi \cosh r} \int dx dy e^{-x^2 - y^2 - \frac{1}{2}\tanh r (x^2 - y^2) + (\lambda - \lambda^*)x - iy(\lambda + \lambda^*)} \\
&= \frac{1}{\pi \cosh r} \int dx \exp\left[-x^2(\tanh r + 1) + x(\lambda - \lambda^*))\right] \\
&\times \int dy \left[-y^2(1 - \tanh r) + -iy(\lambda + \lambda^*))\right] \\
&= \frac{1}{\pi \cosh r} \sqrt{\frac{\pi}{1 + \tanh r}} e^{\frac{(\lambda - \lambda^*)^2}{4(1 + \tanh r)}} \sqrt{\frac{\pi}{\tanh r - 1}} e^{\frac{(\lambda + \lambda^*)^2}{4(\tanh r - 1)}} \\
&= \frac{\exp\left[\frac{1}{4}\frac{\left((1 - \tanh r)(\lambda - \lambda^*)^2 - (1 + \tanh r)((\lambda + \lambda^*)^2)\right)}{1 - \tanh^2 r}\right]}{\cosh r \sqrt{(1 - \tanh^2 r)}} \\
&= \exp\left[\frac{1}{4}\cosh^2 r \left((1 - \tanh r)(\lambda - \lambda^*)^2 - (1 + \tanh r)((\lambda + \lambda^*)^2)\right)\right] \\
&= \exp[-\frac{1}{2}\cosh r \sinh r(\lambda^2 + \lambda^{*2}) - \cosh^2 r|\lambda|^2]
\end{aligned}$$

$$\begin{aligned}
C_N(\lambda) &= C_A(\lambda) e^{|\lambda|^2} \\
&= \exp[-\frac{1}{2}\cosh r \sinh r(\lambda^2 + \lambda^{*2}) - (\cosh^2 r - 1)|\lambda|^2] \\
&= \exp[-\frac{1}{2}\cosh r \sinh r(\lambda^2 + \lambda^{*2}) - (\sinh^2 r)|\lambda|^2]
\end{aligned}$$

$$
\begin{aligned}
W(\alpha) &= \frac{1}{\pi^2}\int d^2\lambda C_N(\lambda)\exp(\lambda^*\alpha-\lambda\alpha^*)e^{-|\lambda|^2/2} \\
&= \frac{1}{\pi^2}\int d^2\lambda \exp[\lambda^*\alpha-\lambda\alpha^*-\frac{1}{2}\cosh r\sinh r(\lambda^2+\lambda^{*2})-(\frac{1}{2}+\sinh^2 r)|\lambda|^2] \\
&= \frac{1}{\pi^2}\int d^2\lambda \exp[\lambda^*\alpha-\lambda\alpha^*-\frac{1}{4}\sinh(2r)(\lambda^2+\lambda^{*2})-\frac{1}{2}\cosh(2r)|\lambda|^2]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{\pi^2}\int dx\exp[-\frac{1}{2}\left(\sinh(2r)+\cosh(2r)\right)x^2+(\alpha-\alpha^*)x] \\
&\times\int dy\exp[-\frac{1}{2}\left(\sinh(2r)-\cosh(2r)\right)y^2-i(\alpha+\alpha^*)x] \\
&= \frac{1}{\pi^2}\sqrt{\frac{\pi}{\frac{1}{2}\left(\sinh(2r)-\cosh(2r)\right)}}\exp\left[\frac{(\alpha-\alpha^*)^2}{4\left(\sinh(2r)+\cosh(2r)\right)}\right] \\
&\times\sqrt{\frac{\pi}{\frac{1}{2}\left(\cosh(2r)-\sinh(2r)\right)}}\exp\left[\frac{-(\alpha+\alpha^*)^2}{4\left(\cosh(2r)-\sinh(2r)\right)}\right] \\
&= \frac{2}{\pi\sqrt{(\cosh^2(2r)-\sinh^2(2r))}}\exp\left[-2\frac{y^2}{e^{2r}}-2\frac{x^2}{e^{-2r}}\right] \\
&= \frac{2}{\pi}\exp\left(-2x^2e^{2r}-2y^2e^{-2r}\right)
\end{aligned}
$$

7.3 Problem 7.3

Displaced squeezed vacuum

$$|\alpha,\xi\rangle = \hat{D}(\alpha)\hat{S}(\xi)|0\rangle \tag{7.3.1}$$

Using the following identities

$$\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a}+\alpha \tag{7.3.2}$$

$$\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{a}\cosh r-\hat{a}^\dagger e^{-2i\varphi}\sinh r \tag{7.3.3}$$

We obtain

$$
\begin{aligned}
\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha)\hat{S}(\xi) &= \hat{a}\cosh r-\hat{a}^\dagger e^{-2i\varphi}\sinh r+\alpha \\
\hat{S}^\dagger(\xi)\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha)\hat{S}(\xi) &= \hat{a}^\dagger\cosh r-\hat{a}e^{2i\varphi}\sinh r+\alpha^*
\end{aligned}
$$

$$
\begin{aligned}
C_N(\lambda) &= \mathrm{Tr}\left(\hat{\rho} e^{\lambda \hat{a}^\dagger} e^{\lambda^* \hat{a}}\right) \\
&= \langle \alpha, \xi | e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} | \alpha, \xi \rangle \\
&= \langle 0 | \hat{S}^\dagger(\xi) \hat{D}^\dagger(\alpha) e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} \hat{D}(\alpha) \hat{S}(\xi) | 0 \rangle \\
&= \langle 0 | \hat{S}^\dagger(\xi) \hat{D}^\dagger(\alpha) e^{\lambda \hat{a}^\dagger} \hat{D}(\alpha) \hat{S}(\xi) \hat{S}^\dagger(\xi) \hat{D}^\dagger(\alpha) e^{-\lambda^* \hat{a}} \hat{D}(\alpha) \hat{S}(\xi) | 0 \rangle \\
&= \langle 0 | e^{\lambda\left(\hat{a}^\dagger \cosh r - \hat{a} e^{2i\varphi} \sinh r + \alpha^*\right)} e^{-\lambda^*\left(\hat{a} \cosh r - \hat{a}^\dagger e^{-2i\varphi} \sinh r + \alpha\right)} | 0 \rangle \\
&= e^{\lambda \alpha^* - \lambda^* \alpha} \langle 0 | e^{\lambda\left(\hat{a}^\dagger \cosh r - \hat{a} e^{2i\varphi} \sinh r\right)} e^{-\lambda^*\left(\hat{a} \cosh r - \hat{a}^\dagger e^{-2i\varphi} \sinh r\right)} | 0 \rangle \\
&= e^{\lambda \alpha^* - \lambda^* \alpha} \langle 0 | e^{\lambda \hat{a}^\dagger \cosh r} e^{-\lambda \hat{a} e^{2i\varphi} \sinh r} e^{\lambda^* \hat{a}^\dagger e^{-2i\varphi} \sinh r} e^{-\lambda^* \hat{a} \cosh r} \\
&\times \exp\left([\lambda \hat{a}^\dagger \cosh r, -\lambda \hat{a} e^{2i\varphi}]\right) \exp\left([\lambda^* \hat{a}^\dagger e^{-2i\varphi} \sinh r, -\lambda^* \hat{a} \cosh r]\right) | 0 \rangle \\
&= e^{\lambda \alpha^* - \lambda^* \alpha} e^{\frac{1}{2} \cosh r \sinh r \left(\lambda^2 e^{2i\varphi} + \lambda^{*2} e^{2i\varphi}\right)} \langle 0 | e^{-\lambda \hat{a} e^{2i\varphi} \sinh r} e^{\lambda^* \hat{a}^\dagger e^{-2i\varphi} \sinh r} | 0 \rangle \\
&= e^{\lambda \alpha^* - \lambda^* \alpha} e^{\frac{1}{2} \cosh r \sinh r \left(\lambda^2 e^{2i\varphi} + \lambda^{*2} e^{2i\varphi}\right)} \langle 0 | e^{-\lambda \hat{a} e^{2i\varphi} \sinh r} e^{\lambda^* \hat{a}^\dagger e^{-2i\varphi} \sinh r} | 0 \rangle \\
&= e^{\lambda \alpha^* - \lambda^* \alpha} e^{\frac{1}{2} \cosh r \sinh r \left(\lambda^2 e^{2i\varphi} + \lambda^{*2} e^{2i\varphi}\right)} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \langle 0 | \frac{\left(-\lambda \hat{a} e^{2i\varphi} \sinh r\right)^n}{\sqrt{n!}} \\
&\times \frac{\left(\lambda^* \hat{a}^\dagger e^{-2i\varphi} \sinh r\right)^{n'}}{\sqrt{n'!}} | 0 \rangle \\
&= e^{\lambda \alpha^* - \lambda^* \alpha} e^{\frac{1}{2} \cosh r \sinh r \left(\lambda^2 e^{2i\varphi} + \lambda^{*2} e^{2i\varphi}\right)} \sum_{n=0}^{\infty} \frac{\left(-|\lambda|^2 \sinh^2 r\right)^n}{n!} \\
&= e^{\lambda \alpha^* - \lambda^* \alpha} e^{\frac{1}{4} \sinh(2r) \left(\lambda^2 e^{2i\varphi} + \lambda^{*2} e^{2i\varphi}\right)} e^{-|\lambda|^2 \sinh^2 r}
\end{aligned}
$$

$$\begin{aligned}
W(\beta) &= \frac{1}{\pi^2}\int d^2\lambda C_N(\lambda)\exp(\lambda^*\beta-\lambda\beta^*)e^{-|\lambda|^2/2} \\
&= \frac{1}{\pi^2}\int d^2\lambda e^{(\lambda^*\beta-\lambda\beta^*)}e^{-|\lambda|^2/2}e^{\lambda\alpha^*-\lambda^*\alpha}e^{\frac{1}{4}\sinh(2r)\left(\lambda^2e^{2i\varphi}+\lambda^{*2}e^{2i\varphi}\right)}e^{-|\lambda|^2\sinh^2 r} \\
&= \frac{1}{\pi^2}\int d^2\lambda e^{\lambda^*(\beta-\alpha)-\lambda(\beta^*-\alpha^*)}e^{\frac{1}{4}\sinh(2r)\left(\lambda^2e^{2i\varphi}+\lambda^{*2}e^{2i\varphi}\right)}e^{-|\lambda|^2(\frac{1}{2}+\sinh^2 r)} \\
&= \frac{1}{\pi^2}\int dx e^{\frac{-x^2}{2}[\sinh(2r)+\cosh(2r)]}e^{x[(\beta-\alpha)-(\beta-\alpha)^*]} \\
&\times\int dy e^{\frac{-y^2}{2}[\cosh(2r)-\sinh(2r)]}e^{iy[(\beta-\alpha)+(\beta-\alpha)^*]} \\
&= \frac{1}{\pi^2}\sqrt{\frac{\pi}{\frac{1}{2}\left(\cosh(2r)+\sinh(2r)\right)}}\exp\left[\frac{1}{2}\frac{\left((\beta-\alpha)-(\beta-\alpha)^*\right)^2}{\left(\cosh(2r)+\sinh(2r)\right)}\right] \\
&\times\sqrt{\frac{\pi}{\frac{1}{2}\left(\cosh(2r)-\sinh(2r)\right)}}\exp\left[\frac{1}{2}\frac{\left((\beta-\alpha)+(\beta-\alpha)^*\right)^2}{\left(\cosh(2r)-\sinh(2r)\right)}\right] \\
&= \frac{2}{\pi}\exp\left(-\frac{1}{2}X^2e^{2r}+\frac{1}{2}Y^2e^{-2r}\right),
\end{aligned}$$

where X the real part of the complex number $\beta-\alpha$), and Y, its imaginary part.

7.4 Problem 7.4

$$\begin{aligned}
\hat{a}|0\rangle &= 0 \\
\hat{S}(\xi)\hat{D}(\alpha)\hat{a}|0\rangle &= 0 \\
\hat{S}(\xi)\hat{D}(\alpha)\hat{a}\hat{D}(-\alpha)\hat{S}(-\xi)\hat{S}(\xi)\hat{D}(\alpha)|0\rangle &= 0 \\
\hat{S}(\xi)(\hat{a}-\alpha)\hat{S}(-\xi)\hat{S}(\xi)\hat{D}(\alpha)|0\rangle &= 0 \\
(\cosh r\hat{a}+e^{i\theta}\sinh r\hat{a}^\dagger-\alpha)\hat{S}(\xi)\hat{D}(\alpha)|0\rangle &= 0 \qquad (7.4.1)
\end{aligned}$$

Let's define the the squeezed coherent state as

$$|\xi,\alpha\rangle = \hat{S}(\xi)\hat{D}(\alpha)|0\rangle, \qquad (7.4.2)$$

$$\begin{aligned}
\mu &= \cosh r, \\
\nu &= e^{i\theta}\sinh r.
\end{aligned}$$

And let write squeezed coherent as an expansion of photon number, namely

$$
\begin{aligned}
|\xi,\alpha\rangle &= \sum_{n=0}^{\infty} c_n |n\rangle \\
(\mu\hat{a} + \nu\hat{a}^\dagger - \alpha)|\xi,\alpha\rangle &= (\mu\hat{a} + \nu\hat{a}^\dagger - \alpha)\sum_{n=0}^{\infty} c_n |n\rangle \\
&= \sum_{n=0}^{\infty} c_n \left(\mu\sqrt{n}|n-1\rangle + \nu\sqrt{n+1}|n+1\rangle - \alpha|n\rangle\right) \\
&= \sum_{n=0}^{\infty} \left(\mu\sqrt{n}c_{n+1} - \alpha c_n + \nu\sqrt{n}c_{n-1}\right)|n\rangle
\end{aligned}
$$

Using equation 7.4.1 we will have the following

$$\sum_{n=0}^{\infty} \left(\mu\sqrt{n}c_{n+1} - \alpha c_n + \nu\sqrt{n}c_{n-1}\right)|n\rangle = 0,$$

which implies

$$\mu\sqrt{n+1}c_{n+1} - \alpha c_n + \nu\sqrt{n}c_{n-1} = 0 \tag{7.4.3}$$

In order to solve the last equation we rewrite c_n as

$$
\begin{aligned}
c_n &= \mathcal{N}\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{n/2} f_n(x) \\
c_{n+1} &= \mathcal{N}\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{n/2}\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{1/2} f_{n+1}(x) \\
c_{n-1} &= \mathcal{N}\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{n/2}\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{-1/2} f_{n-1}(x)
\end{aligned}
$$

into 7.4.3

$$
\begin{aligned}
\mu\sqrt{n+1}\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{1/2} f_{n+1}(x) - \alpha f_n(x) + \nu\sqrt{n}\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{-1/2} f_{n-1}(x) = 0 \\
\mu\sqrt{n+1}f_{n+1}(x) - 2\alpha\left(e^{i\theta}\cosh r\sinh(2r)\right)^{-1/2} f_n(x) + 2\nu f_{n-1}(x) = 0
\end{aligned}
$$

Identifying $x = \alpha\left(e^{i\theta}\cosh r\sinh(2r)\right)^{-1/2}$, and $f_n(x) = H_n(x)/\sqrt{n!}$, where $H_n(x)$ are the Hermite polynomials. Thus

$$
\begin{aligned}
c_n &= \mathcal{N}\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{n/2} H_n(x)/\sqrt{n!} \\
c_0 &= \mathcal{N}
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
c_0 &= \langle 0|\xi,\alpha\rangle \\
&= \langle 0|\hat{S}(\xi)\hat{D}(\alpha)|0\rangle \\
&= \langle -\xi|\alpha\rangle \\
&= \frac{1}{\sqrt{\cosh r}}\exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^2 e^{i\theta}\tanh r\right).
\end{aligned}
$$

Finally we have

$$
c_n = \frac{\exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}\alpha^2 e^{i\theta}\tanh r\right)}{\sqrt{n!\cosh r}}\left(\frac{1}{2}e^{i\theta}\tanh r\right)^{n/2} H_n\left(\alpha\left(e^{i\theta}\cosh r\sinh(2r)\right)^{-1/2}\right).
$$

7.5 Problem 7.5

First we rewrite the state as follows

$$
\begin{aligned}
\hat{a}^\dagger|\alpha\rangle &= \hat{D}(\alpha)\hat{D}(-\alpha)\hat{a}^\dagger\hat{D}(\alpha)|0\rangle && (7.5.1) \\
&= \hat{D}(\alpha)\left(\hat{a}^\dagger + \alpha^*\right)|0\rangle \\
&= \hat{D}(\alpha)\left(|1\rangle + \alpha^*|0\rangle\right), && (7.5.2)
\end{aligned}
$$

where $\hat{D}(\alpha)$ is the displacement operator. Let $|\Psi\rangle$ be the normalized state of the state in Eq. 7.5.1 so that

$$
|\Psi\rangle = \mathcal{N}\hat{a}^\dagger|\alpha\rangle,
$$

where $\mathcal{N}$ is the normalization constant which is given by

$$
\begin{aligned}
\mathcal{N} &= \left[\langle\alpha|\hat{a}^\dagger\hat{a}|\alpha\rangle\right]^{-1/2} \\
&= \left(1+|\alpha|^2\right)^{-1/2}.
\end{aligned}
$$

The normalized state can be rewritten as

$$|\Psi\rangle = \frac{1}{\sqrt{(1+|\alpha|^2)}}\hat{a}^\dagger|\alpha\rangle$$
$$= \frac{1}{\sqrt{(1+|\alpha|^2)}}\hat{D}(\alpha)\left(|1\rangle + \alpha^*|0\rangle\right).$$

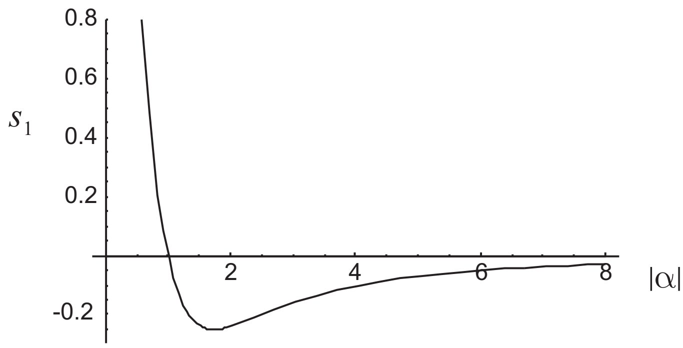
We consider the quadrature squeezing for this state. Numerically one needs to compute the following quantities.

$$\langle\hat{a}\rangle = |\mathcal{N}|^2\alpha\left(2+|\alpha|^2\right)$$
$$\langle\hat{a}^2\rangle = |\mathcal{N}|^2\alpha^2\left(3+|\alpha|^2\right)$$
$$\langle\hat{a}^\dagger\hat{a}\rangle = |\mathcal{N}|^2\left[1+|\alpha|^2\left(3+|\alpha|^2\right)\right].$$

Instead of plotting $\left\langle(\Delta\hat{X}_1)^2\right\rangle$ we have plotted

$$s_1 = 4\left\langle(\Delta\hat{X}_1)^2\right\rangle - 1$$
$$= 2\Re\left(\langle\hat{a}^2\rangle\right) + 2\langle\hat{a}^\dagger\hat{a}\rangle - 2\Re\left(\langle\hat{a}\rangle^2\right) - 2\left|\langle\hat{a}\rangle\right|^2.$$

It is obvious that this state is nonclassical since s_1 goes negative, an indication of squeezing of the field quadrature.



7.6 Problem 7.6

Starting from Eq. (4.120)

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty}\left\{\left[C_e c_n\cos\left(\lambda t\sqrt{n+1}\right) - iC_g c_{n+1}\sin\left(\lambda t\sqrt{n+1}\right)\right]|e\rangle\right.$$
$$\left. + \left[-iC_e c_{n-1}\sin\left(\lambda t\sqrt{n}\right) + C_g c_n\cos\left(\lambda t\sqrt{n}\right)\right]|g\rangle\right\}|n\rangle$$

For the case where the atom is initially at the excited state and the field initially in a coherent state we have

$$C_e = 1,\ C_g = 0,\ \text{and } c_n = e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}},$$

thus

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty}\frac{e^{-|\alpha|^2/2}\alpha^n}{\sqrt{n!}}\left[\cos\left(\lambda t\sqrt{n+1}\right)|e\rangle - i\frac{\sqrt{n}}{\alpha}\sin\left(\lambda t\sqrt{n}\right)|g\rangle\right]|n\rangle.$$

Quadrature operators are defined as

$$\begin{aligned}\hat{X}_1 &= \frac{1}{2}\left(\hat{a}+\hat{a}^\dagger\right),\\ \hat{X}_2 &= \frac{1}{2i}\left(\hat{a}-\hat{a}^\dagger\right).\end{aligned}$$

Numerically, we want to investigate

$$\begin{aligned}\left\langle(\Delta\hat{X}_1)^2\right\rangle &= \frac{1}{4}\left(\langle\hat{a}^2\rangle + \langle\hat{a}^{\dagger 2}\rangle + 2\left\langle\hat{a}^\dagger\hat{a}\right\rangle + 1 - \langle\hat{a}\rangle^2 - \left\langle\hat{a}^\dagger\right\rangle^2 - 2\langle\hat{a}\rangle\left\langle\hat{a}^\dagger\right\rangle\right),\\ \left\langle(\Delta\hat{X}_2)^2\right\rangle &= \frac{1}{4}\left(-\langle\hat{a}^2\rangle - \langle\hat{a}^{\dagger 2}\rangle + 2\left\langle\hat{a}^\dagger\hat{a}\right\rangle + 1 + \langle\hat{a}\rangle^2 + \left\langle\hat{a}^\dagger\right\rangle^2 - 2\langle\hat{a}\rangle\left\langle\hat{a}^\dagger\right\rangle\right)\end{aligned}$$

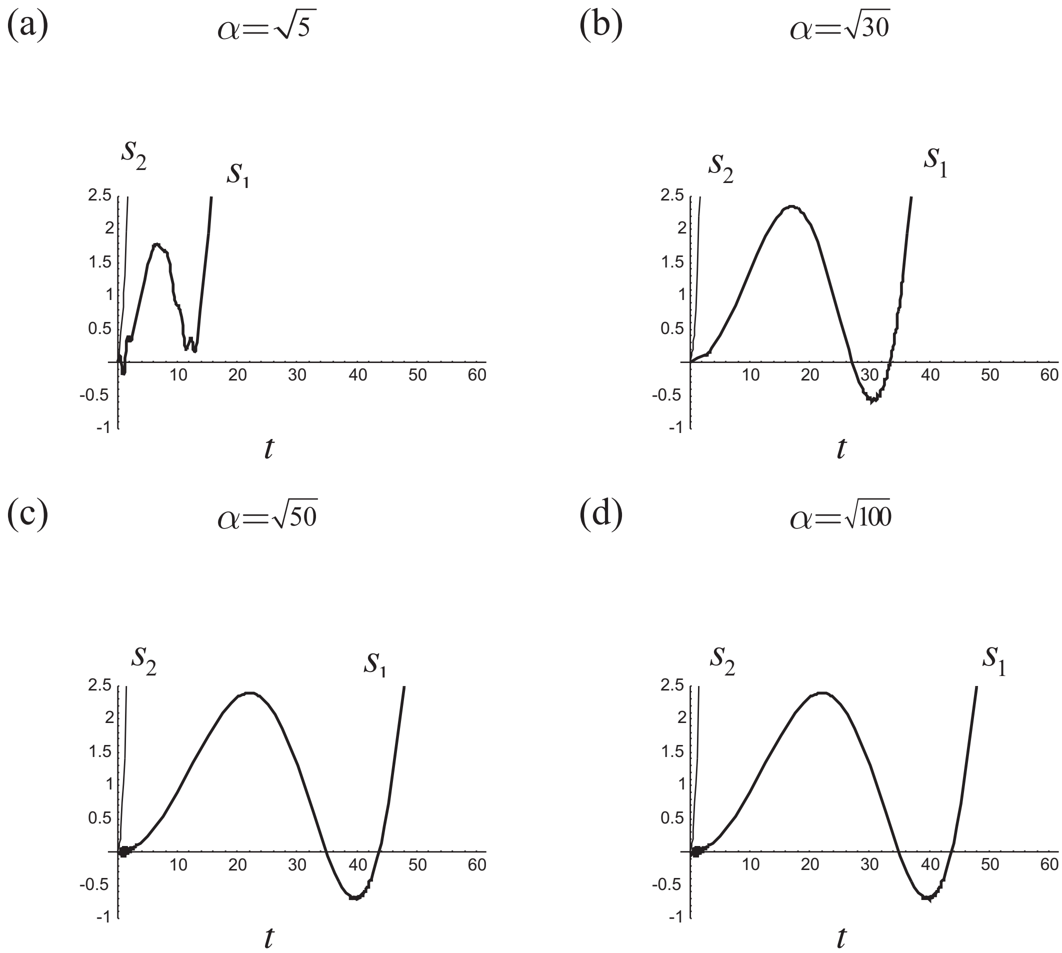
to see if any one of them goes below 1/4. Numerically one needs to compute the following quantities:

$$\begin{aligned}\langle\hat{a}\rangle &= \sum_{n=0}^{\infty}\frac{e^{-|\alpha|^2}\alpha|\alpha|^{2n}}{n!}\Bigg[\cos(\lambda t\sqrt{n+1})\cos(\lambda t\sqrt{n+2})\\ &\qquad + \frac{\sqrt{n(n+1)}}{|\alpha|^2}\sin(\lambda t\sqrt{n})\sin(\lambda t\sqrt{n+1})\Bigg]\\ \langle\hat{a}^2\rangle &= \sum_{n=0}^{\infty}\frac{e^{-|\alpha|^2}\alpha^2|\alpha|^{2n}}{n!}\Bigg[\cos(\lambda t\sqrt{n+1})\cos(\lambda t\sqrt{n+3})\\ &\qquad + \frac{\sqrt{n(n+2)}}{|\alpha|^2}\sin(\lambda t\sqrt{n})\sin(\lambda t\sqrt{n+2})\Bigg]\\ \langle\hat{a}^\dagger\hat{a}\rangle &= \sum_{n=0}^{\infty}\frac{e^{-|\alpha|^2}n|\alpha|^{2n}}{n!}\left[\cos^2\left(\lambda t\sqrt{n+1}\right) + \frac{n}{|\alpha|^2}\sin^2\left(\lambda t\sqrt{n}\right)\right].\end{aligned}$$

Instead of plotting $\left\langle (\Delta\hat{X}_1)^2 \right\rangle$ and $\left\langle (\Delta\hat{X}_2)^2 \right\rangle$ we have plotted

$$
\begin{aligned}
s_1 &= 4\left\langle (\Delta\hat{X}_1)^2 \right\rangle - 1 \\
s_2 &= 4\left\langle (\Delta\hat{X}_2)^2 \right\rangle - 1
\end{aligned}
$$

versus time for different values of α. Obviously if any of the last quantities goes below 0 we have squeezing. In fact the graphs below show squeezing at more than one occasion.



7.7 Problem 7.7

As in the previous problem, we start from Eq. (4.120)

$$
\begin{aligned}
|\Psi(t)\rangle = \sum_{n=0}^{\infty} \Big\{ & \Big[C_e c_n \cos\left(\lambda t\sqrt{n+1}\right) - iC_g c_{n+1} \sin\left(\lambda t\sqrt{n+1}\right)\Big] |e\rangle \\
& + \left[-iC_e c_{n-1} \sin\left(\lambda t\sqrt{n}\right) + C_g c_n \cos\left(\lambda t\sqrt{n}\right)\right] |g\rangle \Big\} |n\rangle,
\end{aligned}
$$

but this time the atom is initially at the excited state and the field initially in a squeezed state $|\alpha, \xi\rangle$ we have

$$
\begin{aligned}
C_e &= 1, \\
C_g &= 0, \\
c_n &= \frac{1}{\sqrt{\cosh r}} \exp[-(|\alpha|^2 + \alpha^{*2} e^{i\theta} \tanh r)/2] \\
&\quad \times \frac{\left(\frac{1}{2} e^{i\theta} \tanh r\right)^{n/2}}{\sqrt{n!}} H_n\left[\gamma(e^{i\theta} \sinh(2r))^{-1/2}\right],
\end{aligned}
$$

where $\gamma = \alpha \cosh r + \alpha^* e^{i\theta} \sinh r$. Now we can write

$$
|\Psi(t)\rangle = \sum_{n=0}^{\infty} \Big[c_n \cos\left(\lambda t\sqrt{n+1}\right) |e\rangle - i c_{n-1} \sin\left(\lambda t\sqrt{n}\right) |g\rangle\Big] |n\rangle.
$$

The atomic inversion for this state is

$$
W(t) = \sum_{n=0}^{\infty} \Big[|c_n|^2 \cos^2\left(\lambda t\sqrt{n+1}\right) - |c_{n-1}|^2 \sin^2\left(\lambda t\sqrt{n}\right)\Big]
$$

7.8 Problem 7.8

a.

$$
\begin{aligned}
\hat{H}_I &= \hbar K \hat{a}^{\dagger 2} \hat{a}^2 \\
\frac{d\hat{a}}{dt} &= \frac{1}{i\hbar}\left[\hat{a}, \hat{H}_I\right] \\
&= -i2K\hat{a}^\dagger \hat{a}^2
\end{aligned}
$$

$$
\begin{aligned}
\hat{a}(t) &= e^{-2iK\hat{a}^\dagger \hat{a} t} \hat{a} \\
&= e^{-2iK\hat{n}t} \hat{a}
\end{aligned}
$$

b.

$$\begin{aligned}\hat{n}(t) &= \hat{a}^\dagger(t)\hat{a}(t) \\ &= \hat{a}^\dagger e^{2iK\hat{a}^\dagger\hat{a}t}e^{-2iK\hat{a}^\dagger\hat{a}t}\hat{a} \\ &= \hat{a}^\dagger\hat{a} = \hat{n}(0)\end{aligned}$$

So if we start with Poissonian photon-counting statistics, it will remain the same for all times.

c.

$$\begin{aligned}\hat{X}_1(t) &= \frac{1}{2}(\hat{a}(t) + \hat{a}^\dagger(t)) \\ \hat{X}_2(t) &= \frac{1}{2i}(\hat{a}(t) - \hat{a}^\dagger(t))\end{aligned}$$

$$\begin{aligned}\hat{a}(t)|\alpha\rangle &= e^{-2iK\hat{n}t}\hat{a}|\alpha\rangle \\ &= e^{-2iK\hat{n}t}\alpha|\alpha\rangle \\ &= \alpha\sum_{n=0}^{\infty} e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}e^{-2iKnt}|n\rangle \\ &= \alpha\sum_{n=0}^{\infty} e^{-|\alpha|^2/2}\frac{\left(\alpha e^{-2iKt}\right)^n}{\sqrt{n!}}|n\rangle \\ &= \alpha\left|\alpha e^{-2iKt}\right\rangle\end{aligned}$$

$$\begin{aligned}\langle\alpha|\hat{a}(t)|\alpha\rangle &= \alpha\left\langle\alpha|\alpha e^{-2iKt}\right\rangle \\ &= \alpha e^{-|\alpha|^2\left(1-e^{-2iKt}\right)}\end{aligned}$$

where we have used

$$\begin{aligned}|\alpha\rangle &= \sum_{n=0}^{\infty} e^{-|\alpha|^2/2}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \\ \langle\beta|\alpha\rangle &= \exp\left(-\frac{|\alpha|^2}{2} - \frac{|\beta|^2}{2} + \beta^*\alpha\right).\end{aligned}$$

$$\begin{aligned}\langle\alpha|\hat{X}_1(t)|\alpha\rangle &= \frac{1}{2}(\langle\alpha|\hat{a}(t)|\alpha\rangle + \langle\alpha|\hat{a}^\dagger(t)|\alpha\rangle) \\ &= \frac{1}{2}\left(\alpha e^{-|\alpha|^2\left(1-e^{-2iKt}\right)} + \alpha^* e^{-|\alpha|^2\left(1-e^{2iKt}\right)}\right) \\ \langle\alpha|\hat{X}_2(t)|\alpha\rangle &= \frac{1}{2i}\left(\alpha e^{-|\alpha|^2\left(1-e^{-2iKt}\right)} - \alpha^* e^{-|\alpha|^2\left(1-e^{2iKt}\right)}\right)\end{aligned}$$

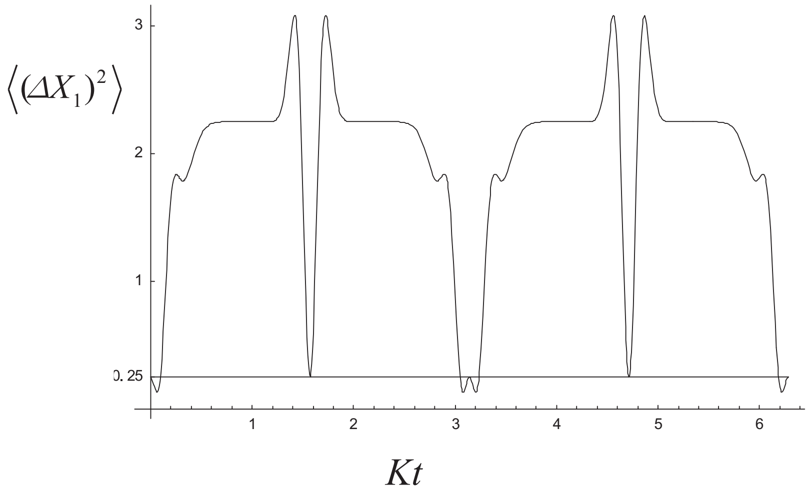
$$
\begin{aligned}
\hat{X}_1^2(t) &= \frac{1}{4}\left(\hat{a}(t)+\hat{a}^\dagger(t)\right)^2 = \frac{1}{4}\left(\hat{a}^2(t)+\hat{a}^{\dagger 2}(t)+2\hat{n}+1\right)\\
\hat{X}_2^2(t) &= -\frac{1}{4}\left(\hat{a}(t)-\hat{a}^\dagger(t)\right)^2 = \frac{1}{4}\left(-\hat{a}^2(t)-\hat{a}^{\dagger 2}(t)+2\hat{n}+1\right)
\end{aligned}
$$

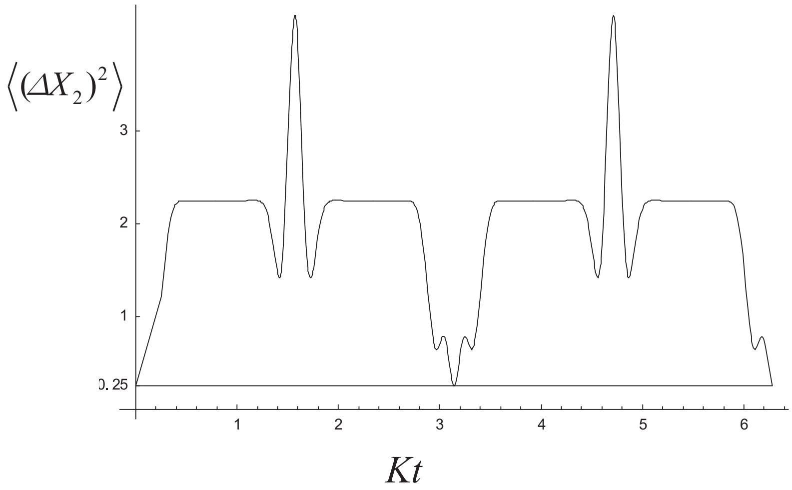
$$
\begin{aligned}
\langle\alpha|\hat{a}^2(t)|\alpha\rangle &= \langle\alpha|e^{-2iK\hat{n}t}\hat{a}e^{-2iK\hat{n}t}\hat{a}|\alpha\rangle\\
&= \alpha\langle\alpha e^{2iKt}|\hat{a}|\alpha e^{-2iKt}\rangle\\
&= \alpha^2 e^{-2iKt}\langle\alpha e^{2iKt}|\alpha e^{-2iKt}\rangle\\
&= \alpha^2 e^{-2iKt}e^{-|\alpha|^2\left(1-e^{-4iKt}\right)}
\end{aligned}
$$

$$
\begin{aligned}
\langle\alpha|\hat{X}_1^2(t)|\alpha\rangle &= \frac{1}{4}\left(\alpha^2 e^{-2iKt}e^{-|\alpha|^2\left(1-e^{-4iKt}\right)}+\alpha^{*2}e^{2iKt}e^{-|\alpha|^2\left(1-e^{4iKt}\right)}+2|\alpha|^2+1\right)\\
\langle\alpha|\hat{X}_2^2(t)|\alpha\rangle &= \frac{1}{4}\left(-\alpha^2 e^{-2iKt}e^{-|\alpha|^2\left(1-e^{-4iKt}\right)}-\alpha^{*2}e^{2iKt}e^{-|\alpha|^2\left(1-e^{4iKt}\right)}+2|\alpha|^2+1\right)
\end{aligned}
$$

$$
\begin{aligned}
\left\langle\left(\Delta\hat{X}_1(t)\right)^2\right\rangle &= \frac{1}{4}\Big[1+2|\alpha|^2\left(1-e^{-2|\alpha|^2(1-\cos 2kt)}\right)\\
&\quad+\alpha^2 e^{-|\alpha|^2}\left(e^{-2iKt+|\alpha|^2e^{-4iKt}}-e^{-|\alpha|^2(1-2e^{-2iKt})}\right)\\
&\quad+\alpha^{*2}e^{-|\alpha|^2}\left(e^{2iKt+|\alpha|^2e^{4iKt}}-e^{-|\alpha|^2(1-2e^{2iKt})}\right)\Big]\\
\left\langle\left(\Delta\hat{X}_2(t)\right)^2\right\rangle &= \frac{1}{4}\Big[1+2|\alpha|^2\left(1-e^{-2|\alpha|^2(1-\cos 2kt)}\right)\\
&\quad-\alpha^2 e^{-|\alpha|^2}\left(e^{-2iKt+|\alpha|^2e^{-4iKt}}-e^{-|\alpha|^2(1-2e^{-2iKt})}\right)\\
&\quad-\alpha^{*2}e^{-|\alpha|^2}\left(e^{2iKt+|\alpha|^2e^{4iKt}}-e^{-|\alpha|^2(1-2e^{2iKt})}\right)\Big]
\end{aligned}
$$

Plotting $\left\langle\left(\Delta\hat{X}_1(t)\right)^2\right\rangle$ and $\left\langle\left(\Delta\hat{X}_2(t)\right)^2\right\rangle$ versus Kt we see that former goes below 1/4 for short time while the latter does not. See graphs below.





7.9 Problem 7.9

Let's $|\Psi_\pm\rangle$ be the normalized real and imaginary states, defined as follows:

$$|\Psi_\pm\rangle = \mathcal{N}_\pm \left(|\alpha\rangle \pm |\alpha^*\rangle\right), \tag{7.9.1}$$

$$\begin{aligned} \langle\Psi_\pm|\Psi_\pm\rangle &= 1 \\ |\mathcal{N}_\pm|^2 \left[2 \pm \left(\langle\alpha|\alpha^*\rangle + \langle\alpha^*|\alpha\rangle\right)\right] &= 1 \\ \mathcal{N}_\pm &= \left[2 \pm \left(\langle\alpha|\alpha^*\rangle + \langle\alpha^*|\alpha\rangle\right)\right]^{-1/2} \end{aligned}$$

$$\langle\alpha|\beta\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2 + \alpha^*\beta\right)$$

$$\mathcal{N}_\pm = \left[2 \pm e^{-|\alpha|^2}\left(e^{\alpha^{*2}} + e^{\alpha^2}\right)\right]^{-1/2} \tag{7.9.2}$$

$$\hat{X}(\vartheta) = \frac{1}{2}\left(\hat{a}e^{i\theta} + \hat{a}^\dagger e^{-i\theta}\right) \tag{7.9.3}$$

$$\begin{aligned}
\left\langle \hat{X}(\vartheta)\right\rangle &= \langle\Psi_\pm|\,\hat{X}(\vartheta)\,|\Psi_\pm\rangle \\
&= \frac{|\mathcal{N}_\pm|^2}{2}\left\{e^{i\vartheta}\left[\alpha + \alpha^* \pm \left(\alpha^*\langle\alpha|\alpha^*\rangle + \alpha\langle\alpha^*|\alpha\rangle\right)\right]\right. \\
&\qquad \left. + e^{i\vartheta}\left[\alpha + \alpha^* \pm \left(\alpha^*\langle\alpha|\alpha^*\rangle + \alpha\langle\alpha^*|\alpha\rangle\right)\right]\right\} \\
&= \frac{|\mathcal{N}_\pm|^2}{2}\left[(\alpha + \alpha^*)\left(e^{i\vartheta} + e^{i\vartheta}\right) \pm \left(e^{i\vartheta} + e^{i\vartheta}\right)\left(\alpha^*\langle\alpha|\alpha^*\rangle + \alpha\langle\alpha^*|\alpha\rangle\right)\right] \\
&= |\mathcal{N}_\pm|^2 \cos\vartheta\left[\alpha + \alpha^* \pm e^{-|\alpha|^2}\left(\alpha^* e^{\alpha^{*2}} + \alpha e^{\alpha^2}\right)\right]
\end{aligned}$$

$$\begin{aligned}
\hat{X}^2(\vartheta) &= \frac{1}{4}\left(\hat{a}e^{i\theta} + \hat{a}^\dagger e^{-i\theta}\right)\left(\hat{a}e^{i\theta} + \hat{a}^\dagger e^{-i\theta}\right) \\
&= \frac{1}{4}\left(\hat{a}^2 e^{i2\theta} + \hat{a}^{\dagger 2} e^{-i2\theta} + \hat{a}\hat{a}^\dagger + \hat{a}^\dagger\hat{a}\right) \\
&= \frac{1}{4}\left(\hat{a}^2 e^{i2\theta} + \hat{a}^{\dagger 2} e^{-i2\theta} + 2\hat{a}\hat{a}^\dagger + 1\right)
\end{aligned}$$

$$\begin{aligned}
\left\langle \hat{X}^2(\vartheta)\right\rangle &= \langle\Psi_\pm|\,\hat{X}^2(\vartheta)\,|\Psi_\pm\rangle \\
&= \frac{|\mathcal{N}_\pm|^2}{4}\left[e^{2i\vartheta}\left(\alpha^2 + \alpha^{*2} \pm \alpha^2\langle\alpha^*|\alpha\rangle \pm \alpha^{*2}\langle\alpha|\alpha^*\rangle\right)\right. \\
&\qquad + e^{-2i\vartheta}\left(\alpha^2 + \alpha^{*2} \pm \alpha^2\langle\alpha^*|\alpha\rangle \pm \alpha^{*2}\langle\alpha|\alpha^*\rangle\right) \\
&\qquad \left. + 2\left(|\alpha|^2 \pm \alpha^2\langle\alpha^*|\alpha\rangle \pm \alpha^{*2}\langle\alpha|\alpha^*\rangle\right) + 1\right] \\
&= \frac{|\mathcal{N}_\pm|^2}{4}\left[2\cos(2\vartheta)\left(\alpha^2 + \alpha^{*2} \pm \alpha^2\langle\alpha^*|\alpha\rangle \pm \alpha^{*2}\langle\alpha|\alpha^*\rangle\right)\right. \\
&\qquad \left. + 2\left(|\alpha|^2 \pm \alpha^2\langle\alpha^*|\alpha\rangle \pm \alpha^{*2}\langle\alpha|\alpha^*\rangle\right) + 1\right] \\
&= \frac{|\mathcal{N}_\pm|^2}{4}\left[1 + 2|\alpha|^2 + 2\cos(2\vartheta)\left(\alpha^2 + \alpha^{*2}\right)\right. \\
&\qquad \left. \pm 2(\cos(2\vartheta) + 1)e^{-|\alpha|^2}\left(\alpha^2 e^{\alpha^{*2}} + \alpha^{*2} e^{\alpha^2}\right)\right]
\end{aligned}$$

$$
\begin{aligned}
C_N(\lambda) &= \mathrm{Tr}\left(\hat{\rho} e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}}\right) \\
&= \langle \Psi_\pm | \, e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} \, |\Psi_\pm\rangle \\
&= |\mathcal{N}_\pm|^2 \left(\langle\alpha| \pm \langle\alpha^*|\right) e^{\lambda \hat{a}^\dagger} e^{-\lambda^* \hat{a}} \left(|\alpha\rangle \pm |\alpha^*\rangle\right) \\
&= |\mathcal{N}_\pm|^2 \left(e^{\lambda\alpha^*}\langle\alpha| \pm e^{\lambda\alpha}\langle\alpha^*|\right)\left(e^{-\lambda^*\alpha}|\alpha\rangle \pm e^{-\lambda^*\alpha^*}|\alpha^*\rangle\right) \\
&= |\mathcal{N}_\pm|^2 \left[e^{\lambda\alpha^* - \lambda\alpha^*} + e^{\lambda\alpha - \lambda^*\alpha^*} \pm \left(e^{\lambda\alpha^* - \lambda^*\alpha^*}\langle\alpha|\alpha^*\rangle + e^{\lambda\alpha - \lambda^*\alpha}\langle\alpha^*|\alpha\rangle\right)\right] \\
&= |\mathcal{N}_\pm|^2 \left[e^{\lambda\alpha^* - \lambda\alpha^*} + e^{\lambda\alpha - \lambda^*\alpha^*} \pm e^{-|\alpha|^2}\left(e^{\alpha^{*2}} e^{\lambda\alpha^* - \lambda^*\alpha^*} + e^{\alpha^2} e^{\lambda\alpha - \lambda^*\alpha}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
W(\lambda) &= \frac{1}{\pi^2}\int d^2\lambda e^{\lambda^*\alpha - \lambda^*\alpha} C_N(\lambda) e^{-|\lambda|^2/2} \\
&= \frac{|\mathcal{N}_\pm|^2}{\pi^2}\int d^2\lambda e^{\lambda^*\alpha - \lambda\alpha^*}\left[e^{\lambda\alpha^* - \lambda\alpha^*}\right. \\
&\quad \left. + e^{\lambda\alpha - \lambda^*\alpha^*} \pm e^{-|\alpha|^2}\left(e^{\alpha^{*2}} e^{\lambda\alpha^* - \lambda^*\alpha^*} + e^{\alpha^2} e^{\lambda\alpha - \lambda^*\alpha}\right)\right] e^{-|\lambda|^2/2} \\
&= \frac{|\mathcal{N}_\pm|^2}{\pi^2}\left[\int e^{-|\lambda|^2/2} d^2\lambda + \int e^{\lambda^*(\alpha - \alpha^*) - \lambda(\alpha - \alpha^*) - |\lambda|^2/2} d^2\lambda\right. \\
&\quad \left. \pm e^{-|\alpha|^2}\left(e^{\alpha^{*2}}\int e^{\lambda^*(\alpha - \alpha^*) - |\lambda|^2/2} d^2\lambda + e^{\alpha^2}\int e^{-\lambda(\alpha - \alpha^*) - |\lambda|^2/2} d^2\lambda\right)\right] \\
&= \frac{|\mathcal{N}_\pm|^2}{\pi^2}\left[2\pi + 2\pi e^{-2(\alpha - \alpha^*)^2} \pm e^{-|\alpha|^2}\left(2\pi e^{\alpha*^2} + 2\pi e^{\alpha^2}\right)\right] \\
&= \frac{2\,|\mathcal{N}_\pm|^2}{\pi}\left[1 + e^{-2(\alpha - \alpha^*)^2} \pm e^{-|\alpha|^2}\left(e^{\alpha^{*2}} + e^{\alpha^2}\right)\right]
\end{aligned}
$$

where we have used the following identity

$$
\int \exp(\alpha x + \alpha^* y - z|\alpha|^2) d^2\alpha = \frac{\pi}{z}\exp\left(\frac{xy}{z}\right). \tag{7.9.4}
$$

7.10 Problem 7.10

$$
\begin{aligned}
\hat{K}_1 &= \frac{1}{2}\left(\hat{a}^{\dagger 2} + \hat{a}^2\right) \\
\hat{K}_2 &= \frac{1}{2i}\left(\hat{a}^{\dagger 2} - \hat{a}^2\right) \\
\hat{K}_3 &= \frac{1}{2}\left(\hat{a}^\dagger \hat{a} + \frac{1}{2}\right)
\end{aligned}
$$

a.

$$\begin{aligned}
\left[\hat{K}_1, \hat{K}_2\right] &= \frac{1}{4i}\left[\hat{a}^{\dagger 2} + \hat{a}^2, \hat{a}^{\dagger 2} - \hat{a}^2\right] \\
&= \frac{1}{4i}\left(\left[\hat{a}^{\dagger 2}, \hat{a}^{\dagger 2} - \hat{a}^2\right] + \left[\hat{a}^2, \hat{a}^{\dagger 2} - \hat{a}^2\right]\right) \\
&= \frac{1}{4i}\left(-\left[\hat{a}^{\dagger 2}, \hat{a}^2\right] + \left[\hat{a}^2, \hat{a}^{\dagger 2}\right]\right) \\
&= -\frac{1}{2i}\left[\hat{a}^{\dagger 2}, \hat{a}^2\right] \\
&= i\frac{1}{2}\left(\hat{a}^{\dagger 2}\hat{a}^2 - \hat{a}^2\hat{a}^{\dagger 2}\right) \\
&= i\frac{1}{2}\left(\hat{a}^{\dagger 2}\hat{a}^2 - \hat{a}(1 + \hat{a}^\dagger\hat{a})\hat{a}^\dagger\right) \\
&= i\frac{1}{2}\left(\hat{a}^{\dagger 2}\hat{a}^2 - \hat{a}\hat{a}^\dagger - (1 + \hat{a}^\dagger\hat{a})(1 + \hat{a}^\dagger\hat{a})\right) \\
&= i\frac{1}{2}\left(\hat{a}^{\dagger 2}\hat{a}^2 - 1 - \hat{a}^\dagger\hat{a} - 1 - 2\hat{a}^\dagger\hat{a} - \hat{a}^\dagger\hat{a}\hat{a}^\dagger\hat{a}\right) \\
&= i\frac{1}{2}\left(\hat{a}^{\dagger 2}\hat{a}^2 - 2 - 3\hat{a}^\dagger\hat{a} - \hat{a}^\dagger(1 + \hat{a}^\dagger\hat{a})\hat{a}\right) \\
&= i\frac{1}{2}\left(\hat{a}^{\dagger 2}\hat{a}^2 - 2 - 4\hat{a}^\dagger\hat{a} - \hat{a}^{\dagger 2}\hat{a}^2\right) \\
&= -4i\hat{K}_3
\end{aligned}$$

$$\begin{aligned}
\left[\hat{K}_2, \hat{K}_3\right] &= \frac{1}{4i}\left[\hat{a}^{\dagger 2} - \hat{a}^2, \hat{a}^\dagger\hat{a} + \frac{1}{2}\right] \\
&= \frac{1}{4i}\left[\hat{a}^{\dagger 2} - \hat{a}^2, \hat{a}^\dagger\hat{a}\right] \\
&= \frac{1}{4i}\left(\left[\hat{a}^{\dagger 2}, \hat{a}^\dagger\hat{a}\right] - \left[\hat{a}^2, \hat{a}^\dagger\hat{a}\right]\right) \\
&= \frac{1}{4i}\left(\hat{a}^\dagger\left[\hat{a}^{\dagger 2}, \hat{a}\right] - \left[\hat{a}^2, \hat{a}^\dagger\right]\hat{a}\right) \\
&= \frac{1}{4i}\left(-2\hat{a}^\dagger\hat{a}^\dagger - 2\hat{a}\hat{a}\right) \\
&= i\frac{1}{2}\left(\hat{a}^{\dagger 2} + \hat{a}^2\right) \\
&= i\hat{K}_1
\end{aligned}$$

$$\begin{aligned}
\left[\hat{K}_3, \hat{K}_1\right] &= \frac{1}{4}\left[\hat{a}^{\dagger 2} + \hat{a}^2, \hat{a}^\dagger \hat{a} + \frac{1}{2}\right] \\
&= \frac{1}{4}\left[\hat{a}^{\dagger 2} + \hat{a}^2, \hat{a}^\dagger \hat{a}\right] \\
&= \frac{1}{4}\left(\left[\hat{a}^{\dagger 2}, \hat{a}^\dagger \hat{a}\right] + \left[\hat{a}^2, \hat{a}^\dagger \hat{a}\right]\right) \\
&= \frac{1}{4}\left(\hat{a}^\dagger \left[\hat{a}^{\dagger 2}, \hat{a}\right] + \left[\hat{a}^2, \hat{a}^\dagger\right] \hat{a}\right) \\
&= \frac{1}{4}\left(-2\hat{a}^\dagger \hat{a}^\dagger + 2\hat{a}\hat{a}\right) \\
&= -i\frac{1}{2i}\left(\hat{a}^{\dagger 2} - \hat{a}^2\right) \\
&= -i\hat{K}_2
\end{aligned}$$

b. According to section (7.1)

$$\left\langle \left(\Delta \hat{A}\right)^2 \right\rangle \left\langle \left(\Delta \hat{B}\right)^2 \right\rangle \geq \frac{1}{4}\left|\left\langle \hat{C} \right\rangle\right|^2, \tag{7.10.1}$$

for any operators $\hat{A}$, $\hat{B}$, and $\hat{C}$ satisfying the following commutation relation

$$\left[\hat{A}, \hat{B}\right] = i\hat{C}. \tag{7.10.2}$$

Applying this to $\hat{K}_1$ and $\hat{K}_2$, we will obtain

$$\left\langle \left(\Delta \hat{K}_1\right)^2 \right\rangle \left\langle \left(\Delta \hat{K}_2\right)^2 \right\rangle \geq 4\left|\left\langle \hat{K}_3 \right\rangle\right|^2.$$

c.

$$\begin{aligned}
\langle \alpha | \hat{K}_1 | \alpha \rangle &= \frac{1}{2}\langle \alpha | \hat{a}^{\dagger 2} + \hat{a}^2 | \alpha \rangle \\
&= \frac{1}{2}\langle \alpha | \hat{a}^{\dagger 2} + \hat{a}^2 | \alpha \rangle \\
&= \frac{1}{2}\left(\alpha^{*2} + \alpha^2\right)
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\hat{K}_2|\alpha\rangle &= \frac{1}{2i}\langle\alpha|\hat{a}^{\dagger 2} - \hat{a}^2|\alpha\rangle \\
&= \frac{1}{2i}\langle\alpha|\hat{a}^{\dagger 2} - \hat{a}^2|\alpha\rangle \\
&= \frac{1}{2i}\left(\alpha^{*2} - \alpha^2\right)
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\hat{K}_3|\alpha\rangle &= \frac{1}{2}\langle\alpha|\hat{a}^{\dagger}\hat{a} + \frac{1}{2}|\alpha\rangle \\
&= \frac{1}{2}\left(|\alpha|^2 + \frac{1}{2}\right)
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\hat{K}_1^2|\alpha\rangle &= \frac{1}{4}\langle\alpha|\left(\hat{a}^{\dagger 2} + \hat{a}^2\right)\left(\hat{a}^{\dagger 2} + \hat{a}^2\right)|\alpha\rangle \\
&= \frac{1}{4}\langle\alpha|\hat{a}^{\dagger 4} + \hat{a}^4 + \hat{a}^{\dagger 2}\hat{a}^2 + \hat{a}^2\hat{a}^{\dagger 2}|\alpha\rangle \\
&= \frac{1}{4}\langle\alpha|\hat{a}^{\dagger 4} + \hat{a}^4 + \hat{a}^{\dagger 2}\hat{a}^2 + \hat{a}^{\dagger 2}\hat{a}^2 + 4\hat{a}^{\dagger}\hat{a} + 2|\alpha\rangle \\
&= \frac{1}{4}\left(2|\alpha|^4 + \alpha^{*4} + \alpha^4 + 4|\alpha|^2 + 2\right)
\end{aligned}$$

$$\begin{aligned}
\langle\alpha|\hat{K}_2^2|\alpha\rangle &= -\frac{1}{4}\langle\alpha|\left(\hat{a}^{\dagger 2} - \hat{a}^2\right)\left(\hat{a}^{\dagger 2} - \hat{a}^2\right)|\alpha\rangle \\
&= -\frac{1}{4}\langle\alpha|\hat{a}^{\dagger 4} + \hat{a}^4 - \hat{a}^{\dagger 2}\hat{a}^2 - \hat{a}^2\hat{a}^{\dagger 2}|\alpha\rangle \\
&= -\frac{1}{4}\langle\alpha|\hat{a}^{\dagger 4} + \hat{a}^4 - \hat{a}^{\dagger 2}\hat{a}^2 - \hat{a}^{\dagger 2}\hat{a}^2 - 4\hat{a}^{\dagger}\hat{a} - 2|\alpha\rangle \\
&= \frac{1}{4}\left(2|\alpha|^4 - \alpha^{*4} - \alpha^4 + 4|\alpha|^2 + 2\right)
\end{aligned}$$

$$\begin{aligned}
\left\langle\left(\Delta\hat{K}_1\right)^2\right\rangle &= \langle\alpha|\hat{K}_1^2|\alpha\rangle - \langle\alpha|\hat{K}_1|\alpha\rangle^2 \\
&= \frac{1}{4}\left(2|\alpha|^4 + \alpha^{*4} + \alpha^4 + 4|\alpha|^2 + 2 - \alpha^{*4} - \alpha^4 - 2|\alpha|^4\right) \\
&= \frac{1}{4}\left(4|\alpha|^2 + 2\right) \\
&= |\alpha|^2 + \frac{1}{2}
\end{aligned}$$

$$\begin{aligned}\left\langle\left(\Delta\hat{K}_2\right)^2\right\rangle &= \langle\alpha|\hat{K}_2^2|\alpha\rangle - \langle\alpha|\hat{K}_2|\alpha\rangle^2 \\ &= \frac{1}{4}\left(2|\alpha|^4 - \alpha^{*4} - \alpha^4 + 4|\alpha|^2 + 2 + \alpha^{*4} + \alpha^4 - 2|\alpha|^4\right) \\ &= \frac{1}{4}\left(4|\alpha|^2 + 2\right) \\ &= |\alpha|^2 + \frac{1}{2}\end{aligned}$$

$$\langle\alpha|\hat{K}_3|\alpha\rangle^2 = \frac{1}{4}\left(|\alpha|^2 + \frac{1}{2}\right)^2$$

Obviously,

$$\left\langle\left(\Delta\hat{K}_1\right)^2\right\rangle\left\langle\left(\Delta\hat{K}_2\right)^2\right\rangle = 4\left|\langle\alpha|\hat{K}_3|\alpha\rangle\right|^2.$$

d. From part c, we can deduce that the squared field quadrature squeezing occurs if $\left\langle\left(\Delta\hat{K}_{1,2}\right)^2\right\rangle < 2\left\langle\hat{K}_3\right\rangle$.

e.

$$\left\langle\left(\Delta\hat{K}_{1,2}\right)^2\right\rangle = \left\langle\hat{K}_{1,2}^2\right\rangle - \left\langle\hat{K}_{1,2}\right\rangle^2.$$

$$\hat{K}_1^2 = \frac{1}{4}\left(\hat{a}^{\dagger 4} + \hat{a}^4 + 2\hat{a}^{\dagger 2}\hat{a}^2 + 4\hat{a}^\dagger\hat{a} + 2\right)$$

Schrödinger cat states are of the form

$$|\Psi(\theta)\rangle = \mathcal{N}\left[|\alpha\rangle + e^{i\theta}|-\alpha\rangle\right],$$

where

$$\mathcal{N} = \left[2 + 2e^{-2|\alpha|^2}\cos\theta\right].$$

$|\psi(0)\rangle$, $|\psi(\pi)\rangle$, and $|\psi(\pi/2)\rangle$ are even, odd and Yurke-Stoler states, respectively. To study the squared field squeezing we determine the following quantities:

$$\begin{aligned}
\left\langle \hat{K}_1 \right\rangle =& \frac{|\mathcal{N}|^2}{2}\left(\alpha^{*2}+\alpha^2\right)\left(2+2e^{-2|\alpha|^2}\cos\theta\right) \\
\left\langle \hat{K}_1^2 \right\rangle =& \frac{|\mathcal{N}|^2}{4}\left[\left(\alpha^{*4}+\alpha^4+2|\alpha|^4+2\right)\left(2+2e^{-2|\alpha|^2}\cos\theta\right)\right. \\
&\left. + 4|\alpha|^2\left(2-2e^{-2|\alpha|^2}\cos\theta\right)\right] \\
\left\langle \hat{K}_2 \right\rangle =& \frac{|\mathcal{N}|^2}{2i}\left(\alpha^{*2}-\alpha^2\right)\left(2+2e^{-2|\alpha|^2}\cos\theta\right) \\
\left\langle \hat{K}_2^2 \right\rangle =& -\frac{|\mathcal{N}|^2}{4}\left[\left(\alpha^{*4}+\alpha^4-2|\alpha|^4-2\right)\left(2+2e^{-2|\alpha|^2}\cos\theta\right)\right. \\
&\left. - 4|\alpha|^2\left(2-2e^{-2|\alpha|^2}\cos\theta\right)\right] \\
\left\langle \hat{K}_3 \right\rangle =& \frac{|\mathcal{N}|^2}{2}|\alpha|^2\left(2-2e^{-2|\alpha|^2}\cos\theta\right)+\frac{1}{4}.
\end{aligned}$$

It is easy to show that $\left\langle \hat{K}_1^2 \right\rangle - \left\langle \hat{K}_1 \right\rangle^2 - 2\left\langle \hat{K}_3 \right\rangle = 0 = \left\langle \hat{K}_2^2 \right\rangle - \left\langle \hat{K}_2 \right\rangle^2 - 2\left\langle \hat{K}_3 \right\rangle$. Thus, none of the states mentioned above is squared field squeezed.

7.11 Problem 7.11

For a coherent state $|\alpha\rangle$ we have

$$\begin{aligned}
\left\langle : (\Delta\hat{X})^2 : \right\rangle &= \langle\alpha| : (\Delta\hat{X})^2 : |\alpha\rangle \\
&= \langle\alpha| : \left(\hat{X} - \langle : \hat{X} : \rangle\right)^2 : |\alpha\rangle \\
&= \langle\alpha| : \hat{X}^2 : |\alpha\rangle - \langle\alpha| : \hat{X} : |\alpha\rangle^2 \\
&= \langle\alpha|\frac{1}{4}\left(\hat{a}^2e^{-2iv}+\hat{a}^{2\dagger}e^{2iv}+2\hat{a}^\dagger\hat{a}\right)|\alpha\rangle - \left[\langle\alpha|\frac{1}{2}\left(\hat{a}e^{-iv}+\hat{a}^\dagger e^{iv}\right)|\alpha\rangle\right]^2 \\
&= \frac{1}{4}\left(\alpha^2e^{-i2v}+\alpha^{2*}e^{i2v}+2|\alpha|^2\right) - \frac{1}{4}\left(\alpha e^{-iv}+\alpha^* e^{iv}\right)^2 \\
&= 0.
\end{aligned}$$

where we have used $\hat{X} = \frac{1}{2}(\hat{a}e^{-iv}+\hat{a}^\dagger e^{iv})$, $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and $: \hat{a}\hat{a}^\dagger : \ = \hat{a}^\dagger\hat{a}$. The generalization to $\left\langle : (\Delta\hat{X})^N : \right\rangle = 0$ is straightforward.

7.12 Problem 7.12

Equation (7.192) is of the form

$$\exp\left(y\Delta\hat{X}\right) =: \exp\left(y\Delta\hat{X}\right) : \exp(y^2/8).$$

The left hand side can be expanded as a

$$\left\langle \exp\left(y\Delta\hat{X}\right)\right\rangle = \sum_{N=0}^{\infty} \frac{y^N}{N!}\left\langle \left(\Delta\hat{X}\right)^N\right\rangle, \tag{7.12.1}$$

and the right hand side as

$$\begin{aligned}
:\left(y\Delta\hat{X}\right): \exp(y^2/8) &= \exp(y^2/8)\sum_{n=0}^{\infty}\left(\frac{y^n}{n!}\right):\left(\Delta\hat{X}\right)^n: \\
&= \sum_{n=0}^{\infty}\frac{(y^2/8)^m}{m!}\sum_{n=0}^{\infty}\left(\frac{y^n}{n!}\right):\left(\Delta\hat{X}\right)^n: \\
&= \sum_{n=0}^{\infty}\varnothing_m\frac{y^m}{2^{3m/2}\left(\frac{m}{2}\right)!}\sum_{n=0}^{\infty}\left(\frac{y^n}{n!}\right):\left(\Delta\hat{X}\right)^n: \\
&= \sum_{n=0}^{\infty}\sum_{n=0}^{\infty}\varnothing_m\frac{y^{(m+n)}}{2^{3m/2}\left(\frac{m}{2}\right)!n!}:\left(\Delta\hat{X}\right)^n: \qquad (7.12.2)
\end{aligned}$$

where the symbol $\varnothing_n$ is defined as

$$\varnothing_n = \begin{cases} 0 & \text{for } n \text{ odd} \\ 1 & \text{for } n \text{ even} \end{cases} \tag{7.12.3}$$

Using the following transformation identity

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{n,m} = \sum_{p=0}^{\infty}\sum_{q=0}^{p} a_{q,p-q}, \tag{7.12.4}$$

we can rewrite

$$\begin{aligned}
:\left(y\Delta\hat{X}\right): \exp(y^2/8) &= \sum_{p=0}^{\infty}\sum_{q=0}^{p}\varnothing_q\frac{y^p}{2^{3q/2}\left(\frac{q}{2}\right)!(p-q)!}:\left(\Delta\hat{X}\right)^{(p-q)}: \\
&= \sum_{N=0}^{\infty}\frac{y^p}{N!}\sum_{q=0}^{N}\varnothing_q\frac{N!}{2^{3q/2}\left(\frac{q}{2}\right)!(N-q)!}:\left(\Delta\hat{X}\right)^{(N-q)}:
\end{aligned}$$

Equating coefficients of like powers in equations7.12.1 and 7.12.2 we will have

$$\left\langle \left(y\Delta\hat{X} \right)^N \right\rangle = \sum_{q=0}^{N} \o_q \frac{N!}{2^{3q/2}\left(\frac{q}{2}\right)!(N-q)!} \left\langle : \left(\Delta\hat{X} \right)^{(N-q)} : \right\rangle .$$

Expanding this equation leads to Eq. (7.194).

7.13 Problem 7.13

Intrinsic N[th] order squeezing exist if $\left\langle : \left(\Delta\hat{X}\right)^N : \right\rangle < 0$ where $\left(\Delta\hat{X}\right)^N = \left(\hat{X} - \left\langle \Delta\hat{X} \right\rangle\right)^N$ and where $\hat{X} = \frac{1}{2}\left(\hat{a} + \hat{a}^\dagger\right)$. In terms of the P function we can write

$$\left\langle : \left(\Delta\hat{X}\right)^N : \right\rangle = \frac{1}{2^N} \int d^2\alpha P(\alpha) \left[\alpha + \alpha^* - \langle\hat{a}\rangle - \langle\hat{a}^\dagger\rangle\right]^N$$

To have the left hand side less than zero, with N even, $P(\alpha)$ must take on negative values in some region of phase space. Note that if N is odd, the left hand side could be negative even though $P(\alpha)$ is positive definite. Thus only for even N is higher order squeezing a non-classical effect.

7.14 Problem 7.14

The conditions for higher-order squeezing in a broadband field is obtained the same way we have obtained Equation (7.196), except a constant C must inserted in order to satisfy the inequality in Eq. (7.206). The rest follows exactly in the same fashion and leads

$$\left\langle \left(\Delta\hat{X}_i^{(C)}\right)^{2l} \right\rangle < (2l-1)!! \left(\frac{C}{4}\right)^l ,$$

where $i = 1$ or 2. Also notice that for the broadband case Eq. (7.194) must be adjusted to

$$\begin{aligned} \left\langle \left(\Delta\hat{X}_i^{(C)}\right)^N \right\rangle &= \left\langle : \left(\Delta\hat{X}_i^{(C)}\right)^N : \right\rangle + \frac{N^{(2)}}{1!}\left(\frac{C}{8}\right) \left\langle : \left(\Delta\hat{X}_i^{(C)}\right)^{N-2} : \right\rangle \\ &+ \frac{N^{(4)}}{1!}\left(\frac{C}{8}\right)^2 \left\langle : \left(\Delta\hat{X}_i^{(C)}\right)^{N-4} : \right\rangle + \cdots \\ &+ \begin{cases} (N-1)!! & N \text{ even}, \\ 1 & N \text{ odd}. \end{cases} \end{aligned}$$

7.15 Problem 7.15

Pair coherent state $|\eta, q\rangle$ is defined as

$$\hat{a}\hat{b}|\eta, q\rangle = \eta|\eta, q\rangle \tag{7.15.1}$$

$$\left(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\right)|\eta, q\rangle = q|\eta, q\rangle \tag{7.15.2}$$

In general we can expand pair coherent state as any two-mode state as

$$|\eta, q\rangle = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_{m,n}|m, n\rangle.$$

Eq. 7.15.2 can be written now as

$$\begin{aligned}
\left(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\right)|\eta, q\rangle &= q|\eta, q\rangle \\
\left(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b}\right)\sum_{m=0}^{\infty}\sum_{m=0}^{\infty} c_{m,n}|m, n\rangle &= \sum_{m=0}^{\infty}\sum_{m=0}^{\infty} qc_{m,n}|m, n\rangle \\
\sum_{m=0}^{\infty}\sum_{m=0}^{\infty}(m-n)c_{m,n}|m, n\rangle &= \sum_{m=0}^{\infty}\sum_{m=0}^{\infty} qc_{m,n}|m, n\rangle.
\end{aligned}$$

Obviously from the above equality we infer that $m - n = q$, so $c_{m,n}$ depends only on m and q. That why we will drop the n subscript and we write the pair coherent state as

$$|\eta, q\rangle = \sum_{m=0}^{\infty} c_n|n+q, n\rangle. \tag{7.15.3}$$

From equation 7.15.1we have

$$\begin{aligned}
\hat{a}\hat{b}|\eta, q\rangle &= \eta|\eta, q\rangle \\
\sum_{n=0}^{\infty} c_n\hat{a}\hat{b}|n, n+q\rangle &= \sum_{n=0}^{\infty} c_n\eta|n, n+q\rangle \\
\sum_{n=1}^{\infty} c_n\sqrt{n(n+q)}|n-1, n+q-1\rangle &= \sum_{n=0}^{\infty} c_n\eta|n, n+q\rangle \\
\sum_{n=0}^{\infty} c_{n+1}\sqrt{(n+1)(n+q+1)}|n, n+q\rangle &= \sum_{n=0}^{\infty} c_n\eta|n, n+q\rangle.
\end{aligned}$$

The last equality leads to

$$c_n = c_{n-1}\frac{\eta}{\sqrt{n(n+q)}} = \cdots = c_0\frac{\eta^n\sqrt{q!}}{\sqrt{n!(n+q)!}},$$

so we have

$$|\eta, q\rangle = \sum_{n=0}^{\infty} c_0\frac{\eta^n\sqrt{q!}}{\sqrt{n!(n+q)!}}|n, n+q\rangle.$$

$$\sum_{n=0}^{\infty}|c_0|^2\frac{|\eta|^{2n}q!}{n!(n+q)!} = |c_0|^2 q!|\eta|^{-q}I_q(2|\eta|) = 1 \tag{7.15.4}$$

$$c_0 = \sqrt{\frac{|\eta|^q}{q!I_q(2|\eta|)}} \tag{7.15.5}$$

$$|\eta, q\rangle = \sqrt{\frac{|\eta|^q}{I_q(2|\eta|)}}\sum_{n=0}^{\infty}\frac{\eta^n}{\sqrt{n!(n+q)!}}|n, n+q\rangle.$$

7.16 Problem 7.16

Two-mode squeezed vacuum states

$$|\xi\rangle_2 = \hat{S}_2(\xi)|0,0\rangle$$

$$\begin{aligned}
\left\langle \hat{a}^{\dagger 2}\hat{a}^2\right\rangle &= {}_2\langle\xi|\hat{a}^{\dagger 2}\hat{a}^2|\xi\rangle_2 \\
&= \langle 0,0|\hat{S}_2^\dagger(\xi)\hat{a}^{\dagger 2}\hat{a}^2\hat{S}_2(\xi)|0,0\rangle \\
&= \langle 0,0|\hat{S}_2^\dagger(\xi)\hat{a}^{\dagger 2}\hat{S}_2(\xi)\hat{S}_2^\dagger(\xi)\hat{a}^2\hat{S}_2(\xi)|0,0\rangle \\
&= \langle 0,0|\left(\hat{a}^\dagger\cosh r - e^{-i\theta}\hat{b}\sinh r\right)^2\left(\hat{a}\cosh r - e^{i\theta}\hat{b}^\dagger\sinh r\right)^2|0,0\rangle \\
&= \sinh^2 r\langle 0,1|\left(\hat{a}^\dagger\cosh r - e^{-i\theta}\hat{b}\sinh r\right)\left(\hat{a}\cosh r - e^{i\theta}\hat{b}^\dagger\sinh r\right)|0,1\rangle \\
&= 2\sinh^4 r,
\end{aligned}$$

where we have used $\hat{S}_2(\xi)\hat{S}_2^\dagger(\xi) = I$ and $\hat{S}_2(\xi)\hat{a}\hat{S}_2^\dagger(\xi) = \hat{a}^\dagger \cosh r - e^{-i\theta}\hat{b}\sinh r$.

$$
\begin{aligned}
\left\langle \hat{b}^{\dagger 2}\hat{b}^2 \right\rangle &= {}_2\langle\xi|\hat{b}^{\dagger 2}\hat{b}^2|\xi\rangle_2 \\
&= \langle 0,0|\hat{S}_2^\dagger(\xi)\hat{b}^{\dagger 2}\hat{b}^2\hat{S}_2(\xi)|0,0\rangle \\
&= \langle 0,0|\hat{S}_2^\dagger(\xi)\hat{b}^{\dagger 2}\hat{S}_2(\xi)\hat{S}_2^\dagger(\xi)\hat{b}^2\hat{S}_2(\xi)|0,0\rangle \\
&= \langle 0,0|\left(\hat{b}^\dagger \cosh r - e^{-i\theta}\hat{a}\sinh r\right)^2\left(\hat{b}\cosh r - e^{i\theta}\hat{a}^\dagger \sinh r\right)^2|0,0\rangle \\
&= \sinh^2 r\langle 1,0|\left(\hat{b}^\dagger \cosh r - e^{-i\theta}\hat{a}\sinh r\right)\left(\hat{b}\cosh r - e^{i\theta}\hat{a}^\dagger \sinh r\right)|1,0\rangle \\
&= 2\sinh^4 r.
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b} \right\rangle &= {}_2\langle\xi|\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}|\xi\rangle_2 \\
&= \langle 0,0|\hat{S}_2^\dagger(\xi)\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\hat{S}_2(\xi)|0,0\rangle \\
&= \langle 0,0|\hat{S}_2^\dagger(\xi)\hat{a}^\dagger\hat{S}_2(\xi)\hat{S}_2^\dagger(\xi)\hat{a}\hat{b}^\dagger\hat{S}_2(\xi)\hat{S}_2^\dagger(\xi)\hat{b}\hat{S}_2(\xi)|0,0\rangle \\
&= \langle 0,0|\left(\hat{a}^\dagger \cosh r - e^{-i\theta}\hat{b}\sinh r\right)\hat{S}_2^\dagger(\xi)\hat{a}\hat{b}^\dagger\hat{S}_2(\xi)\left(\hat{b}\cosh r - e^{i\theta}\hat{a}^\dagger \sinh r\right)|0,0\rangle \\
&= \sinh^2 r\langle 0,1|\hat{S}_2^\dagger(\xi)\hat{a}\hat{S}_2(\xi)\hat{S}_2^\dagger(\xi)\hat{b}^\dagger\hat{S}_2(\xi)|1,0\rangle \\
&= \sinh^2 r\langle 0,1|\left(\hat{a}\cosh r - e^{i\theta}\hat{b}^\dagger \sinh r\right)\left(\hat{b}^\dagger \cosh r - e^{-i\theta}\hat{a}\sinh r\right)|1,0\rangle \\
&= \sinh^2 r\left(\sinh^2 r + \cosh^2 r\right)
\end{aligned}
$$

The inequality

$$\left\langle \hat{a}^{\dagger 2}\hat{a}^2 \right\rangle \left\langle \hat{b}^{\dagger 2}\hat{b}^2 \right\rangle \geq \left\langle \hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b} \right\rangle$$

is violated for $r = 0.7$ for example.

For a pair coherent state we have

$$
\begin{aligned}
|\eta, q\rangle &= \sqrt{\frac{|\eta|^q}{I_q(2|\eta|)}} \sum_{n=0}^{\infty} \frac{\eta^n}{\sqrt{n!(n+q)!}} |n, n+q\rangle \\
\hat{a}^2 |\eta, q\rangle &= \sqrt{\frac{|\eta|^q}{I_q(2|\eta|)}} \sum_{n=2}^{\infty} \frac{\eta^n \sqrt{n(n-1)}}{\sqrt{n!(n+q)!}} |n, n+q\rangle \\
\langle \eta, q | \hat{a}^{\dagger 2} \hat{a}^2 | \eta, q\rangle &= \frac{|\eta|^q}{I_q(2|\eta|)} \sum_{n=2}^{\infty} \frac{|\eta|^{2n} n(n-1)}{n!(n+q)!} \\
\langle \eta, q | \hat{b}^{\dagger 2} \hat{b}^2 | \eta, q\rangle &= \frac{|\eta|^q}{I_q(2|\eta|)} \sum_{n=0}^{\infty} \frac{|\eta|^{2n} (n+q)(n+q-1)}{n!(n+q)!} \\
&= \frac{|\eta|^q}{I_q(2|\eta|)} \sum_{n=0}^{\infty} \frac{|\eta|^{2n}}{n!(n+q-2)!} \\
\langle \eta, q | \hat{a}^{\dagger} \hat{a} \hat{b}^{\dagger} \hat{b} | \eta, q\rangle &= \frac{|\eta|^q}{I_q(2|\eta|)} \sum_{n=0}^{\infty} \frac{|\eta|^{2n} n(n+q)}{n!(n+q)!}
\end{aligned}
$$

Again the inequality

$$
\left\langle \hat{a}^{\dagger 2} \hat{a}^2 \right\rangle \left\langle \hat{b}^{\dagger 2} \hat{b}^2 \right\rangle \geq \left\langle \hat{a}^{\dagger} \hat{a} \hat{b}^{\dagger} \hat{b} \right\rangle
$$

is violated, for example $|\eta| = 0.7$ and $q = 1$.

7.17 Problem 7.17

$$
\begin{aligned}
|\eta, q\rangle &= \sqrt{\frac{|\eta|^q}{I_q(2|\eta|)}} \sum_{n=0}^{\infty} \frac{\eta^n}{\sqrt{n!(n+q)!}} |n, n+q\rangle \\
\hat{\rho} &= \frac{|\eta|^q}{I_q(2|\eta|)} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{\eta^n \eta^{*n'}}{\sqrt{n!(n+q)!n'!(n'+q)!}} |n, n+q\rangle \langle n', n'+q|
\end{aligned}
$$

$$
\begin{aligned}
\hat{\rho}_b &= \mathrm{Tr}_a \hat{\rho} \\
&= \sum_{m=0}^{\infty} {}_b\langle m|\hat{\rho}|m\rangle_a \\
&= \frac{|\eta|^q}{I_q(2|\eta|)} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{n'=0}^{\infty} \frac{\eta^n \eta^{*n'}}{\sqrt{n!(n+q)!n'!(n'+q)!}} {}_a\langle m|n, n+q\rangle\langle n', n+q|m\rangle_a \\
&= \frac{|\eta|^q}{I_q(2|\eta|)} \sum_{n=0}^{\infty} \frac{|\eta|^{2n}}{n!(n+q)!} |n+q\rangle_{bb}\langle n+q|.
\end{aligned}
$$

Since $\hat{\rho}_b$ is diagonalized, the von Neumann entropy is easily found to be

$$
\begin{aligned}
S(\hat{\rho}_b) &= -\mathrm{Tr}[\hat{\rho}_b \ln \hat{\rho}_b] \\
&= -\sum_k (\rho_b)_{kk} \ln(\rho_b)_{kk} \\
&= -\frac{|\eta|^q}{I_q(2|\eta|)} \sum_n \frac{|\eta|^{2n}}{n!(n+q)!} \ln\left(\frac{|\eta|^{2n+q}}{I_q(2|\eta|)n!(n+q)!}\right).
\end{aligned}
$$

7.18 Problem 7.18

$$
\begin{aligned}
|\mathrm{in}\rangle &= |\alpha\rangle_a |\xi\rangle_b \\
&= \hat{D}(\alpha)\hat{S}(\xi)|0\rangle \\
|\mathrm{out}\rangle &= \hat{U}_{\mathrm{MZI}}|\mathrm{in}\rangle \\
\hat{U}_{\mathrm{MZI}} &= \hat{U}_{\mathrm{BS2}}\hat{U}_{\mathrm{PS}}\hat{U}_{\mathrm{BS1}} \\
\hat{U}_{\mathrm{BS1}} &= e^{-i\pi \hat{J}_x/2} \\
\hat{U}_{\mathrm{BS2}} &= e^{-i\pi \hat{J}_x/2} \\
\hat{U}_{\mathrm{PS}} &= e^{-i\phi \hat{J}_z}
\end{aligned}
$$

$$\begin{aligned}
\left\langle \hat{J}_z \right\rangle &= \left\langle \text{out} \left| \hat{J}_z \right| \text{out} \right\rangle \\
&= \left\langle \text{in} \left| \hat{U}^\dagger_{\text{BS1}} \hat{U}^\dagger_{\text{PS}} \hat{U}^\dagger_{\text{BS2}} \hat{J}_z \hat{U}_{\text{BS2}} \hat{U}_{\text{PS}} \hat{U}_{\text{BS1}} \right| \text{in} \right\rangle \\
&= \left\langle \text{in} \left| e^{i\pi \hat{J}_x/2} e^{i\phi \hat{J}_z} e^{i\pi \hat{J}_x/2} \hat{J}_z e^{-i\pi \hat{J}_x/2} e^{-i\phi \hat{J}_z} e^{-i\pi \hat{J}_x/2} \right| \text{in} \right\rangle \\
&= \left\langle \text{in} \left| e^{i\pi \hat{J}_x/2} e^{i\phi \hat{J}_z} \hat{J}_y e^{-i\phi \hat{J}_z} e^{-i\pi \hat{J}_x/2} \right| \text{in} \right\rangle \\
&= \left\langle \text{in} \left| e^{i\pi \hat{J}_x/2} \left(-\sin\phi \hat{J}_x + \cos\phi \hat{J}_z \right) e^{-i\pi \hat{J}_x/2} \right| \text{in} \right\rangle \\
&= \left\langle \text{in} \left| \left(-\sin\phi \hat{J}_x + \cos\phi \hat{J}_z \right) \right| \text{in} \right\rangle \\
&= -\sin\phi \left\langle \text{in} \left| \hat{J}_x \right| \text{in} \right\rangle + \cos\phi \left\langle \text{in} \left| \hat{J}_z \right| \text{in} \right\rangle
\end{aligned}$$

$$\begin{aligned}
\left\langle \hat{J}_z^2 \right\rangle &= \left\langle \text{out} \left| \hat{J}_z^2 \right| \text{out} \right\rangle \\
&= \left\langle \text{in} \left| \hat{U}^\dagger_{\text{BS1}} \hat{U}^\dagger_{\text{PS}} \hat{U}^\dagger_{\text{BS2}} \hat{J}_z \hat{U}_{\text{BS2}} \hat{U}_{\text{PS}} \hat{U}_{\text{BS1}} \hat{U}^\dagger_{\text{BS1}} \hat{U}^\dagger_{\text{PS}} \hat{U}^\dagger_{\text{BS2}} \hat{J}_z \hat{U}_{\text{BS2}} \hat{U}_{\text{PS}} \hat{U}_{\text{BS1}} \right| \text{in} \right\rangle \\
&= \left\langle \text{in} \left| \left(-\sin\phi \hat{J}_x + \cos\phi \hat{J}_z \right)^2 \right| \text{in} \right\rangle \\
&= \left\langle \text{in} \left| \sin^2\phi \hat{J}_x^2 + \cos^2\phi \hat{J}_z^2 - \cos\phi \sin\phi \left(\hat{J}_x \hat{J}_z + \hat{J}_z \hat{J}_x \right) \right| \text{in} \right\rangle
\end{aligned}$$

$$\begin{aligned}
\left\langle \text{in} \left| \hat{J}_x \right| \text{in} \right\rangle &= \frac{1}{2} \left\langle \text{in} \left| \hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger \right| \text{in} \right\rangle \\
&= \frac{1}{2} \left\langle 0 \left| \hat{S}^\dagger(\xi) \hat{D}^\dagger(\alpha) \left(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger \right) \hat{S}(\xi) \hat{D}(\alpha) \right| 0 \right\rangle \\
&= \frac{1}{2} \left\langle 0 \left| \left((\hat{a}^\dagger + \alpha^*)(\cosh r \hat{b} + e^{i\varphi} \sinh r \hat{b}^\dagger) \right. \right. \right. \\
&\quad \left. \left. \left. + (\hat{a} + \alpha)(\cosh r \hat{b}^\dagger + e^{-i\varphi} \sinh r \hat{b}) \right) \right| 0 \right\rangle \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\left\langle \text{in} \left| \hat{J}_x^2 \right| \text{in} \right\rangle &= \frac{1}{4} \left\langle \text{in} \left| \left(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger \right) \left(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger \right) \right| \text{in} \right\rangle \\
&= \frac{1}{4} \left\langle 0 \left| \hat{S}^\dagger(\xi) \hat{D}^\dagger(\alpha) \left(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger \right) \left(\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger \right) \hat{S}(\xi) \hat{D}(\alpha) \right| 0 \right\rangle \\
&= \frac{1}{4} \left\langle 0 \left| \Big((\hat{a}^\dagger + \alpha^*)(\cosh r \hat{b} + e^{i\varphi} \sinh r \hat{b}^\dagger) + (\hat{a} + \alpha)(\cosh r \hat{b}^\dagger + e^{-i\varphi} \sinh r \hat{b}) \Big) \right. \right. \\
&\quad \times \left. \left. \Big((\hat{a}^\dagger + \alpha^*)(\cosh r \hat{b} + e^{i\varphi} \sinh r \hat{b}^\dagger) + (\hat{a} + \alpha)(\cosh r \hat{b}^\dagger + e^{-i\varphi} \sinh r \hat{b}) \Big) \right| 0 \right\rangle \\
&= \frac{1}{4} \left\langle 0 \left| \Big(\alpha^* \cosh r \hat{b} + (\hat{a} + \alpha) e^{-i\varphi} \sinh r \hat{b} \Big) \right. \right. \\
&\times \left. \left. \Big((\hat{a}^\dagger + \alpha^*) e^{i\varphi} \sinh r \hat{b}^\dagger + \alpha \cosh r \hat{b}^\dagger \Big) \right| 0 \right\rangle \\
&= \frac{1}{4} \left(|\alpha|^2 \cosh^2 r + (1 + |\alpha|^2) \sinh^2 r \right) \\
&= \frac{1}{4} \left[|\alpha|^2 \left(\cosh^2 r + \sinh^2 r \right) + \sinh^2 r \right]
\end{aligned}$$

$$\begin{aligned}
\left\langle \text{in} \left| \hat{J}_z \right| \text{in} \right\rangle &= \frac{1}{2} \left\langle \text{in} \left| \hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \right| \text{in} \right\rangle \\
&= \frac{1}{2} \left\langle 0 \left| \hat{S}^\dagger(\xi) \hat{D}^\dagger(\alpha) \left(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \right) \hat{S}(\xi) \hat{D}(\alpha) \right| 0 \right\rangle \\
&= \frac{1}{2} \left\langle 0 \left| \left((\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) \right) \right| 0 \right\rangle \\
&\quad - \frac{1}{2} \left\langle 0 \left| \Big((\cosh r \hat{b}^\dagger + e^{-i\varphi} \sinh r \hat{b})(\cosh r \hat{b} + e^{i\varphi} \sinh r \hat{b}^\dagger) \Big) \right| 0 \right\rangle \\
&= \frac{1}{2} \left(|\alpha|^2 - \sinh^2 r \right)
\end{aligned}$$

$$
\begin{aligned}
\left\langle \text{in} \left| \hat{J}_z^2 \right| \text{in} \right\rangle &= \frac{1}{4} \left\langle \text{in} \left| \left(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \right)^2 \right| \text{in} \right\rangle \\
&= \frac{1}{4} \left\langle 0 \left| \hat{S}^\dagger(\xi) \hat{D}^\dagger(\alpha) \left(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \right)^2 \hat{S}(\xi) \hat{D}(\alpha) \right| 0 \right\rangle \\
&= \frac{1}{4} \langle 0| \Big((\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) - (\cosh r \hat{b}^\dagger + e^{-i\varphi} \sinh r \hat{b})(\cosh r \hat{b} + e^{i\varphi} \sinh r \hat{b}^\dagger) \Big) \\
&\times \Big((\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) - (\cosh r \hat{b}^\dagger + e^{-i\varphi} \sinh r \hat{b})(\cosh r \hat{b} + e^{i\varphi} \sinh r \hat{b}^\dagger) \Big) |0\rangle \\
&= \frac{1}{4} \langle 0| \Big(\alpha^*(\hat{a} + \alpha) - e^{-i\varphi} \sinh r \hat{b}(\cosh r \hat{b} + e^{i\varphi} \sinh r \hat{b}^\dagger) \Big) \\
&\times \Big((\hat{a}^\dagger + \alpha^*)\alpha - (\cosh r \hat{b}^\dagger + e^{-i\varphi} \sinh r \hat{b}) e^{i\varphi} \sinh r \hat{b}^\dagger \Big) |0\rangle \\
&= \frac{1}{4} \left(|\alpha|^4 + |\alpha|^2 (1 - \sinh^2 r) + 2 \sinh^2 r \cosh^2 r + \sinh^4 r \right)
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \text{in} \left| \hat{J}_x \hat{J}_z \right| \text{in} \right\rangle &= \frac{1}{4} \left\langle \text{in} \left| \left(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger \right) \left(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \right) \right| \text{in} \right\rangle \\
&= \frac{1}{4} \left\langle 0 \left| \hat{S}^\dagger(\xi) \hat{D}^\dagger \left(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger \right) \left(\hat{a}^\dagger \hat{a} - \hat{b}^\dagger \hat{b} \right) \hat{S}(\xi) \hat{D}(\alpha) \right| 0 \right\rangle \\
&= \frac{1}{4} \langle 0| \Big((\hat{a}^\dagger + \alpha^*)(\cosh r \hat{b} + e^{i\varphi} \sinh r \hat{b}^\dagger) + (\hat{a} + \alpha)(\cosh r \hat{b}^\dagger + e^{-i\varphi} \sinh r \hat{b}) \Big) \\
&\times \Big((\hat{a}^\dagger + \alpha^*)(\hat{a} + \alpha) - (\cosh r \hat{b}^\dagger + e^{-i\varphi} \sinh r \hat{b})(\cosh r \hat{b} + e^{i\varphi} \sinh r \hat{b}^\dagger) \Big) |0\rangle \\
&= \frac{1}{4} \langle 0| \Big(\alpha^* \cosh r \hat{b} + e^{-i\varphi} \sinh r (\hat{a} + \alpha) \hat{b} \Big) \\
&\times \Big(\alpha(\hat{a}^\dagger + \alpha^*) - (\cosh r \hat{b}^\dagger + e^{-i\varphi} \sinh r \hat{b}) e^{i\varphi} \sinh r \hat{b}^\dagger \Big) |0\rangle \\
&= 0
\end{aligned}
$$

$$
\left\langle \text{in} \left| \hat{J}_x \hat{J}_z \right| \text{in} \right\rangle = \left\langle \text{in} \left| \hat{J}_z \hat{J}_x \right| \text{in} \right\rangle = 0
$$

$$\begin{aligned}
\left\langle\left(\Delta\hat{J}_z\right)^2\right\rangle &= \left\langle\hat{J}_z^2\right\rangle - \left\langle\hat{J}_z\right\rangle^2 \\
&= \left\langle \text{in}\left|\sin^2\phi\hat{J}_x^2 + \cos^2\phi\hat{J}_z^2 - \cos\phi\sin\phi\left(\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x\right)\right|\text{in}\right\rangle \\
&\quad - \left(-\sin\phi\left\langle \text{in}\left|\hat{J}_x\right|\text{in}\right\rangle + \cos\phi\left\langle \text{in}\left|\hat{J}_z\right|\text{in}\right\rangle\right)^2 \\
&= \sin^2\phi\left\langle \text{in}\left|\hat{J}_x^2\right|\text{in}\right\rangle + \cos^2\phi\left\langle \text{in}\left|\hat{J}_z^2\right|\text{in}\right\rangle - \cos^2\phi\left\langle \text{in}\left|\hat{J}_z\right|\text{in}\right\rangle^2 \\
&= \frac{1}{4}\left[\sin^2\phi\left(|\alpha|^2(\cosh^2 r + \sinh^2 r) + \sinh^2 r\right) + \cos^2\phi\left(|\alpha|^2 2\cosh^2 r\sinh^2 r\right)\right]
\end{aligned}$$

For $|\alpha|^2 \gg \sinh^2 r$ and $\theta \to \pi/2$ we have

$$\begin{aligned}
\Delta\phi &= \sqrt{(\Delta\hat{J}_z)^2}/\left|\partial\langle\hat{J}_z\rangle/\partial\phi\right| \\
&= e^{-r}/\sqrt{|\alpha|^2}
\end{aligned}$$

7.19 Problem 7.19

$$|\text{in}\rangle = \mathcal{N}|\alpha\rangle_a(|\beta\rangle_b \pm |-\beta\rangle_b)$$

Where

$$\mathcal{N} = \frac{1}{\sqrt{2}}\left(1 \pm e^{-2|\beta|^2}\right)^{-1/2}$$

$$\begin{aligned}
|\text{out}\rangle &= \hat{U}_{\text{MZI}}|0\rangle \\
\hat{U}_{\text{MZI}} &= \hat{U}_{\text{BS2}}\hat{U}_{\text{PS}}\hat{U}_{\text{BS1}} \\
\hat{U}_{\text{BS1}} &= e^{-i\pi\hat{J}_x/2} \\
\hat{U}_{\text{BS2}} &= e^{-i\pi\hat{J}_x/2} \\
\hat{U}_{\text{PS}} &= e^{-i\phi\hat{J}_z}
\end{aligned}$$

From the previous problem

$$\begin{aligned}
\left\langle\hat{J}_z\right\rangle &= -\sin\phi\left\langle \text{in}\left|\hat{J}_x\right|\text{in}\right\rangle + \cos\phi\left\langle \text{in}\left|\hat{J}_z\right|\text{in}\right\rangle \\
\left\langle\hat{J}_z^2\right\rangle &= \left\langle \text{in}\left|\sin^2\phi\hat{J}_x^2 + \cos^2\phi\hat{J}_z^2 - \cos\phi\sin\phi\left(\hat{J}_x\hat{J}_z + \hat{J}_z\hat{J}_x\right)\right|\text{in}\right\rangle
\end{aligned}$$

$$\begin{aligned}\left\langle \mathrm{in} \left| \hat{J}_x \right| \mathrm{in} \right\rangle &= \frac{|\mathcal{N}|^2}{2} \langle\alpha|(\langle\beta| \pm \langle-\beta|)[\hat{a}^\dagger \hat{b} + \hat{a}\hat{b}^\dagger]|\alpha\rangle(|\beta\rangle \pm |-\beta\rangle) \\ &= \frac{|\mathcal{N}|^2}{2} \langle\alpha|(\langle\beta| \pm \langle-\beta|) \left[\beta\hat{a}^\dagger|\alpha\rangle(|\beta\rangle \mp |-\beta\rangle) + \alpha\hat{b}^\dagger|\alpha\rangle(|\beta\rangle \pm |-\beta\rangle)\right] \\ &= 0\end{aligned}$$

because $(\langle\beta| \pm \langle-\beta|)(|\beta\rangle \mp |-\beta\rangle) = 0$.

$$\begin{aligned}\left\langle \mathrm{in} \left| \hat{J}_z \right| \mathrm{in} \right\rangle &= \frac{|\mathcal{N}|^2}{2} \langle\alpha|(\langle\beta| \pm \langle-\beta|)[\hat{a}^\dagger \hat{a} + \hat{b}^\dagger\hat{b}]|\alpha\rangle(|\beta\rangle \pm |-\beta\rangle) \\ &= \left(|\alpha|^2 - |\beta|^2\right)/2\end{aligned}$$

$$\begin{aligned}\left\langle \mathrm{in} \left| \hat{J}_x^2 \right| \mathrm{in} \right\rangle &= \frac{|\mathcal{N}|^2}{4} \\ &\times \langle\alpha|(\langle\beta| \pm \langle-\beta|) \left[\hat{a}^\dagger\hat{a}^\dagger\hat{b}\hat{b} + \hat{b}^\dagger\hat{b}^\dagger\hat{a}\hat{a} + \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} + 2\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\right] |\alpha\rangle(|\beta\rangle \pm |-\beta\rangle) \\ &= \left(|\alpha|^2 + |\beta|^2 + 2|\alpha|^2|\beta|^2 + \alpha^2\beta^{*2} + \alpha^{*2}\beta^2\right)/4\end{aligned}$$

$$\begin{aligned}\left\langle \mathrm{in} \left| \hat{J}_z^2 \right| \mathrm{in} \right\rangle &= \frac{|\mathcal{N}|^2}{4} \\ &\times \langle\alpha|(\langle\beta| \pm \langle-\beta|) \left[\hat{a}^\dagger\hat{a}^\dagger\hat{a}\hat{a} + \hat{b}^\dagger\hat{b}^\dagger\hat{b}\hat{b} + \hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b} - 2\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}\right] |\alpha\rangle(|\beta\rangle \pm |-\beta\rangle) \\ &= \left(|\alpha|^2 + |\beta|^2 + |\alpha|^4 + |\beta|^4 - 2|\alpha|^2|\beta|^2\right)/4\end{aligned}$$

$$\left\langle \mathrm{in} \left| \hat{J}_x \hat{J}_z \right| \mathrm{in} \right\rangle = \left\langle \mathrm{in} \left| \hat{J}_z \hat{J}_x \right| \mathrm{in} \right\rangle = 0$$

$$\begin{aligned}\left\langle \left(\Delta\hat{J}_z\right)^2 \right\rangle &= \left\langle \hat{J}_z^2 \right\rangle - \left\langle \hat{J}_z \right\rangle^2 \\ &= \sin^2\phi \left(|\alpha|^2 + |\beta|^2 + 2|\alpha|^2|\beta|^2 + \alpha^2\beta^{*2} + \alpha^{*2}\beta^2\right)/4 \\ &+ \cos^2\phi \left(|\alpha|^2 + |\beta|^2 + |\alpha|^4 + |\beta|^4 - 2|\alpha|^2|\beta|^2\right)/4 - \cos^2\phi \left(|\alpha|^2 - |\beta|^2\right)^2/4 \\ &= \left[|\alpha|^2 + |\beta|^2 + (\alpha\beta^* + \alpha^*\beta)^2 \sin^2\phi\right]/4\end{aligned}$$

$$\begin{aligned}\Delta\phi &= \frac{\Delta J_z}{\left|\partial \left\langle \hat{J}_z \right\rangle / \partial\phi\right|} \\ &= \frac{\sqrt{|\alpha|^2 + |\beta|^2 + (\alpha\beta^* + \alpha^*\beta)^2 \sin^2\phi}}{(|\alpha|^2 - |\beta|^2)\,|\sin\phi|}.\end{aligned}$$

For $\beta = 0$ we regain the standard quantum limit.

7.20 Problem 7.20

$$\hat{H} = \hbar\omega_p \hat{a}^\dagger \hat{a} + \hbar\omega_b \hat{b}^\dagger \hat{b} + \hbar\omega_c \hat{c}^\dagger \hat{c} + i\hbar\chi^{(2)}\left(\hat{a}\hat{b}\hat{c}^\dagger - \hat{a}^\dagger\hat{b}^\dagger\hat{c}\right)$$

a. Using the parametric approximation, assuming that the pump field to be a strong coherent state of the form $|\gamma e^{-i\omega_p t}\rangle$, we rewrite the hamiltonian as

$$\hat{H}^{(PA)} = \hbar\omega_p \hat{a}^\dagger \hat{a} + \hbar\omega_b \hat{b}^\dagger \hat{b} + \hbar\omega_c \hat{c}^\dagger \hat{c} + i\hbar\left(\eta\hat{b}\hat{c}^\dagger e^{-i\omega_p t} - \eta^*\hat{b}^\dagger\hat{c}e^{i\omega_p t}\right).$$

Given that $\omega_p = \omega_b - \omega_c = 0$ for $\omega_b = \omega_c$, the interaction picture Hamiltonian has this expression

$$\hat{H}_I = -i\hbar\left(\eta\hat{b}^\dagger\hat{c} - \eta^*\hat{b}\hat{c}^\dagger\right),$$

where

$$\eta = \chi^{(2)}\gamma.$$

b. For simplicity let assume that η is real, so $\eta = \eta^*$. The evolution operator is then

$$\begin{aligned}
\hat{U}_{fc} &= \exp\left(-i\hat{H}_I t/\hbar\right) \\
&= \exp\left(t\eta(\hat{b}^\dagger\hat{c} - \hat{b}\hat{c}^\dagger)\right) \\
&= \exp\left(i2t\eta\hat{J}_2\right)
\end{aligned}$$

Given that

$$\hat{b}(0) = \hat{b},$$

$$\left[\hat{J}_2, \hat{b}\right] = i\frac{\hat{c}}{2},$$

and

$$\left[\hat{J}_2, \left[\hat{J}_2, \hat{b}\right]\right] = \frac{\hat{b}}{4},$$

we will have

$$
\begin{aligned}
\hat{b}(t) &= \hat{U}_{fc}\hat{b}\hat{U}_{fc}^{\dagger} \\
&= e^{i2t\eta\hat{J}_2}\hat{b}e^{-i2t\eta\hat{J}_2} \\
&= \hat{b} + i2t\eta\left[\hat{J}_2, \hat{b}\right] + \frac{(i2t\eta)2}{2!}\left[\hat{J}_2, \left[\hat{J}_2, \hat{b}\right]\right] + \cdots \\
&= \cos(2\eta t)\hat{b} + i\sin(2\eta t)\hat{c}.
\end{aligned}
$$

Using the same procedure, and using

$$\hat{c}(0) = \hat{c},$$

$$\left[\hat{J}_2, \hat{c}\right] = -i\frac{\hat{b}}{2},$$

and

$$\left[\hat{J}_2, \left[\hat{J}_2, \hat{c}\right]\right] = \frac{\hat{c}}{4},$$

we will have

$$
\begin{aligned}
\hat{c}(t) &= \hat{U}_{fc}\hat{c}\hat{U}_{fc}^{\dagger} \\
&= e^{i2t\eta\hat{J}_2}\hat{c}e^{-i2t\eta\hat{J}_2} \\
&= \hat{c} + i2t\eta\left[\hat{J}_2, \hat{c}\right] + \frac{(i2t\eta)^2}{2!}\left[\hat{J}_2, \left[\hat{J}_2, \hat{c}\right]\right] + \cdots \\
&= \cos(2\eta t)\hat{b} - i\sin(2\eta t)\hat{c}.
\end{aligned}
$$

$$
\begin{aligned}
\hat{U}_{fc}(t)|0\rangle_b|N\rangle_c &= \hat{U}_{fc}(t)\frac{\hat{c}^{\dagger N}}{\sqrt{N!}}|0\rangle_b|0\rangle_c \\
&= \hat{U}_{fc}(t)\frac{\hat{c}^{\dagger N}}{\sqrt{N!}}\hat{U}^{\dagger}_{fc}(t)\hat{U}_{fc}(t)|0\rangle_b|0\rangle_c \\
&= \frac{\hat{c}^{\dagger N}(t)}{\sqrt{N!}}|0\rangle_b|0\rangle_c \\
&= \frac{1}{\sqrt{N!}}\left(\cos(2\eta t)\hat{b}^{\dagger} + i\sin(2\eta t)\hat{c}^{\dagger}\right)^N|0\rangle_b|0\rangle_c \\
&= \frac{1}{\sqrt{N!}}\sum_{q=0}^{N} i^{N-q}\begin{pmatrix} N \\ q \end{pmatrix}\cos^q(2\eta t)\hat{b}^{\dagger q}\sin^{N-q}(2\eta t)\hat{c}^{\dagger(N-q)}|0\rangle_b|0\rangle_c \\
&= \frac{1}{\sqrt{N!}}\sum_{q=0}^{N} i^{N-q}\begin{pmatrix} N \\ q \end{pmatrix}\cos^q(2\eta t)\sin^{N-q}(2\eta t)\sqrt{q!(N-q)!}|q\rangle_b|N-q\rangle_c \\
&= \sum_{q=0}^{N} i^{N-q}\sqrt{\begin{pmatrix} N \\ q \end{pmatrix}}\cos^q(2\eta t)\sin^{N-q}(2\eta t)|q\rangle_b|N-q\rangle_c.
\end{aligned}
$$

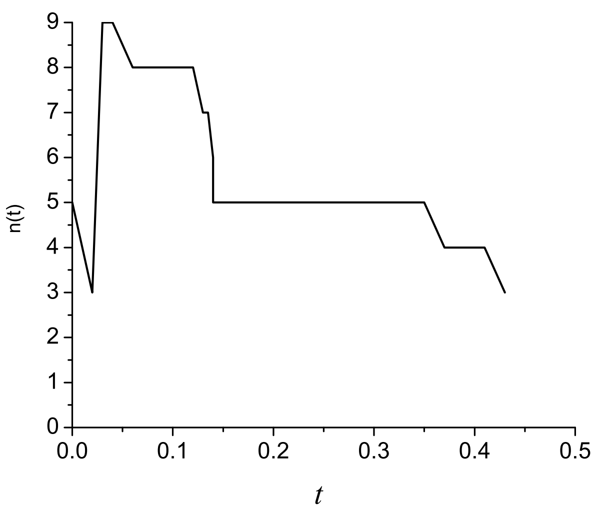
$$
\begin{aligned}
P_{n_1,n_2} &= \left|\langle n_1|\langle n_2|\hat{U}_{fc}(t)|0\rangle_b|N\rangle_c\right|^2 \\
&= \left|\sqrt{\begin{pmatrix} N \\ q \end{pmatrix}}\cos^q(2\eta t)\sin^{N-q}(2\eta t)\right|^2\delta_{n_2,N-n_1} \\
&= \begin{pmatrix} N \\ q \end{pmatrix}\cos^{2q}(2\eta t)\sin^{2(N-q)}(2\eta t)\delta_{n_2,N-n_1}
\end{aligned}
$$

Chapter 8

Dissipative Interactions

8.1 Problem 8.1

The graph below is a plot of the expected photon number of a state that has undergone many quantum jumps.



8.2 Problem 8.2

$$
\begin{aligned}
\bar{n} &= \langle \hat{n} \rangle \\
&= \frac{1}{2} \left(\langle 0| + \langle 10| \right) \hat{n} \left(|0\rangle + |10\rangle \right) \\
&= 5
\end{aligned}
$$

After quantum jump where a single photon has been emitted the (normalized) state becomes

$$|9\rangle$$

and

$$\bar{n} = 9.$$

"Classically" it does not make sense, but this state is a non-classical one.

8.3 Problem 8.3

Let

$$\hat{\rho} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{m,n}(t) |m\rangle\langle n|$$

$$\frac{d\hat{\rho}}{dt} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{d\rho_{m,n}(t)}{dt} |m\rangle\langle n|$$

$$
\begin{aligned}
\hat{a}\hat{\rho}\hat{a}^\dagger &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{m,n}(t) \hat{a} |m\rangle\langle n| \hat{a}^\dagger \\
&= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \rho_{m,n}(t) \sqrt{mn} |m-1\rangle\langle n-1| \\
&= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \rho_{m+1,n+1}(t) \sqrt{(m+1)(n+1)} |m\rangle\langle n|
\end{aligned}
$$

$$\hat{a}^\dagger\hat{a}\hat{\rho} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\rho_{m,n}(t)\hat{a}^\dagger\hat{a}|m\rangle\langle n|$$
$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\rho_{m,n}(t)m|m\rangle\langle n|$$

$$\hat{\rho}\hat{a}^\dagger\hat{a} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\rho_{m,n}(t)n|m\rangle\langle n|.$$

Eq. (8.25) is equivalent to

$$\frac{d\rho_{mn}}{dt} = \frac{\gamma}{2}\left(2\sqrt{(m+1)(n+1)}\rho_{m+1,n+1}(t) - (n+m)\rho_{m,n}(t)\right)$$

8.4 Problem 8.4

In general a density operator has the following form

$$\hat{\rho} = \sum_{m,n=0}^{\infty}\rho_{m,n}|m\rangle\langle n|,$$

and the corresponding characteristic function would be:

$$C_W(\alpha) = \mathrm{Tr}\left\{\hat{\rho}\hat{D}(\alpha)\right\}$$
$$= \sum_{n'}\langle n'|\hat{\rho}\hat{D}(\alpha)|n'\rangle$$
$$= \sum_{n'}\langle n'|\sum_{m,n=0}^{\infty}\rho_{m,n}|m\rangle\langle n|\hat{D}(\alpha)|n'\rangle$$
$$= \sum_{m,n=0}^{\infty}\rho_{m,n}\langle n|\hat{D}(\alpha)|m\rangle$$

It is important to compute $\langle m|\hat{D}(\alpha)|n\rangle$. There are many ways to do so, but we follow the expansion one:

$$
\begin{aligned}
\langle m|\hat{D}(\alpha)|n\rangle &= \langle m|e^{\alpha\hat{a}^\dagger-\alpha^*\hat{a}}|n\rangle \\
&= e^{-|\alpha|^2/2}\langle m|e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|n\rangle \\
&= e^{-|\alpha|^2/2}\left(\sum_{m'=0}^{m}\langle m|\frac{(\alpha\hat{a}^\dagger)^{m'}}{m'!}\right)\left(\sum_{n'=0}^{n}\frac{(-\alpha^*\hat{a})^{n'}}{n'!}|n\rangle\right) \\
&= e^{-|\alpha|^2/2}\left(\sum_{m'=0}^{m}\langle m-m'|\frac{\alpha^{m'}}{m'!}\sqrt{\frac{m!}{(m-m')!}}\right)\left(\sum_{n'=0}^{n}\frac{(-1)^{n'}\alpha^{*n'}}{n'!}\sqrt{\frac{n!}{(n-n')!}}|n-n
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-|\alpha|^2/2}\sum_{m'=0}^{m}\sum_{n'=0}^{n}\frac{(-1)^{n'}\alpha^{m'}\alpha^{*n'}}{m'!n'!}\sqrt{\frac{m!n!}{(m-m')!(n-n')!}}\delta_{m-m',n-n'} \\
&= e^{-|\alpha|^2/2}\sqrt{\frac{m!}{n!}}\sum_{m'=0}^{m}\frac{(-1)^{(n-m+m')}\alpha^{m'}\alpha^{*(n-m+m')}}{m'!(n-m+m')!}\frac{n!}{(m-m')!} \\
&= e^{-|\alpha|^2/2}(-1)^{(n-m)}\alpha^{*(n-m)}\sqrt{\frac{m!}{n!}}\sum_{m'=0}^{m}\frac{(-1)^{m'}|\alpha|^{2m'}}{m'!(n-m+m')!}\frac{(n-m+m)!}{(m-m')!} \\
&= e^{-|\alpha|^2/2}(-1)^{(n-m)}\alpha^{*(n-m)}\sqrt{\frac{m!}{n!}}L_m^{n-m}(|\alpha|^2),
\end{aligned}
$$

where L_m^k is the associated Laguerre polynomials.

8.5 Problem 8.5

The master equation (8.26) is equivalent to

$$\dot{\rho}_{m,n}(t) = \frac{\gamma}{2}\left(2\sqrt{(m+1)(n+1)}\rho_{m+1,n+1}(t) - (m+n)\rho_{m,n}(t)\right),$$

where

$$\hat{\rho}(t) = \sum_{m=0}\sum_{n=0}\rho_{m,n}(t)|m\rangle\langle n|$$

Solving numerically that equation using Mathematica, we display in graph (a) the photon probability

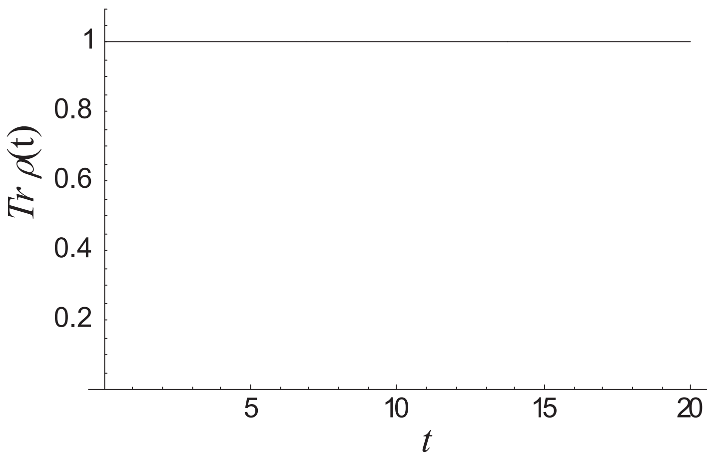
$$P_n(t) = \mathrm{Tr}\hat{\rho}(t) = \sum_{n=0}\rho_{n,n}(t),$$

and in graph (b) the plot of

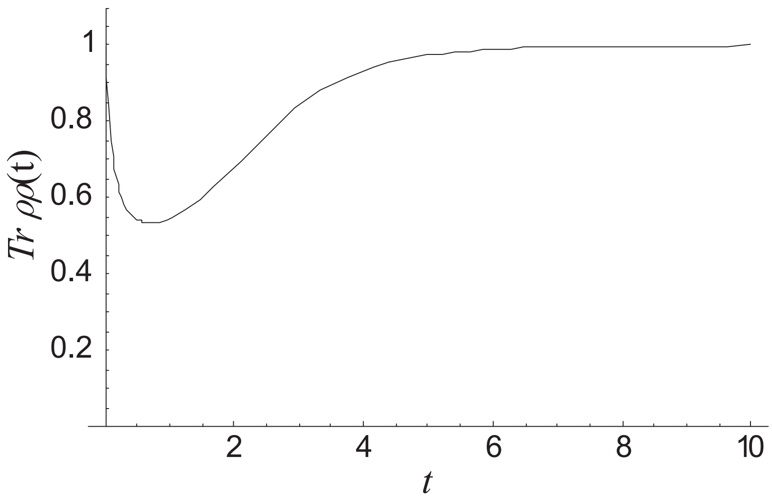
$$\mathrm{Tr}\hat{\rho}^2(t) = \sum_{n=0}\sum_{m=0} |\rho_{n,m}(t)|^2(t).$$

Notice that that in graph (b) the state decoheres into a statistical mixture and then to a vacuum state. Graph (a) shows that the probability maintains the value of unity.

(a)



(b)



8.6 Problem 8.6

$$\hat{\rho} = \sum_{m,n}^{\infty} \rho_{m,n} |m\rangle\langle n|$$
$$\frac{d\hat{\rho}}{dt} = \sum_{m,n}^{\infty} \frac{d\rho_{m,n}}{dt} |m\rangle\langle n|$$

$$\begin{aligned}\hat{a}^\dagger \hat{a}\hat{\rho} &= \sum_{m,n}^{\infty} \rho_{m,n} \hat{a}^\dagger \hat{a} |m\rangle\langle n| \\ &= \sum_{m,n}^{\infty} \rho_{m,n} m |m\rangle\langle n|\end{aligned}$$

$$\begin{aligned}\hat{\rho}\hat{a}^\dagger \hat{a} &= \sum_{m,n}^{\infty} \rho_{m,n} |m\rangle\langle n| \hat{a}^\dagger \hat{a} \\ &= \sum_{m,n}^{\infty} \rho_{m,n} n |m\rangle\langle n|\end{aligned}$$

$$\begin{aligned}\hat{a}\hat{\rho}\hat{a}^\dagger &= \sum_{m,n}^{\infty} \rho_{m,n} \hat{a} |m\rangle\langle n| \hat{a}^\dagger \\ &= \sum_{m,n=1}^{\infty} \rho_{m,n} \sqrt{mn} |m-1\rangle\langle n-1| \\ &= \sum_{m,n=0}^{\infty} \rho_{(m+1),(n+1)} \sqrt{(m+1)(n+1)} |m\rangle\langle n|\end{aligned}$$

$$\frac{d\hat{\rho}}{dt} = \frac{\gamma}{2}\left[2\hat{a}\hat{\rho}\hat{a}^\dagger + \hat{a}^\dagger \hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger \hat{a}\right]$$

is equivalent to

$$\frac{d\rho_{m,n}(t)}{dt} = \frac{\gamma}{2}\left[2\sqrt{(m+1)(n+1)}\rho_{(m+1),(n+1)}(t) - (m+n)\rho_{m,n}(t)\right].$$

$$\rho_{m,n}(t) = \exp\left(-\frac{\gamma t(m+n)}{2}\right)\sum_l \left(\frac{(m+l)!(n+l)!}{m!n!}\right)^{1/2} \frac{\left(1-e^{-\gamma t}\right)^l}{l!}\rho_{m+l,n+l}(0)$$

$$\begin{aligned}
\frac{d\rho_{m,n}(t)}{dt} &= -\frac{\gamma(m+n)}{2}\rho_{m,n}(t) \\
&+ \exp\left(-\frac{\gamma t(m+n)}{2}\right)\sum_l \left(\frac{(m+l)!(n+l)!}{m!n!}\right)^{1/2} \frac{l\gamma e^{-\gamma t}\left(1-e^{-\gamma t}\right)^{l-1}}{l!}\rho_{m+l,n+l}(0) \\
&= -\frac{\gamma(m+n)}{2}\rho_{m,n}(t) + \gamma\exp\left(-\frac{\gamma t(m+n)}{2}\right) \\
&\times \sum_l \left(\frac{(m+l)!(n+l)!}{m!n!}\right)^{1/2} \frac{e^{-\gamma t}\left(1-e^{-\gamma t}\right)^{l-1}}{(l-1)!}\rho_{m+l,n+l}(0) \\
&= -\frac{\gamma(m+n)}{2}\rho_{m,n}(t) + \gamma\sqrt{(m+1)(n+1)}\exp\left(-\frac{\gamma t(m+n+2)}{2}\right) \\
&\times \sum_l \left(\frac{(m+1+l)!(n+1+l)!}{(m+1)!(n+1)!}\right)^{1/2} \frac{\left(1-e^{-\gamma t}\right)^{l-1}}{(l)!}\rho_{m+1+l,n+1+l}(0) \\
&= \frac{\gamma}{2}\left(2\sqrt{(m+1)(n+1)}\rho_{m+1,n+1}(t) - (m+n)\rho_{m,n}(t)\right)
\end{aligned}$$

8.7 Problem 8.7

For $|\alpha\rangle$ as an initial state

$$\begin{aligned}
\hat{\rho}(0) &= |\alpha\rangle\langle\alpha| \\
\rho_{m,n}(0) &= e^{-|\alpha|^2}\frac{\alpha^m\alpha^{*n}}{\sqrt{m!n!}} \\
\rho_{m+l,n+l}(0) &= e^{-|\alpha|^2}\frac{\alpha^m\alpha^{*n}|\alpha|^{2l}}{\sqrt{(m+l)!(n+1)!}}
\end{aligned}$$

$$
\begin{aligned}
\rho_{m,n}(t) = e^{-\frac{\gamma t(m+n)}{2}} & \sum_l \left(\frac{(m+l)!(n+l)!}{m!n!} \right)^{1/2} \frac{(1-e^{-\gamma t})^l}{l!} \\
& \times e^{-|\alpha|^2} \frac{\alpha^m \alpha^{*n} |\alpha|^{2l}}{\sqrt{(m+l)!(n+1)!}} \\
& = e^{-\frac{\gamma t(m+n)}{2}} e^{-|\alpha|^2} \frac{\alpha^m \alpha^{*n}}{\sqrt{m!n!}} \sum_l \frac{\left(|\alpha|^2(1-e^{-\gamma t})\right)^l}{l!} \\
& = e^{-\frac{\gamma t(m+n)}{2}} e^{-|\alpha|^2} \frac{\alpha^m \alpha^{*n}}{\sqrt{m!n!}} \exp\left(|\alpha|^2(1-e^{-\gamma t})\right) \\
& = e^{-\frac{\gamma t(m+n)}{2}} \frac{\alpha^m \alpha^{*n}}{\sqrt{m!n!}} e^{-|\alpha|^2 e^{-\gamma t}} \\
& = e^{-|\alpha|^2 e^{-\gamma t}} \frac{\left(\alpha e^{-\frac{\gamma t}{2}}\right)^m \left(\alpha^* e^{-\frac{\gamma t}{2}}\right)^n}{\sqrt{m!n!}}
\end{aligned}
$$

which simply means that

$$
\hat{\rho}(t) = |\alpha e^{-\gamma t/2}\rangle\langle\alpha e^{-\gamma t/2}| \tag{8.7.1}
$$

For $N[|\alpha\rangle + |-\alpha\rangle]$

$$
\begin{aligned}
\hat{\rho}(0) &= |N|^2 \left[|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha| + |-\alpha\rangle\langle\alpha| + |\alpha\rangle\langle-\alpha| \right] \\
&= |N|^2 \sum \frac{\alpha^m \alpha^{*n}}{\sqrt{m!n!}} \left[1 + (-1)^{m+n} + (-1)^m + (-1)^n \right] |m\rangle\langle n|
\end{aligned}
$$

$$
\begin{aligned}
\rho_{m,n}(0) &= |N|^2 \frac{\alpha^m \alpha^{*n}}{\sqrt{m!n!}} \left[1 + (-1)^{m+n} + (-1)^m + (-1)^n \right] \\
\rho_{m+l,n+l}(0) &= |N|^2 \frac{\alpha^m \alpha^{*n} |\alpha|^{2l}}{\sqrt{(m+l)!(n+l)!}} \left[1 + (-1)^{m+n} + (-1)^l((-1)^m + (-1)^n) \right]
\end{aligned}
$$

$$
\begin{aligned}
\rho_{m,n}(t) &= e^{\frac{-\gamma t(m+n)}{2}} \sum_l \left(\frac{(m+l)!(n+l)!}{n!m!}\right)^{1/2} \frac{(1-e^{-\gamma t})^l}{l!} \rho_{m+l,n+l}(0) \\
&= e^{\frac{-\gamma t(m+n)}{2}} \sum_l \left(\frac{(m+l)!(n+l)!}{n!m!}\right)^{1/2} \frac{(1-e^{-\gamma t})^l}{l!} |N|^2 \frac{\alpha^m \alpha^{*n} |\alpha|^{2l}}{\sqrt{(m+l)!(n+l)!}} \\
&\times \left[1 + (-1)^{m+n} + (-1)^l((-1)^m + (-1)^n)\right] \\
&= |N|^2 e^{\frac{-\gamma t(m+n)}{2}} \frac{\alpha^m \alpha^{*n}}{\sqrt{m!n!}} \sum_l \\
&\times \left[\frac{(|\alpha|^2(1-e^{-\gamma t}))^l}{l!}\left(1+(-1)^{m+n}\right) + \frac{(-|\alpha|^2(1-e^{-\gamma t}))^l}{l!}((-1)^m + (-1)^n)\right] \\
&= |N|^2 e^{\frac{-\gamma t(m+n)}{2}} \frac{\alpha^m \alpha^{*n}}{\sqrt{m!n!}} \left[e^{|\alpha|^2(1-e^{-\gamma t})}\left(1+(-1)^{m+n}\right) + e^{-|\alpha|^2(1-e^{-\gamma t})}((-1)^m + (-1)^n)\right]
\end{aligned}
$$

Thus

$$
\hat{\rho}(t) = |N|^2[e^{|\alpha|^2(1-e^{-\gamma t})}\left(|\alpha e^{-\gamma t/2}\rangle\langle \alpha e^{-\gamma t/2}| + |-\alpha e^{-\gamma t/2}\rangle\langle -\alpha e^{-\gamma t/2}|\right) \tag{8.7.2}
$$

$$
+ e^{-|\alpha|^2(1-e^{-\gamma t})}\left(|-\alpha e^{-\gamma t/2}\rangle\langle \alpha e^{-\gamma t/2}| + |\alpha e^{-\gamma t/2}\rangle\langle -\alpha e^{-\gamma t/2}|\right)] \tag{8.7.3}
$$

8.8 Problem 8.8

Eq. (8.34) is equivalent to

$$
\begin{aligned}
\dot{\rho}_{m,n}(t) &= -iG\left(\sqrt{m+1}\rho_{m+1,n}(t) + \sqrt{m}\rho_{m,n-1}(t) - \sqrt{n+1}\rho_{m,n+1}(t) - \sqrt{n}\rho_{m,n-1}(t)\right) \\
&+ \frac{\gamma}{2}\left(2\sqrt{(m+1)(n+1)}\rho_{m+1,n+1}(t) - (m+n)\rho_{m,n}(t)\right),
\end{aligned}
$$

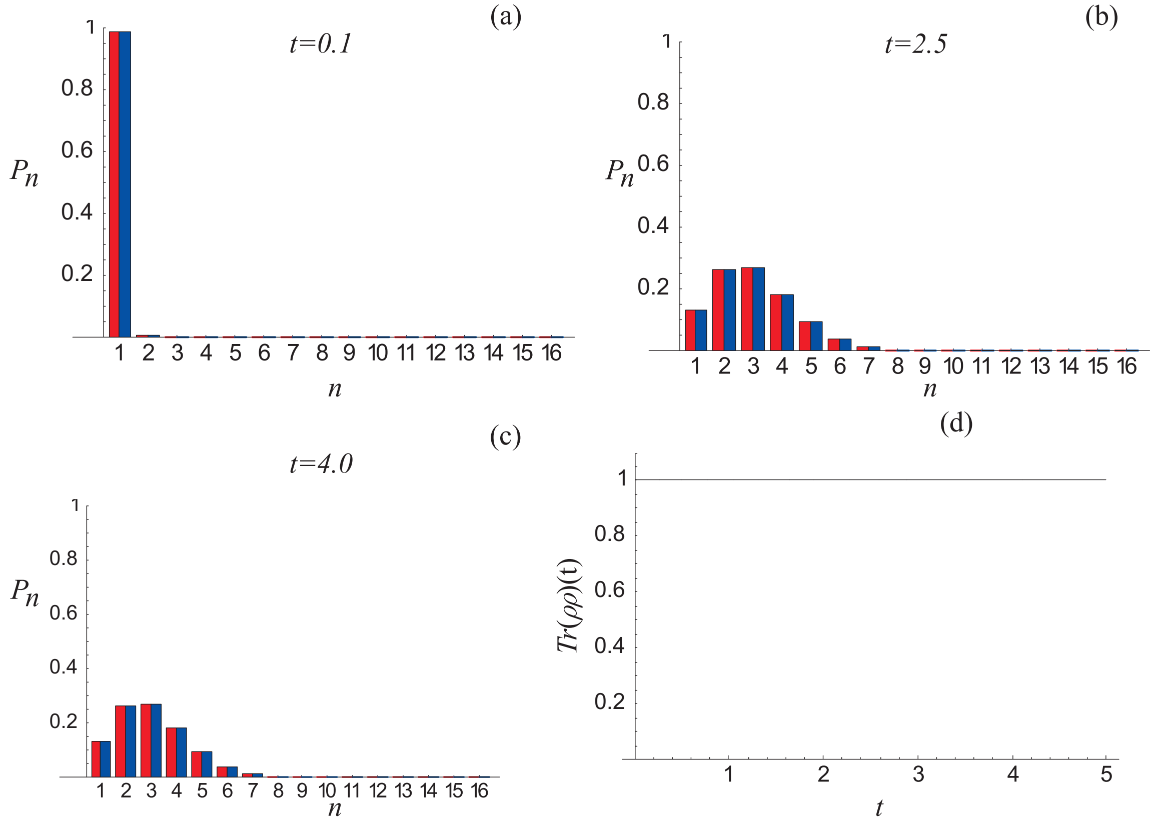
where

$$
\hat{\rho}(t) = \sum_{m=0}\sum_{n=0}\rho_{m,n}(t)|m\rangle\langle n|
$$

Solving numerically that equation using Mathematica, we display graphs below: The first three ones are bar-chart graphs of the photon number distributions at different times, on the same graphs we plot the photon distributions for coherent states where the average photon number is taken as

$$
|\alpha|^2 = \sum_{n=0} n\rho_{n,n}(t).
$$

It is clear from the plots that the state is a coherent state. In Graph c we plot $\mathbf{Tr}\hat{\rho}^2$ versus time. It shows that the evolving state is a pure state at all time, since $\mathbf{Tr}\hat{\rho}^2(t) = 1$.



Chapter 9

Optical Test of Quantum Mechanics

9.1 Problem 9.1

$$\begin{aligned}|\Psi_0\rangle &= |0\rangle_s|0\rangle_i,\\ \hat{H}_I &= \hbar\eta(\hat{a}_s^\dagger\hat{a}_i^\dagger + \hat{a}_s\hat{a}_i)\end{aligned}$$

$$\begin{aligned}\hat{H}_I|\Psi_0\rangle &= \hbar\eta(\hat{a}_s^\dagger\hat{a}_i^\dagger + \hat{a}_s\hat{a}_i)|0\rangle_s|0\rangle_i\\ &= \hbar\eta|1\rangle_s|1\rangle_i\end{aligned}$$

$$\begin{aligned}\hat{H}_I^2|\Psi_0\rangle &= \hat{H}_I\hbar\eta|1\rangle_s|1\rangle_i\\ &= (\hbar\eta)^2(\hat{a}_s^\dagger\hat{a}_i^\dagger + \hat{a}_s\hat{a}_i)|1\rangle_s|1\rangle_i\\ &= (\hbar\eta)^2(2|2\rangle_s|2\rangle_i + |0\rangle_s|0\rangle_i)\end{aligned}$$

$$\begin{aligned}|\Psi\rangle &= \left[1 - it\hat{H}_I/\hbar + (-it\hat{H}_I/\hbar)^2/2\right]|\Psi_0\rangle\\ &= (1 - i\frac{t}{\hbar}\hat{H}_I + \frac{t^2}{2\hbar^2}\hat{H}_I^2)|0\rangle_s|0\rangle_i\\ &= |0\rangle_s|0\rangle_i - i\mu|1\rangle_s|1\rangle_i - \frac{\mu^2}{2}\left(2|2\rangle_s|2\rangle_i + |0\rangle_s|0\rangle_i\right)\\ &= (1-\mu^2/2)|0\rangle_s|0\rangle_i - i\mu|1\rangle_s|1\rangle_i - \mu^2|2\rangle_s|2\rangle_i\end{aligned}$$

Following Equation (6.17),

$$
\begin{aligned}
|2\rangle_s|2\rangle_i = \frac{1}{2}\hat{a}_s^{\dagger 2}\hat{a}_i^{\dagger 2}|0\rangle \quad &\overset{BS}{\longrightarrow} \quad \frac{1}{8}\left(\hat{a}_2^\dagger + i\hat{a}_3^\dagger\right)^2\left(i\hat{a}_2^\dagger + \hat{a}_3^\dagger\right)^2|0\rangle \\
&= -\frac{1}{8}\left(\hat{a}_2^\dagger + i\hat{a}_3^\dagger\right)^2\left(\hat{a}_2^\dagger - i\hat{a}_3^\dagger\right)^2|0\rangle \\
&= -\frac{1}{8}\left(\hat{a}_2^{\dagger 2} + \hat{a}_3^{\dagger 2}\right)^2|0\rangle \\
&= -\frac{1}{8}\left(\hat{a}_2^{\dagger 4} + 2\hat{a}_2^{\dagger 2}\hat{a}_3^{\dagger 2} + \hat{a}_3^{\dagger 4}\right)|0\rangle \\
&= -\frac{1}{8}\left(\sqrt{4!}|4\rangle_2|0\rangle_3 + 4|2\rangle_2|2\rangle_3 + \sqrt{4!}|0\rangle_2|4\rangle_3\right) \\
&= -\frac{\sqrt{6}}{4}|4\rangle_2|0\rangle_3 - \frac{1}{2}|2\rangle_2|2\rangle_3 - \frac{\sqrt{6}}{4}|0\rangle_2|4\rangle_3
\end{aligned}
$$

Using simple binomial distribution we would obtain

$$
\frac{1}{16}\left(|4\rangle_2|0\rangle_3 + 4|3\rangle_2|1\rangle_3 + 6|2\rangle_2|2\rangle_3 + 4|1\rangle_2|3\rangle_3 + |0\rangle_2|4\rangle_3\right),
$$

for a classical case.

9.2 Problem 9.2

Repeating the same procedures as in section 9.6 except we define

$$
\begin{aligned}
S &= X_1X_2 - X_1X_2' + X_1'X_2 + X_1'X_2' \\
&= X_1(X_2 - X_2') + X_1'(X_2 + X_2') = \pm 2,
\end{aligned}
$$

$$
-2 \leq C_{\rm CV}(\theta,\phi) - C_{\rm CV}(\theta,\phi') + C_{\rm CV}(\theta',\phi) + C_{\rm CV}(\theta',\phi') \leq +2
$$

Again

$$
C_{\rm CV}(\theta,\phi) = -\cos[2(\theta-\phi)]
$$

For $\theta = 0$, $\phi = 1.17$, $\theta' = 2.34$ and $\phi' = 3.51$, $S = 2.8273$. So Bell's inequality is violated.

9.3 Problem 9.3

Repeating the same procedures as in section 9.6 except we define

$$\begin{aligned} S &= X_1X_2 - X_1X_2' + X_1'X_2 + X_1'X_2' \\ &= X_1(X_2 - X_2') + X_1'(X_2 + X_2') = \pm 2, \\ C_{CV} &= \cos[2(\theta - \phi)]. \end{aligned}$$

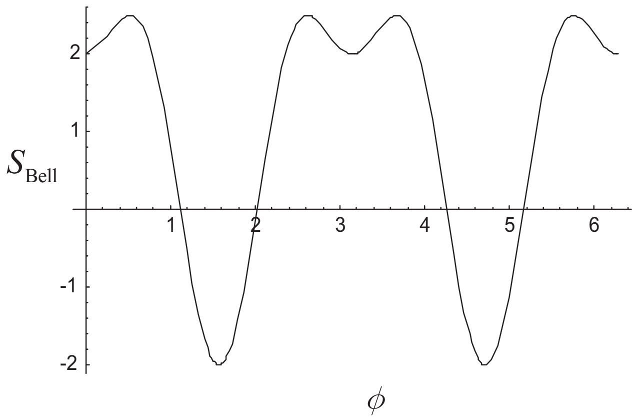
and

$$-2 \leq C_{\rm CV}(\theta,\phi) - C_{\rm CV}(\theta,\phi') + C_{\rm CV}(\theta',\phi) + C_{\rm CV}(\theta',\phi') \leq +2$$

For

$$\theta = 0 \quad, \phi' = 2\phi, \quad and \quad \theta' = \phi,$$

the last inequality is violated. See graph below.



9.4 Problem 9.4

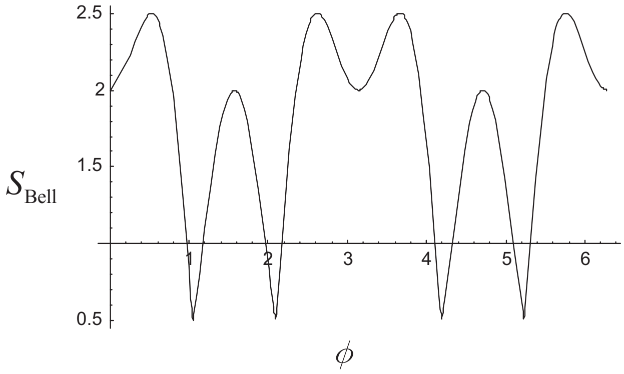
$$C_{HV}(\theta,\phi) = \int d\lambda \rho(\lambda) A(\theta,\lambda) B(\phi,\lambda)$$

$$
\begin{aligned}
|C_{HV}(\theta,\phi) - C_{HV}(\theta,\phi')| &= \int d\lambda\rho(\lambda)A(\theta,\lambda)B(\phi,\lambda) - \int d\lambda\rho(\lambda)A(\theta,\lambda)B(\phi',\lambda) \\
&= \int d\lambda\rho(\lambda)A(\theta,\lambda)B(\phi,\lambda) \\
&\pm \int d\lambda\rho(\lambda)A(\theta,\lambda)A(\theta',\lambda)B(\phi,\lambda)B(\phi',\lambda) \\
&- \int d\lambda\rho(\lambda)A(\theta,\lambda)B(\phi',\lambda) \\
&\mp \int d\lambda\rho(\lambda)A(\theta,\lambda)A(\theta',\lambda)B(\phi,\lambda)B(\phi',\lambda) \\
&= \int d\lambda\rho(\lambda)A(\theta,\lambda)B(\phi,\lambda)\left[1 \pm A(\theta',\lambda)B(\phi',\lambda)\right] \\
&- \int d\lambda\rho(\lambda)A(\theta,\lambda)B(\phi',\lambda)\left[1 \pm A(\theta',\lambda)B(\phi,\lambda)\right] \\
|C_{HV}(\theta,\phi) - C_{HV}(\theta,\phi')| &\leq \int d\lambda\rho(\lambda)\left[1 \pm A(\theta',\lambda)B(\phi',\lambda)\right] \\
&\quad + \int d\lambda\rho(\lambda)\left[1 \pm A(\theta',\lambda)B(\phi,\lambda)\right] \\
&= 2 \pm \int d\lambda\rho(\lambda)A(\theta',\lambda)B(\phi',\lambda) \pm \int d\lambda\rho(\lambda)A(\theta',\lambda)B(\phi,\lambda) \\
&= 2 \pm C_{HV}(\theta',\phi') \pm C_{HV}(\theta',\phi) \\
&\leq 2 + |C_{HV}(\theta',\phi') + C_{HV}(\theta',\phi)| \\
S_{\text{Bell}} &\leq 2
\end{aligned}
$$

For

$$\theta = 0 \quad, \phi' = 2\phi, \quad and \quad \theta' = \phi,$$

the last inequality is violated. See graph below.


2.5
2
1.5
S_{Bell}
0.5
1
2
3
4
5
6
ϕ

Chapter 10

Experiments in Cavity QED and with Trapped Ions

10.1 Problem 10.1

The radius of a Rydberg atom scales as $n^2 a_0$. On the other hand the dipole operator is defined as $\hat{d} = q\hat{r}$. It is clear that the dipole moment goes as n^2.

10.2 Problem 10.2

Using the standard steps of linear algebra we can determine the eigenvalue of the matrix:

$$\begin{pmatrix} 0 & 0 & i\Omega_0/2 \\ 0 & -\omega_0/Q & -i\Omega_0/2 \\ i\Omega_0 & -i\Omega_0 & -\omega_0/2Q \end{pmatrix}.$$

In order to find the eigenvalue we have to solve Λ such the determinant of the following matrix vanishes.

$$\det\begin{pmatrix} -\Lambda & 0 & i\Omega_0/2 \\ 0 & -\omega_0/Q - \Lambda & -i\Omega_0/2 \\ i\Omega_0 & -i\Omega_0 & -\omega_0/2Q - \Lambda \end{pmatrix} = 0$$

$$\Lambda\left[\left(\frac{\omega_0}{Q}+\Lambda\right)\left(\frac{\omega_0}{2Q}+\Lambda\right)+\frac{\Omega_0^2}{2}\right]+\frac{\Omega_0^2}{2}\left(\frac{\omega_0}{Q}+\Lambda\right) = 0$$

$$\Lambda\left(\frac{\omega_0}{Q}+\Lambda\right)\left(\frac{\omega_0}{2Q}+\Lambda\right)+\Omega_0^2\left(\frac{\omega_0}{2Q}+\Lambda\right) = 0$$

$$\left(\Lambda+\frac{\omega_0}{2Q}\right)\left(\Lambda^2+\frac{\omega_0}{Q}\Lambda+\Omega_0^2\right) = 0.$$

The last equation has three possible solutions:

$$\Lambda_0 = -\frac{\omega_0}{2Q}$$

$$\Lambda_\pm = -\frac{\omega_0}{2Q} \pm \frac{\omega_0}{2Q}\left(1-\frac{4\Omega_0^2 Q^2}{\omega_0^2}\right).$$

10.3 Problem 10.3

For an atom prepared in the superposition state

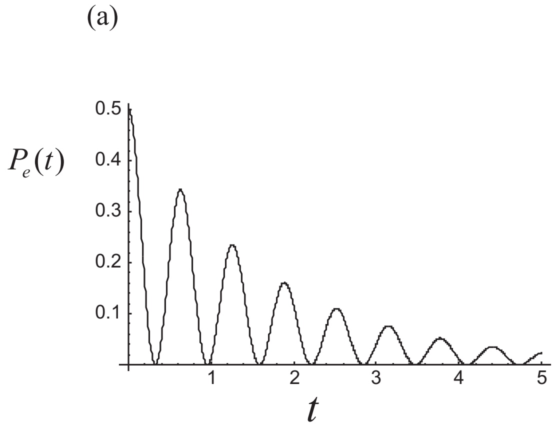
$$|\psi_{\text{atom}}\rangle = \frac{1}{\sqrt{2}}(|e\rangle + e^{i\varphi}|g\rangle),$$

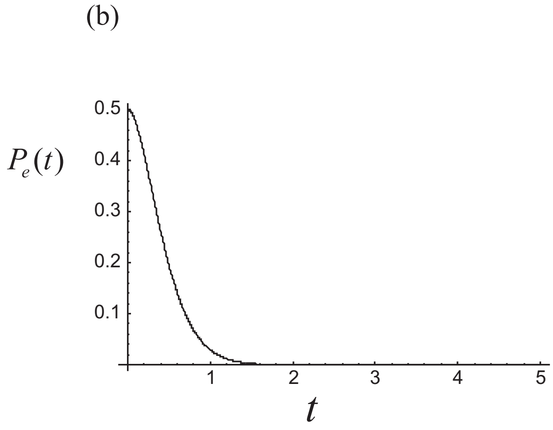
injected into a cavity whose field is initially in a vacuum, the initial conditions become

$$\rho_{11} = \frac{1}{2},$$
$$\rho_{22} = 0,$$
$$\rho_{12} = 0,$$
$$\rho_{33} = \frac{1}{2}.$$

Notice that the initial conditions do not depend on the relative phase φ.

To obtain the time evolution for the excited state population one can numerically solve the system of equations in Eq. (10.17) with the initial conditions mentioned above. In graphs a and b we plot $P_e(t)$ for high and low Q cavities, respectively.





10.4 Problem 10.4

$$\hat{\rho} = \sum_{m,n} |m\rangle\langle n| \otimes [\rho_{em,n}|e\rangle\langle e| + \rho_{gm,n}|g\rangle\langle g| + \rho_{egm,n}|e\rangle\langle g| + \rho_{gem,n}|g\rangle\langle e|]$$

where $\rho_{gem,n} = \rho^*_{egn,m}$. Equation (10.16) has the form

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H}_I, \hat{\rho}\right] - \frac{\kappa}{2}\left(\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a}\right) + \kappa\hat{a}\hat{\rho}\hat{a}^\dagger \tag{10.4.1}$$

$$\frac{d\hat{\rho}}{dt} = \sum_{m,n} |m\rangle\langle n| \otimes [\dot{\rho}_{em,n}|e\rangle\langle e| + \dot{\rho}_{gm,n}|g\rangle\langle g| + \dot{\rho}_{egm,n}|e\rangle\langle g| + \dot{\rho}_{gem,n}|g\rangle\langle e|]$$

$$\begin{aligned}\left(\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a}\right) = \sum_{m,n}&(m+n)|m\rangle\langle n| \\ &\otimes [\rho_{em,n}|e\rangle\langle e| + \rho_{gm,n}|g\rangle\langle g| + \rho_{egm,n}|e\rangle\langle g| + \rho_{gem,n}|g\rangle\langle e|]\end{aligned}$$

$$\begin{aligned}\hat{a}\hat{\rho}\hat{a}^\dagger &= \sum_{m,n} \sqrt{mn}|m-1\rangle\langle n-1| \\ &\quad \otimes [\rho_{em,n}|e\rangle\langle e| + \rho_{gm,n}|g\rangle\langle g| + \rho_{egm,n}|e\rangle\langle g| + \rho_{gem,n}|g\rangle\langle e|] \\ &= \sum_{m,n} \sqrt{(m+1)(n+1)}|m\rangle\langle n| \\ &\quad \otimes \left[\rho_{e(m+1),(n+1)}|e\rangle\langle e| + \rho_{g(m+1),(n+1)}|g\rangle\langle g| + \rho_{eg(m+1),(n+1)}|e\rangle\langle g| + \rho_{ge(m+1),(n+1)}|g\rangle\langle e|\right]\end{aligned}$$

$$\hat{H}_I\hat{\rho} = \sum_{m,n}\lambda\left(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-\right)|m\rangle\langle n| \otimes \left[\rho_{em,n}|e\rangle\langle e| + \rho_{gm,n}|g\rangle\langle g| + \rho_{egm,n}|e\rangle\langle g| + \rho_{gem,n}|g\rangle\langle e|\right]$$

10.5 Problem 10.5

Let's write the normalized state as

$$|\text{sup}\rangle = \mathcal{N}\left(|\alpha e^{i\phi}\rangle + |\alpha e^{-i\phi}\rangle\right).$$

We have

$$\begin{aligned}
\langle\text{sup}|\text{sup}\rangle &= |\mathcal{N}|^2\left(\langle\alpha e^{i\phi}|\alpha e^{i\phi}\rangle + \langle\alpha e^{-i\phi}|\alpha e^{-i\phi}\rangle + \langle\alpha e^{i\phi}|\alpha e^{-i\phi}\rangle + \langle\alpha e^{-i\phi}|\alpha e^{i\phi}\rangle\right) \\
&= |\mathcal{N}|^2\left(2 + e^{-|\alpha|^2+|\alpha|^2e^{2i\phi}} + e^{-|\alpha|^2+|\alpha|^2e^{-2i\phi}}\right) \\
&= |\mathcal{N}|^2\left(2 + e^{-|\alpha|^2(1-\cos 2\phi)}e^{i|\alpha|^2\sin 2\phi} + e^{-|\alpha|^2(1-\cos 2\phi)}e^{-i|\alpha|^2\sin 2\phi}\right) \\
&= |\mathcal{N}|^2\left[2 + e^{-|\alpha|^2(1-\cos 2\phi)}\left(e^{i|\alpha|^2\sin 2\phi} + e^{-i|\alpha|^2\sin 2\phi}\right)\right] \\
&= 2\,|\mathcal{N}|^2\left[1 + e^{-|\alpha|^2(1-\cos 2\phi)}\cos\left(|\alpha|^2\sin 2\phi\right)\right], \\
&= 1
\end{aligned}$$

so that

$$\mathcal{N} = \frac{1}{\sqrt{2}}\left[1 + e^{-|\alpha|^2(1-\cos 2\phi)}\cos\left(|\alpha|^2\sin 2\phi\right)\right]^{-1/2}.$$

Initially the density operator can be expressed as

$$\begin{aligned}
\hat{\rho}(0) &= |\mathcal{N}|^2\left[|\alpha e^{i\phi}\rangle\langle\alpha e^{i\phi}| + |\alpha e^{-i\phi}\rangle\langle\alpha e^{-i\phi}| + |\alpha e^{i\phi}\rangle\langle\alpha e^{-i\phi}| + |\alpha e^{-i\phi}\rangle\langle\alpha e^{i\phi}|\right] \\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\rho_{m,n}(0),
\end{aligned}$$

where

$$\rho_{m,n}(0) = |\mathcal{N}|^2\,e^{-|\alpha|^2}\frac{\alpha^m\alpha^{*n}}{\sqrt{m!n!}}\left[e^{i(m-n)\phi} + e^{-i(m-n)\phi} + e^{-i(m+n)\phi} + e^{i(m+n)\phi}\right].$$

Using Eq. (8.39) one can show that

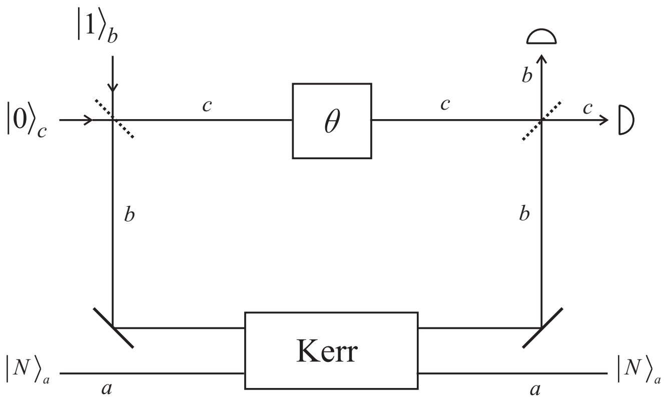
$$\begin{aligned}
\rho_{m,n}(t) = |\mathcal{N}|^2\,e^{-|\alpha|^2}e^{-\gamma t(m+n)/2}\frac{\alpha^m\alpha^{*n}}{\sqrt{m!n!}}&\left[e^{|\alpha|^2\left(1-e^{-\gamma t}\right)}\left(e^{i(m-n)\phi} + e^{-i(m-n)\phi}\right)\right. \\
&\left. + e^{-i(m+n)\phi}e^{|\alpha|^2e^{-i2\phi}\left(1-e^{-\gamma t}\right)} + e^{i(m+n)\phi}e^{|\alpha|^2e^{i2\phi}\left(1-e^{-\gamma t}\right)}\right],
\end{aligned}$$

which can can be written in terms of coherent states as

$$\begin{aligned}\hat{\rho}(t) = |\mathcal{N}|^2 \Big[& \left|\alpha e^{i\phi}e^{-\gamma t}\right\rangle \left\langle \alpha e^{i\phi}e^{-\gamma t}\right| + \left|\alpha e^{-i\phi}e^{-\gamma t}\right\rangle \left\langle \alpha e^{-i\phi}e^{-\gamma t}\right| \\ & + e^{|\alpha|^2 e^{-i2\phi}\left(1-e^{-\gamma t}\right)} \left|\alpha e^{i\phi}e^{-\gamma t}\right\rangle \left\langle \alpha e^{-i\phi}e^{-\gamma t}\right| \\ & + e^{|\alpha|^2 e^{i2\phi}\left(1-e^{-\gamma t}\right)} \left|\alpha e^{-i\phi}e^{-\gamma t}\right\rangle \left\langle \alpha e^{i\phi}e^{-\gamma t}\right| \Big].\end{aligned}$$

As in section 8.5, one studies the decay of the "off-diagonal" terms: $e^{|\alpha|^2 e^{i2\phi}\left(1-e^{-\gamma t}\right)}$ and $e^{|\alpha|^2 e^{-i2\phi}\left(1-e^{-\gamma t}\right)}$. In short time $e^{|\alpha|^2 e^{\pm i2\phi}\left(1-e^{-\gamma t}\right)} \approx e^{-|\alpha|^2\gamma t\cos(2\phi)}e^{\pm i|\alpha|^2\gamma t\sin(2\phi)}$, so the decoherence time is given by $T_{\text{decoh}} = 1/(\gamma|\alpha|^2\cos(2\phi))$.

10.6 Problem 10.6



In the figure above we have depicted a possible QND device to measure photon number for optical fields. The math is a follows:

$$\begin{aligned}|in\rangle &= |N\rangle_a|1\rangle_b|0\rangle_c \\ |out\rangle &= \hat{U}_{\text{BS2}}\hat{U}_{\text{PS}}\hat{U}_{\text{Kerr}}\hat{U}_{\text{BS1}}|in\rangle \\ &= \hat{U}_{\text{BS2}}\hat{U}_{\text{PS}}\hat{U}_{\text{Kerr}}\hat{U}_{\text{BS1}}|N\rangle_a|1\rangle_b|0\rangle_c \\ &= \frac{1}{\sqrt{2}}\hat{U}_{\text{BS2}}\hat{U}_{\text{PS}}\hat{U}_{\text{Kerr}}\left(|1\rangle_b|0\rangle_c + i|0\rangle_b|1\rangle_c\right)|N\rangle_a \\ &= \frac{1}{\sqrt{2}}\hat{U}_{\text{BS2}}\left(e^{i\chi Nt}|1\rangle_b|0\rangle_c + ie^{i\theta}|0\rangle_b|1\rangle_c\right)|N\rangle_a \\ &= \frac{1}{2}\left[e^{i\chi Nt}(|1\rangle_b|0\rangle_c + i|0\rangle_b|1\rangle_c) + ie^{i\theta}(|0\rangle_b|1\rangle_c + i|1\rangle_b|0\rangle_c)\right]|N\rangle_a \\ &= \frac{1}{2}\left[(e^{i\chi Nt} - e^{i\theta})|1\rangle_b|0\rangle_c + i(e^{i\chi Nt} + e^{i\theta})|0\rangle_b|1\rangle_c\right]|N\rangle_a\end{aligned}$$

The probabilities that we detect $|1\rangle_b|0\rangle_c$ and $|0\rangle_b|1\rangle_c$ are

$$\begin{aligned} P_{(|1\rangle_b|0\rangle_c)} &= \frac{1}{2}(1+\cos(\theta+\chi Nt)) \\ P_{(|0\rangle_b|1\rangle_c)} &= \frac{1}{2}(1-\cos(\theta+\chi Nt)). \end{aligned}$$

The oscillation of these probabilities determines N, and notice that $|N\rangle_a$ is not demolished after each measurement.

10.7 Problem 10.7

From (10.66) we have

$$\hat{H}_I = \mathcal{D}E_0 e^{i\varphi} e^{i\omega_L t} \exp\left[-i\eta\left(\hat{a}e^{ivt} + \hat{a}^\dagger e^{-ivt}\right)\right] \hat{\sigma}_- e^{-i\omega_0 t} + H.c.$$

As η is small, we expand to second order

$$\begin{aligned} \exp\left[-i\eta\left(\hat{a}e^{ivt} + \hat{a}^\dagger e^{-ivt}\right)\right] &\approx 1 - i\eta\left(\hat{a}e^{ivt} + \hat{a}^\dagger e^{-ivt}\right) - \frac{\eta^2}{2}\left(\hat{a}e^{ivt} + \hat{a}^\dagger e^{-ivt}\right)^2 \\ &= 1 - i\eta\left(\hat{a}e^{ivt} + \hat{a}^\dagger e^{-ivt}\right) - \frac{\eta^2}{2}\left(\hat{a}^2 e^{i2vt} + \hat{a}^{\dagger 2} e^{-i2vt} + \hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger\right) \\ &= 1 - i\eta\left(\hat{a}e^{ivt} + \hat{a}^\dagger e^{-ivt}\right) - \frac{\eta^2}{2}\left(\hat{a}^2 e^{i2vt} + \hat{a}^{\dagger 2} e^{-i2vt} + 2\hat{a}^\dagger\hat{a} + 1\right). \end{aligned}$$

Thus

$$\begin{aligned} \hat{H}_I &= \mathcal{D}E_0 e^{i\varphi} e^{i\omega_L t} \\ &\times \left[1 - i\eta\left(\hat{a}e^{ivt} + \hat{a}^\dagger e^{-ivt}\right) - \frac{\eta^2}{2}\left(\hat{a}^2 e^{i2vt} + \hat{a}^{\dagger 2} e^{-i2vt} + 2\hat{a}^\dagger\hat{a} + 1\right)\right] \hat{\sigma}_- e^{-i\omega_0 t} + H.c. \end{aligned}$$

We choose $\omega_L = \omega_0$ and throw away all terms oscillating as $e^{i(\omega_L \pm v)t}$ and $e^{i(\omega_L \pm 2v)t}$, to get

$$\begin{aligned} \hat{H}_I &= \mathcal{D}E_0 e^{i\varphi}\left[1 - \frac{\eta^2}{2}\left(2\hat{a}^\dagger\hat{a} + 1\right)\right]\hat{\sigma}_- + H.c \\ &= \mathcal{D}E_0\left(1 - \frac{\eta^2}{2}\right)\left(e^{i\varphi}\hat{\sigma}_- + e^{-i\varphi}\hat{\sigma}_+\right) - \mathcal{D}E_0\eta^2\left(e^{i\varphi}\hat{\sigma}_- + e^{-i\varphi}\hat{\sigma}_+\right)\hat{a}^\dagger\hat{a} \\ &= \hat{H}^{(1)} + \hat{H}^{(2)}, \end{aligned}$$

where

$$\hat{H}^{(1)} = \mathcal{D}E_0\left(1-\frac{\eta^2}{2}\right)\left(e^{i\varphi}\hat{\sigma}_- + e^{-i\varphi}\hat{\sigma}_+\right)$$
$$\hat{H}^{(1)} = -\mathcal{D}E_0\eta^2\left(e^{i\varphi}\hat{\sigma}_- + e^{-i\varphi}\hat{\sigma}_+\right)\hat{a}^\dagger\hat{a}.$$

It is clear that $\left[\hat{H}^{(1)}, \hat{H}^{(2)}\right] = 0$, so we can work in picture where the effective Hamiltonian is given by

$$\hat{H}_{eff} = \hbar\chi\hat{a}^\dagger\hat{a}\left(e^{i\varphi}\hat{\sigma}_- + e^{-i\varphi}\hat{\sigma}_+\right),$$

where $\chi = \mathcal{D}E_0\eta^2/\hbar$.

For convenience we now set $\varphi = 0$.

$$\hat{H}_{eff} = \hbar\chi\hat{a}^\dagger\hat{a}\left(\hat{\sigma}_- + \hat{\sigma}_+\right).$$

If the center of mass motion state of the ion is a state of n phonons, $|n\rangle$, then the dressed state $|n\pm\rangle$ are given by

$$|n\pm\rangle = \frac{1}{\sqrt{2}}\left[|n\rangle\left(|e\rangle \pm |g\rangle\right)\right]$$

with corresponding energy eigenvalues

$$E_{n\pm} = \pm\hbar\chi n.$$

Suppose now that the initial state is $|g\rangle|\alpha\rangle$. Then

$$|\Psi(t)\rangle = e^{-i\hat{H}_{eff}t/\hbar}|g\rangle|\alpha\rangle$$

with

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle.$$

We can write in terms of the dressed states

$$|g\rangle|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle|g\rangle$$
$$= \frac{1}{\sqrt{2}}e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}(|n+\rangle - |n-\rangle).$$

$$\begin{aligned}
|\Psi(t)\rangle &= e^{-i\hat{H}_{eff}t/h}|g\rangle|\alpha\rangle \\
&= \frac{1}{\sqrt{2}} e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \left(e^{-i\chi n t}|n+\rangle - e^{i\chi n t}|n-\rangle \right) \\
&= \frac{1}{2} \left(\left|\alpha e^{-i\chi t}\right\rangle (|e\rangle + |g\rangle) - \left|\alpha e^{i\chi t}\right\rangle (|e\rangle - |g\rangle) \right) \\
&= |g\rangle|S_+\rangle + |e\rangle|S_-\rangle,
\end{aligned}$$

where

$$|S_\pm\rangle = \frac{1}{2}\left[\left|\alpha e^{-i\phi/2}\right\rangle \pm \left|\alpha e^{i\phi/2}\right\rangle\right]$$

and where $\phi = 2\chi t$. The internal states of the ion are generally entangled with the vibrational state of the center of mass. Note that at $\phi = \pi$ we have $|S_\pm\rangle = \frac{1}{2}\left[|-i\alpha\rangle \pm |i\alpha\rangle\right]$, even and odd cat states.

Chapter 11

Applications of Entanglement

11.1 Problem 11.1

$$|\Psi\rangle = \frac{1}{2\sqrt{2}}\left(1 - e^{2i\theta}\right)(|2\rangle|0\rangle - |0\rangle|2\rangle) + \frac{i}{2}\left(1 + e^{2i\theta}\right)|1\rangle|1\rangle \tag{11.1.1}$$

$$\begin{aligned}\hat{\Pi}_b|\Psi\rangle &= \hat{\Pi}_b\left[\frac{1}{2\sqrt{2}}\left(1 - e^{2i\theta}\right)(|2\rangle|0\rangle - |0\rangle|2\rangle) + \frac{i}{2}\left(1 + e^{2i\theta}\right)|1\rangle|1\rangle\right] \\ &= \frac{1}{2\sqrt{2}}\left(1 - e^{2i\theta}\right)(|2\rangle|0\rangle - |0\rangle|2\rangle) - \frac{i}{2}\left(1 + e^{2i\theta}\right)|1\rangle|1\rangle\end{aligned}$$

$$\begin{aligned}\left\langle\hat{\Pi}_b\right\rangle &= \left\langle\Psi\left|\hat{\Pi}_b\right|\Psi\right\rangle \\ &= \frac{1}{4}\left(\left|1 - e^{2i\theta}\right|^2 - \left|1 + e^{2i\theta}\right|^2\right) \\ &= -\cos(2\theta)\end{aligned}$$

11.2 Problem 11.2

$$\begin{aligned}|in\rangle &= |2\rangle_a|2\rangle_b \\ &= \frac{\hat{a}^{\dagger 2}}{\sqrt{2}}\frac{\hat{b}^{\dagger 2}}{\sqrt{2}}|0\rangle_a|0\rangle_b \\ &= \frac{1}{2}\hat{a}^{\dagger 2}\hat{b}^{\dagger 2}|0\rangle_a|0\rangle_b\end{aligned}$$

$$\begin{aligned}
\hat{U}_{\rm BS1}\hat{a}^\dagger\hat{U}^\dagger_{\rm BS1} &= \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger + i\hat{b}^\dagger\right)\\
\hat{U}_{\rm BS1}\hat{b}^\dagger\hat{U}^\dagger_{\rm BS1} &= \frac{1}{\sqrt{2}}\left(i\hat{a}^\dagger + \hat{b}^\dagger\right)
\end{aligned}$$

$$\begin{aligned}
\hat{U}_{\rm BS1}|in\rangle &= \frac{1}{2}\hat{U}_{\rm BS1}\hat{a}^{\dagger 2}\hat{b}^{\dagger 2}|0\rangle_a|0\rangle_b\\
&= \frac{1}{8}\left(\hat{a}^\dagger + i\hat{b}^\dagger\right)^2\left(i\hat{a}^\dagger + \hat{b}^\dagger\right)^2|0\rangle_a|0\rangle_b\\
&= -\frac{1}{8}\left[\left(\hat{a}^\dagger + i\hat{b}^\dagger\right)\left(\hat{a}^\dagger - i\hat{b}^\dagger\right)\right]^2|0\rangle_a|0\rangle_b\\
&= -\frac{1}{8}\left(\hat{a}^{\dagger 2} + \hat{b}^{\dagger 2}\right)^2|0\rangle_a|0\rangle_b\\
&= -\frac{1}{8}\left(\hat{a}^{\dagger 4} + 2\hat{a}^{\dagger 2}\hat{b}^{\dagger 2} + \hat{b}^{\dagger 4}\right)|0\rangle_a|0\rangle_b\\
&= -\frac{1}{8}\left(\sqrt{4!}|4\rangle_a|0\rangle_b + 2\sqrt{2!}\sqrt{2!}|2\rangle_a|2\rangle_b + \sqrt{4!}|0\rangle_a|4\rangle_b\right)\\
&= -\frac{1}{8}\left(\sqrt{4!}|4\rangle_a|0\rangle_b + 4|2\rangle_a|2\rangle_b + \sqrt{4!}|0\rangle_a|4\rangle_b\right)
\end{aligned}$$

$$\begin{aligned}
\hat{U}_{\rm PS}\hat{U}_{\rm BS1}|in\rangle &= -\hat{U}_{\rm PS}\frac{1}{8}\left(\sqrt{4!}|4\rangle_a|0\rangle_b + 4|2\rangle_a|2\rangle_b + \sqrt{4!}|0\rangle_a|4\rangle_b\right)\\
&= -\frac{1}{8}\left[\sqrt{4!}\left(|4\rangle_a|0\rangle_b + e^{i4\theta}|0\rangle_a|4\rangle_b\right) + 4e^{i2\theta}|2\rangle_a|2\rangle_b\right]
\end{aligned}$$

$$\begin{aligned}
\hat{U}_{\rm BS2}|4\rangle|0\rangle &= \frac{1}{\sqrt{4!}}\hat{U}_{\rm BS1}\hat{a}^{\dagger 4}|0\rangle_a|0\rangle_b\\
&= \frac{1}{4\sqrt{4!}}\left(\hat{a}^\dagger + i\hat{b}^\dagger\right)^4|0\rangle_a|0\rangle_b\\
&= \frac{1}{4\sqrt{4!}}\left(\hat{a}^{\dagger 4} + i4\hat{a}^{\dagger 3}\hat{b}^\dagger - 6\hat{a}^{\dagger 2}\hat{b}^{\dagger 2} - i4\hat{a}^\dagger\hat{b}^{\dagger 3} + \hat{b}^{\dagger 4}\right)|0\rangle_a|0\rangle_b\\
&= \frac{1}{4\sqrt{4!}}\left[\sqrt{4!}\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\ \right) + i4\sqrt{3!}\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) - 12|2\rangle_a|2\rangle_b\right]
\end{aligned}$$

Also

$$\hat{U}_{\rm BS2}|0\rangle|4\rangle = \frac{1}{4\sqrt{4!}}\left[\sqrt{4!}\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\ \right) - i4\sqrt{3!}\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) - 12|2\rangle_a|2\rangle_b\right]$$

Also

$$\hat{U}_{\text{BS2}}\left(|4\rangle|0\rangle + e^{i4\theta}|0\rangle|4\rangle\right) = \frac{1}{4\sqrt{4!}}\Big[\sqrt{4!}\left(1+e^{i4\theta}\right)\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\ \right)$$
$$+i4\sqrt{3!}\left(1-e^{i4\theta}\right)\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) - 12\left(1+e^{i4\theta}\right)|2\rangle_a|2\rangle_b\Big]$$

$$\hat{U}_{\text{BS2}} e^{i2\theta}|2\rangle|2\rangle = -\frac{1}{8}e^{i2\theta}\left(\sqrt{4!}(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b) + 4|2\rangle_a|2\rangle_b\right)$$

$$\begin{aligned}
|out\rangle &= \hat{U}_{\text{BS2}}\hat{U}_{\text{PS}}\hat{U}_{\text{BS1}}|in\rangle \\
&= -\frac{1}{8}\hat{U}_{\text{BS2}}\left[\sqrt{4!}\left(|4\rangle_a|0\rangle_b + e^{i4\theta}|0\rangle_a|4\rangle_b\right) + 4e^{i2\theta}|2\rangle_a|2\rangle_b\right] \\
&= -\frac{1}{8}\Bigg\{\frac{1}{4}\Big[\sqrt{4!}\left(1+e^{i4\theta}\right)\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\ \right) \\
&\quad +i4\sqrt{3!}\left(1-e^{i4\theta}\right)\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) - 12\left(1+e^{i4\theta}\right)|2\rangle_a|2\rangle_b\Big] \\
&\quad -\frac{1}{2}e^{i2\theta}\left(\sqrt{4!}(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b) + 4|2\rangle_a|2\rangle_b\right)\Bigg\} \\
&= -\frac{1}{8}\Bigg\{\left[\frac{\sqrt{4!}}{4}\left(1+e^{i4\theta}\right) - \frac{\sqrt{4!}}{2}e^{i2\theta}\right]\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\ \right) \\
&\quad + i\sqrt{3!}\left(1-e^{i4\theta}\right)\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) - \left[3\left(1+e^{i4\theta}\right) + 2e^{i2\theta}\right]|2\rangle_a|2\rangle_b\Bigg\}\,.
\end{aligned}$$

$$\begin{aligned}
\hat{\Pi}_b|out\rangle = -\frac{1}{8}\Bigg\{&\left[\frac{\sqrt{4!}}{4}\left(1+e^{i4\theta}\right) - \frac{\sqrt{4!}}{2}e^{i2\theta}\right]\left(|4\rangle_a|0\rangle_b + |0\rangle_a|4\rangle_b\ \right) \\
&- i\sqrt{3!}\left(1-e^{i4\theta}\right)\left(|3\rangle_a|1\rangle_b - |1\rangle_a|3\rangle_b\right) - \left[3\left(1+e^{i4\theta}\right) + 2e^{i2\theta}\right]|2\rangle_a|2\rangle_b\Bigg\}
\end{aligned}$$

$$\begin{aligned}
\langle out|\hat{\Pi}_b|out\rangle &= \frac{1}{64}\left\{2\left|\frac{\sqrt{4!}}{4}\left(1+e^{i4\theta}\right) - \frac{\sqrt{4!}}{2}e^{i2\theta}\right|^2 - 2\left|\sqrt{3!}\left(1-e^{i4\theta}\right)\right|^2 + \left|3\left(1+e^{i4\theta}\right) + 2e^{i2\theta}\right|^2\right\} \\
&= \frac{1}{4}\left(1 + 3\cos(4\theta)\right)
\end{aligned}$$

where

$$\Delta\hat{\Pi}_b = \sqrt{1 - \left\langle\hat{\Pi}_b\right\rangle^2} \tag{11.2.1}$$

$$
\begin{aligned}
\Delta\theta &= \Delta\hat{\Pi}_b / \left| \frac{\partial \left\langle \hat{\Pi}_b \right\rangle}{\partial\theta} \right| \\
&= \frac{\sqrt{1 - \frac{1}{16}(1 + 3\cos(4\theta))^2}}{4|\sin(4\theta)|}
\end{aligned}
$$

11.3 Problem 11.3

$$
\begin{aligned}
|\psi_N\rangle &= \frac{1}{\sqrt{2}} \left(|N\rangle_a |0\rangle_b + e^{i\Phi_N} |N\rangle_a |0\rangle_b \right) \\
\hat{U}_{PS} |\psi_N\rangle &= \frac{1}{\sqrt{2}} \left(|N\rangle_a |0\rangle_b + e^{i(\Phi_N + N\theta)} |N\rangle_a |0\rangle_b \right) \\
|out\rangle &= \hat{U}_{\text{BS2}} \hat{U}_{PS} |\psi_N\rangle = e^{i\frac{\pi}{2}\hat{J}_x} \hat{U}_{PS} |\psi_N\rangle
\end{aligned}
$$

$$
\begin{aligned}
\left\langle \hat{\Pi}_b \right\rangle &= \langle out | e^{i\pi \hat{b}^\dagger \hat{b}} | out\rangle = \langle out | e^{i\pi(\hat{J}_0 - \hat{J}_3)} | out \rangle \\
&= \langle \psi_N | \hat{U}^\dagger_{\text{BS2}} e^{-i\frac{\pi}{2}\hat{J}_x} e^{i\pi(\hat{J}_0 - \hat{J}_3)} e^{i\frac{\pi}{2}\hat{J}_x} \hat{U}_{\text{BS2}} |\psi_N\rangle \\
&= \langle \psi_N | \hat{U}^\dagger_{\text{BS2}} e^{i\pi\hat{J}_0} e^{-i\pi\hat{J}_2} \hat{U}_{\text{BS2}} |\psi_N\rangle \\
&= \frac{1}{2} \left({}_a\langle N| {}_b\langle 0| + e^{-i(\Phi_N + N\theta)} {}_a\langle 0| {}_b\langle N| \right) e^{i\pi\hat{J}_0} e^{-i\pi\hat{J}_2} \left(|N\rangle_a |0\rangle_b + e^{i(\Phi_N + N\theta)} |0\rangle_a |N\rangle_b \right) \\
&= \frac{1}{2} \left({}_a\langle N| {}_b\langle 0| + e^{-i(\Phi_N + N\theta)} {}_a\langle 0| {}_b\langle N| \right) e^{i\pi N} \left(|0\rangle_a |N\rangle_b + (-1)^N e^{i(\Phi_N + N\theta)} |N\rangle_a |0\rangle_b \right) \\
&= \begin{cases} \cos(\Phi_N - N\theta) & \text{for even } \; N, \\ -\sin(\Phi_N - N\theta) & \text{for odd } N. \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\Delta\theta &= \frac{\Delta\hat{\Pi}_b}{\left| \partial \langle \hat{\Pi}_b \rangle / \partial\theta \right|} \\
&= \frac{\sqrt{1 - \left\langle \hat{O} \right\rangle^2}}{\left| \partial \langle \hat{\Pi}_b \rangle / \partial\theta \right|} \\
&= \frac{1}{N}.
\end{aligned}
$$

Where we have used the following identities

$$e^{i\pi \hat{J}_2}|n\rangle_a|m\rangle_b = (-1)^m|m\rangle_a|n\rangle_b$$
$$e^{-i\frac{\pi}{2}\hat{J}_x}e^{-i\pi\hat{J}_3}e^{i\frac{\pi}{2}\hat{J}_x} = e^{-i\pi\hat{J}_2}.$$

11.4 Problem 11.4

$$\hat{\rho}_{AB} = \frac{1}{2}(|0\rangle_A|0\rangle_{BA}\langle 0|_A\langle 0| + |0\rangle_A|0\rangle_{BA}\langle 0|_A\langle 0|)$$

Assume Alice has an unknown state $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$ that we want to teleport to Bob. We will follow the same procedure in the text. Basically forming the following density operator

$$\hat{\rho} = |\psi\rangle\langle\psi| \otimes \hat{\rho}_{AB}$$

and do the measurements along $|\Phi^{\pm}\rangle$ and $|\Psi^{\pm}\rangle$. According to the output we would apply the appropriate operator to retrieve the unknown state $|\psi\rangle$. After measurements we found that

$$\langle\Psi^{\pm}|\hat{\rho}|\Psi^{\pm}\rangle = |c_0|^2|0\rangle_A\ {}_A\langle 0| + |c_1|^2|1\rangle_A\ {}_A\langle 1|$$
$$\langle\Phi^{\pm}|\hat{\rho}|\Phi^{\pm}\rangle = |c_1|^2|0\rangle_A\ {}_A\langle 0| + |c_0|^2|1\rangle_A\ {}_A\langle 1|$$

which is a statistical mixture for all possible cases. Obviously teleporting a state with the shared statistical mixture is impossible.

11.5 Problem 11.5

Let

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A|0\rangle_B|0\rangle_C - |1\rangle_A|1\rangle_B|1\rangle_C\right)$$

be the shared state, and let

$$|\psi\rangle = c_0|0\rangle + c_1|1\rangle$$

be the unknown state that Alice wants to teleport to both Bob and Claire. Following the procedure introduced in the text we have

$$
\begin{aligned}
|\Psi_{\mathrm{ABC}}\rangle &= |\psi\rangle|\Psi\rangle \\
&= \frac{1}{\sqrt{2}}\left(c_0|0\rangle + c_1|1\rangle\right)\left(|0\rangle_A|0\rangle_B|0\rangle_C - |1\rangle_A|1\rangle_B|1\rangle_C\right) \\
&= \frac{1}{\sqrt{2}}(c_0|0\rangle|0\rangle_A|0\rangle_B|0\rangle_C + c_1|1\rangle|0\rangle_A|0\rangle_B|0\rangle_C \\
7 &- c_0|0\rangle|1\rangle_A|1\rangle_B|1\rangle_C - c_1|1\rangle|1\rangle_A|1\rangle_B|1\rangle_C) \\
&= \frac{1}{2}\left|\Phi^+\right\rangle\left(c_0|0\rangle_B|0\rangle_C - c_1|1\rangle_B|1\rangle_C\right) + \frac{1}{2}\left|\Phi^-\right\rangle\left(c_0|0\rangle_B|0\rangle_C + c_1|1\rangle_B|1\rangle_C\right) \\
&+ \frac{1}{2}\left|\Psi^+\right\rangle\left(c_1|0\rangle_B|0\rangle_C + c_0|1\rangle_B|1\rangle_C\right) + \frac{1}{2}\left|\Psi^-\right\rangle\left(c_1|0\rangle_B|0\rangle_C - c_0|1\rangle_B|1\rangle_C\right).
\end{aligned}
$$

Clearly a measurement along the $|\Psi^{\pm}\rangle$ or $|\Phi^{\pm}\rangle$ will collapse state $|\Psi_{\mathrm{ABC}}\rangle$ into an entangled state between Bob and Claire of the form $\left(c_{\substack{0\\1}}|0\rangle_B|0\rangle_C \pm c_{\substack{1\\0}}|1\rangle_B|1\rangle_C\right)$. So $|\psi\rangle$ is not teleported to Bob and Claire at the same time. Also notice that Bob and Claire share the information about $|\psi\rangle$.

11.6 Problem 11.6

It is easy to show that

$$
\begin{aligned}
\hat{U}_H|0\rangle &= \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right]|0\rangle \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right) \\
\hat{U}_H|1\rangle &= \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right]|1\rangle \\
&= \frac{1}{\sqrt{2}}\left(|0\rangle - |1\rangle\right).
\end{aligned}
$$

In deed the unitary operator of the Hadamard gate can be represented as $\hat{U}_H = \frac{1}{\sqrt{2}}\left[|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| - |1\rangle\langle 1|\right]$

11.7 Problem 11.7

$$\hat{X} = |0\rangle\langle 1| + |1\rangle\langle 0|$$

$$\hat{U}_{\text{C-NOT}}|x\rangle|y\rangle = |x\rangle|\text{mod}_2(x+y)\rangle$$

equivalently we can write

$$\hat{U}_{\text{C-NOT}}|0\rangle|y\rangle = |0\rangle|y\rangle$$

and

$$\hat{U}_{\text{C-NOT}}|1\rangle|y\rangle = |1\rangle\left|\text{mod}_2(1+y)\right\rangle .$$

Now let investigate the following representation of the C-NOT gate

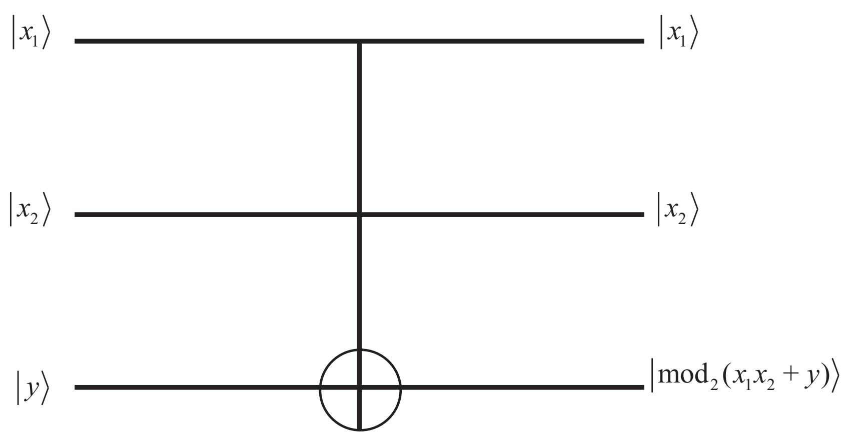
$$\begin{aligned}\hat{U}'_{\text{C-NOT}} &= |0\rangle\langle 0| \otimes \hat{I} + |1\rangle\langle 1| \otimes \hat{X} \\ &= |0\rangle\langle 0| \otimes (|0\rangle\langle 0| + |1\rangle\langle 1|) + |1\rangle\langle 1| \otimes (|0\rangle\langle 1| + |1\rangle\langle 0|)\end{aligned}$$

It is easy to see that

$$\begin{aligned}\hat{U}'_{\text{C-NOT}}|0\rangle|y\rangle &= |0\rangle|y\rangle \\ \hat{U}'_{\text{C-NOT}}|1\rangle|y\rangle &= |1\rangle\left|\text{mod}_2(1+y)\right\rangle .\end{aligned}$$

Obviously $\hat{U}_{\text{C-NOT}}$ and $\hat{U}'_{\text{C-NOT}}$ are identical.

11.8 Problem 11.8



Let $\hat{U}_{TG}$ be a unitary transformation such that

$$\hat{U}_{TG}|x_1\rangle|x_2\rangle|y\rangle = |x_1\rangle|x_2\rangle|\mathrm{mod}_2(x_1x_2+y)\rangle$$

which we can rewrite as

$$\hat{U}_{TG} = \hat{U}_{CCN} = |0\rangle_{a\ a}\langle 0| \otimes \hat{I}_b \otimes \hat{I}_c + |1\rangle_{a\ a}\langle 1| \otimes \hat{U}_{CN},$$

where $\hat{U}_{CN} = |0\rangle_{b\ b}\langle 0| \otimes \hat{I}_c + |1\rangle_{b\ b}\langle 1| \otimes \hat{X}_c$. Obviously $\hat{U}_{TG}$ is a controlled-controlled-not gate.

11.9 Problem 11.9

The Toffoli gate is a 3-qubit gate given by

$$\hat{U}_{TG}|y\rangle|x_1\rangle|x_2\rangle = |\mathrm{mod}_2(x_1x_2+y)\rangle|x_1\rangle|x_2\rangle,$$

where we have set the first qubit as target. The qubits are identified as in Eq. (11.45) by

$$|y\rangle|x_1\rangle|x_2\rangle = |\ ,\ \rangle_{a,b}|\ ,\ \rangle_{c,d}|\ ,\ \rangle_{e,f},$$

where

$$|0\rangle|0\rangle|0\rangle = |0,1\rangle_{a,b}|0,1\rangle_{c,d}|0,1\rangle_{e,f},$$
$$|0\rangle|0\rangle|1\rangle = |0,1\rangle_{a,b}|0,1\rangle_{c,d}|1,0\rangle_{e,f},$$
$$\text{etc.}$$

We assume the interaction among the modes in the Kerr medium as

$$\hat{U}_{\mathrm{Kerr}}(\pi) = \exp\left(i\pi\hat{b}^\dagger\hat{b}\hat{c}^\dagger\hat{c}\hat{e}^\dagger\hat{e}\right).$$

It is clear that only modes b, c, and e are coupled. Taking into account the action of both beam splitters, we can write the unitary transformation representing the Toffoli gate as

$$\hat{U}_{\mathrm{Kerr}}(\pi) = \exp\left[i\frac{\pi}{2}\hat{c}^\dagger\hat{c}\hat{e}^\dagger\hat{e}\left(\hat{a}^\dagger\hat{a}+\hat{b}^\dagger\hat{b}\right)\right]\exp\left[i\frac{\pi}{2}\hat{c}^\dagger\hat{c}\hat{e}^\dagger\hat{e}\left(\hat{a}^\dagger\hat{a}-\hat{b}^\dagger\hat{b}\right)\right].$$

We know that only the input states $|y\rangle|1\rangle|1\rangle$ will create a transformation. In fact, it is easy to show that

$$\begin{aligned}\hat{U}_{\mathrm{TG}}|0\rangle|1\rangle|1\rangle &= \hat{U}_{\mathrm{TG}}|0,1\rangle_{a,b}|1,0\rangle_{c,d}|1,0\rangle_{e,f}\\ &= |1,0\rangle_{a,b}|1,0\rangle_{c,d}|1,0\rangle_{e,f}\\ &= -|1\rangle|1\rangle|1\rangle\end{aligned}$$

and

$$\begin{aligned}
\hat{U}_{\mathrm{TG}}|1\rangle|1\rangle|1\rangle &= \hat{U}_{\mathrm{TG}}|1,0\rangle_{a,b}|1,0\rangle_{c,d}|1,0\rangle_{e,f} \\
&= |0,1\rangle_{a,b}|1,0\rangle_{c,d}|1,0\rangle_{e,f} \\
&= |0\rangle|1\rangle|1\rangle.
\end{aligned}$$

Thus we have designed an optical realization of the Toffoli gate, apart from an irrelevant phase factor.

11.10 Problem 11.10

Suppose $|\phi_1\rangle$ and $|\phi_2\rangle$ are orthogonal states, the 2 qubits, that can be cloned according to

$$\begin{aligned}
\hat{U}|\phi_1\rangle|0\rangle &= |\phi_1\rangle|\phi_1\rangle \\
\hat{U}|\phi_2\rangle|0\rangle &= |\phi_2\rangle|\phi_2\rangle
\end{aligned}$$

where $\hat{U}$ is the supposed unitary cloning operator. Now consider the superposition

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|\phi_1\rangle + |\phi_2\rangle\right),$$

If U is a unitary cloning operator we should have

$$\begin{aligned}
\hat{U}|\psi\rangle|0\rangle &= |\psi\rangle|\psi\rangle \\
&= \frac{1}{2}(|\phi_1\rangle|\phi_1\rangle + |\phi_2\rangle|\phi_2\rangle + |\phi_1\rangle|\phi_2\rangle + |\phi_2\rangle|\phi_1\rangle)
\end{aligned}$$

But

$$\begin{aligned}
\hat{U}|\psi\rangle|0\rangle &= \frac{1}{\sqrt{2}}(\hat{U}|\phi_1\rangle|0\rangle + \hat{U}|\phi_2\rangle|0\rangle) \\
&= \frac{1}{\sqrt{2}}(|\phi_1\rangle|\phi_1\rangle + |\phi_2\rangle|\phi_2\rangle) \neq |\psi\rangle|\psi\rangle
\end{aligned}$$

Thus $\hat{U}$ does not exist.